Dynamic Networks: Recent Results and Applications

Anna Nagurney

John F. Smith Memorial Professor
Isenberg School of Management
University of Massachusetts - Amherst

RADCLIFFE INSTITUTE FOR ADVANCED STUDY
HARVARD UNIVERSITY

Exploratory Seminar

Dynamic Networks: Behavior, Optimization, and Design

October 20-21, 2006

Co-organizer -Professor David Parkes - Harvard University

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks
http://supernet.som.umass.edu

Funding for this Exploratory Seminar provided by:

RADCLIFFE INSTITUTE FOR ADVANCED STUDY
HARVARD UNIVERSITY

We thank the Radcliffe Institute for Advanced Study for its support and for help in the organization of this seminar.

Funding for Anna Nagurney's research has been provided by:

National Science Foundation

AT&T Foundation

John F. Smith Memorial Fund - University of Massachusetts at Amherst

THE ROCKEFELLER FOUNDATION

RADCLIFFE INSTITUTE FOR ADVANCED STUDY

HARVARD UNIVERSITY

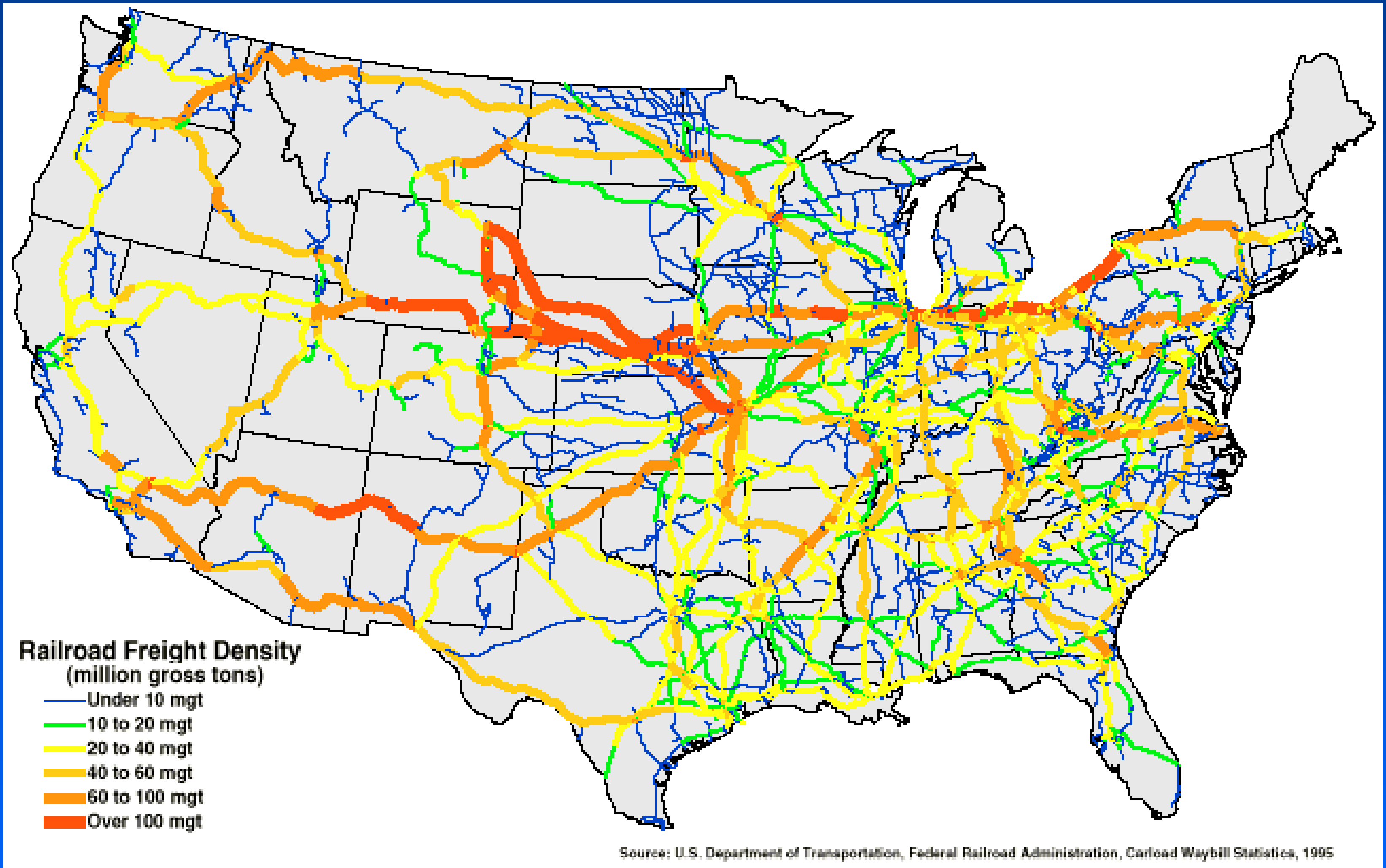
Outline of Presentation:

- Background
- Brief History of the Science of Networks
- Interdisciplinary Impact of Networks
- The Braess Paradox
- Methodological Tools
- Some Interesting Applications
- The Time-Dependent Braess Paradox
- The Internet as a Dynamic Network
- New Challenges and Opportunities: Unification of Evolutionary Variational Inequalities and Projected Dynamical Systems

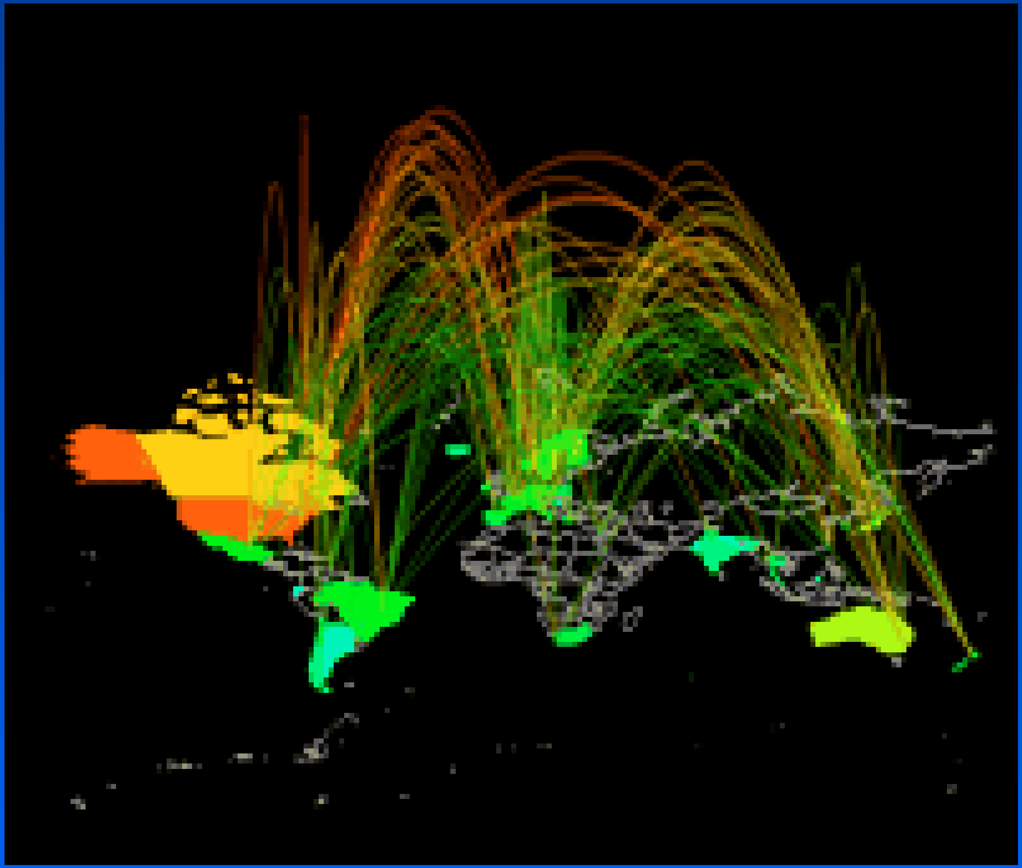
Components of Common Physical Networks

Network System	Nodes	Links	Flows
Transportation	Intersections, Homes, Workplaces, Airports, Railyards	Roads, Airline Routes, Railroad Track	Automobiles, Trains, and Planes,
Manufacturing and logistics	Workstations, Distribution Points	Processing, Shipment	Components, Finished Goods
Communication	Computers, Satellites, Telephone Exchanges	Fiber Optic Cables Radio Links	Voice, Data, Video
Energy	Pumping Stations, Plants	Pipelines, Transmission Lines	Water, Gas, Oil, Electricity

US Railroad Freight Flows

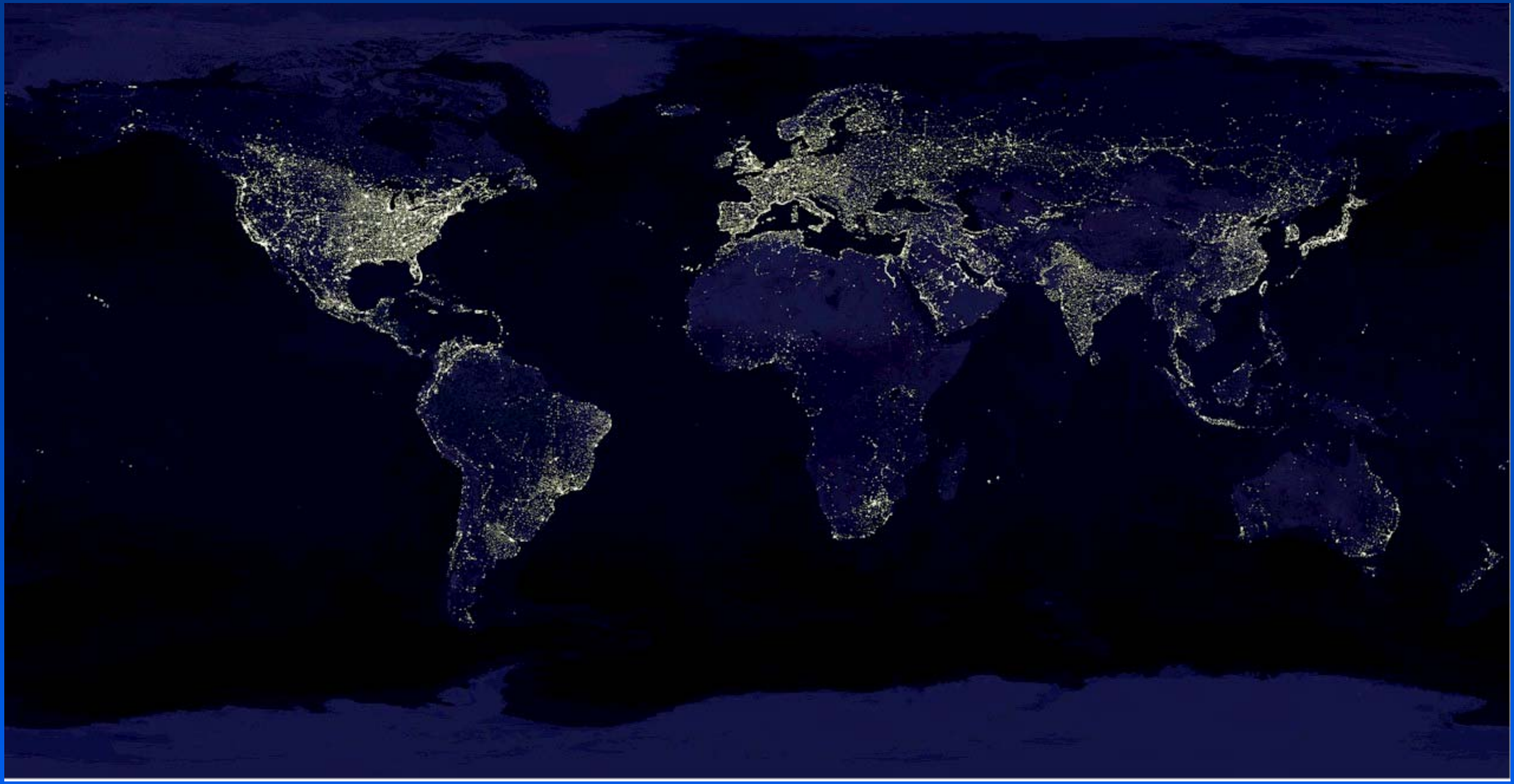


Internet Traffic Flows Over One 2 Hour Period



from Stephen Eick, Visual Insights

Electricity is Modernity



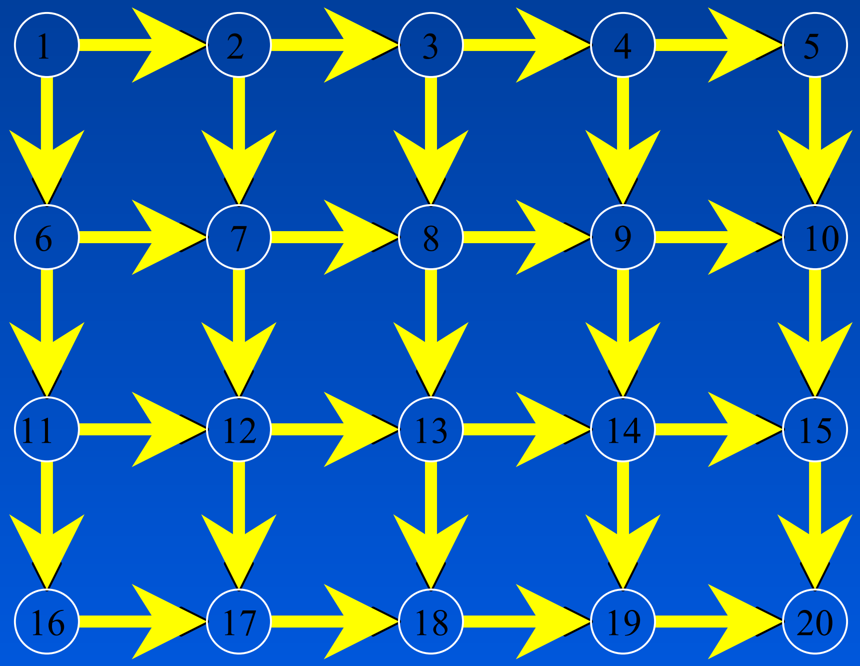
The scientific study of networks involves:

- how to **model** such applications as **mathematical entities**,
- how to **study the models** qualitatively,
- how to design **algorithms** to solve the resulting models.

The basic components of networks are:

- Nodes
- Links or arcs
- Flows

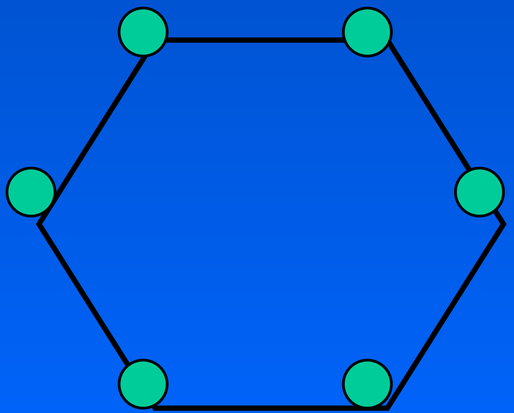
Nodes Links Flows



Brief History of the Science of Networks

1736 - Euler - the earliest paper on graph theory - Konigsberg bridges problem.

1758 - Quesnay in his *Tableau Economique* introduced a graph to depict the circular flow of financial funds in an economy.



1781 - Monge, who had worked under Napoleon Bonaparte, publishes what is probably the first paper on transportation in minimizing cost.

1838 - Cournot states that a competitive price is determined by the intersection of supply and demand curves in the context of spatially separate markets in which transportation costs are included.

1841 - Kohl considered a two node, two route transportation network problem.

1845 - Kirchhoff wrote Laws of Closed Electric Circuits.

1920 - Pigou studied a transportation network system of two routes and noted that the decision-making behavior of the users on the network would result in different flow patterns.

1936 - Konig published the first book on graph theory.

1939, 1941, 1947 - Kantorovich, Hitchcock, and Koopmans considered the network flow problem associated with the classical minimum cost transportation problem and provided insights into the special network structure of these problems, which yielded special-purpose algorithms.

1948, 1951 - Dantzig published the simplex method for linear programming and adapted it for the classical transportation problem.

1951 - Enke showed that spatial price equilibrium problems can be solved using electronic circuits

1952 - Copeland in his book asked, *Does money flow like water or electricity?*

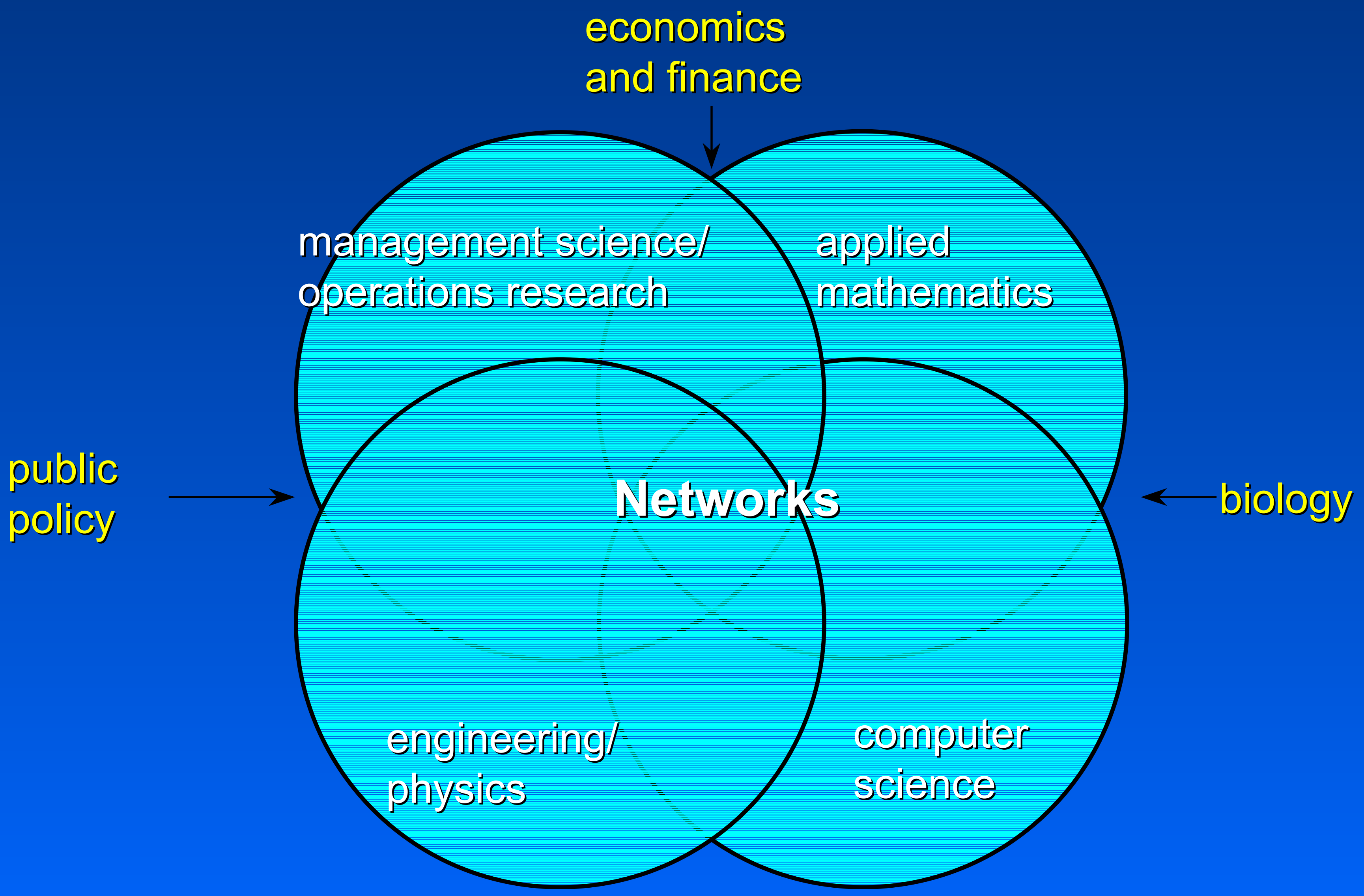
1952 - Samuelson gave a rigorous mathematical formulation of spatial price equilibrium and emphasized the network structure.

1956 - Beckmann, McGuire, and Winsten in their book, *Studies in the Economics of Transportation*, provided a rigorous treatment of congested urban transportation systems under different behavioral mechanisms due to Wardrop (1952).

1962 - Ford and Fulkerson publish *Flows in Networks*.

1969 - Dafermos and Sparrow coined the terms *user-optimization* and system-optimization and develop algorithms for the computation of solutions that exploit the network structure of transportation problems.

Networks in Different Disciplines



Interdisciplinary Impact of Networks

Management Science

Networks

Economics

Interregional Trade

General Equilibrium

Industrial Organization

Portfolio Optimization

Flow of Funds Accounting

Engineering

Energy

Manufacturing

Telecommunications

Transportation

Sociology

Social Networks

Organizational Theory

Computer Science

Routing Algorithms

Biology

DNA Sequencing

Targeted Cancer Therapy

Characteristics of Networks Today

- *large-scale nature* and complexity of network topology;
- *congestion*;
- alternative behavior of users of the network, which may lead to *paradoxical phenomena*;
- the *interactions among networks* themselves such as in transportation versus telecommunications;
- *policies* surrounding networks today may have a *major impact* not only economically but also *socially, politically, and security-wise*.

- *alternative behaviors of the users of the network*
  - system-optimized versus
  - user-optimized (network equilibrium),

  which may lead to

  paradoxical phenomena.

Transportation science has historically been the discipline that has pushed the frontiers in terms of methodological developments for such problems (which are often large-scale) beginning with the work of Beckmann, McGuire, and Winsten (1956).

Definition: Transportation Network Equilibrium

A route flow pattern $x^* \in K$ is said to be a transportation network equilibrium (according to Wardrop's (1952) first principle) if only the minimum cost routes are used (that is, have positive flow) for each O/D pair. The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every path $p \in P_w$:

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \textit{if} \quad x_p^* > 0, \\ \geq 0, & \textit{if} \quad x_p^* = 0. \end{cases}$$

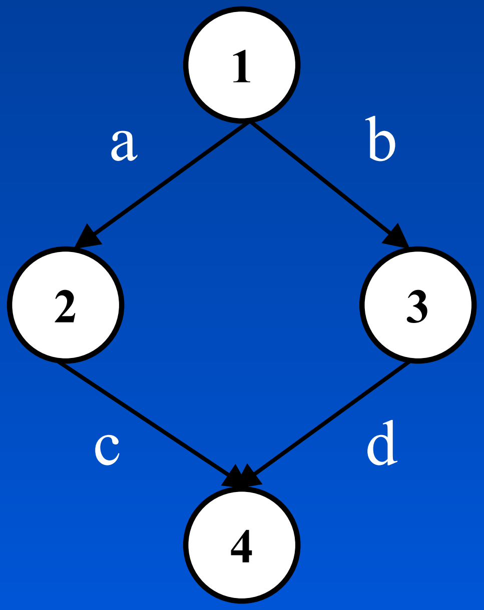
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of **6,** the equilibrium path flows are $\mathbf{x_{p_1}{}^* = x_{p_2}{}^* = 3}$ and

The equilibrium path travel cost is

$\mathbf{C_{p_1} = C_{p_2} = 83}$.



$$\mathbf{c_a(f_a)=10\, f_a \quad c_b(f_b) = f_b+50}$$

$$\mathbf{c_c(f_c) = f_c+50 \quad c_d(f_d) = 10\, f_d}$$

Adding a Link
Increases Travel Cost for All!

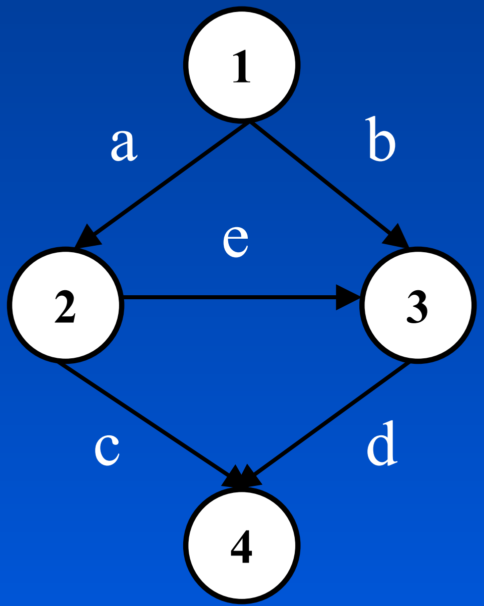
Adding a new link creates a new path **p_3=(a,e,d).**

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path **p_3, C_{p_3}=70.**

The new equilibrium flow pattern network is

$x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2$.

The equilibrium path travel costs: **$C_{p_1} = C_{p_2} = C_{p_3} = 92$.**



$c_e(f_e) = f_e + 10$

The 1968 Braess article has been translated from German to English and appears as

On a Paradox of Traffic Planning

by Braess, Nagurney, Wakolbinger

in the November 2005 issue of *Transportation Science.*

The tools that we are using in our Dynamic Network research include:

- network theory
- optimization theory
- game theory
- variational inequality theory
- evolutionary variational inequality theory
- projected dynamical systems theory
- double-layered dynamics theory
- network visualization tools.

Dafermos (1980) showed that the transportation network equilibrium (also referred to as user-optimization) conditions as formulated by Smith (1979) were a finite-dimensional variational inequality.

In 1993, Dupuis and Nagurney proved that the set of solutions to a variational inequality problem coincided with the set of solutions to a projected dynamical system (PDS) in R^n.

In 1996, Nagurney and Zhang published **Projected Dynamical Systems and Variational Inequalities**.

In 2002, Cojocaru proved the 1993 result for Hilbert Spaces.

VI Formulation of Transportation Network Equilibrium (Dafermos (1980), Smith (1979))

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

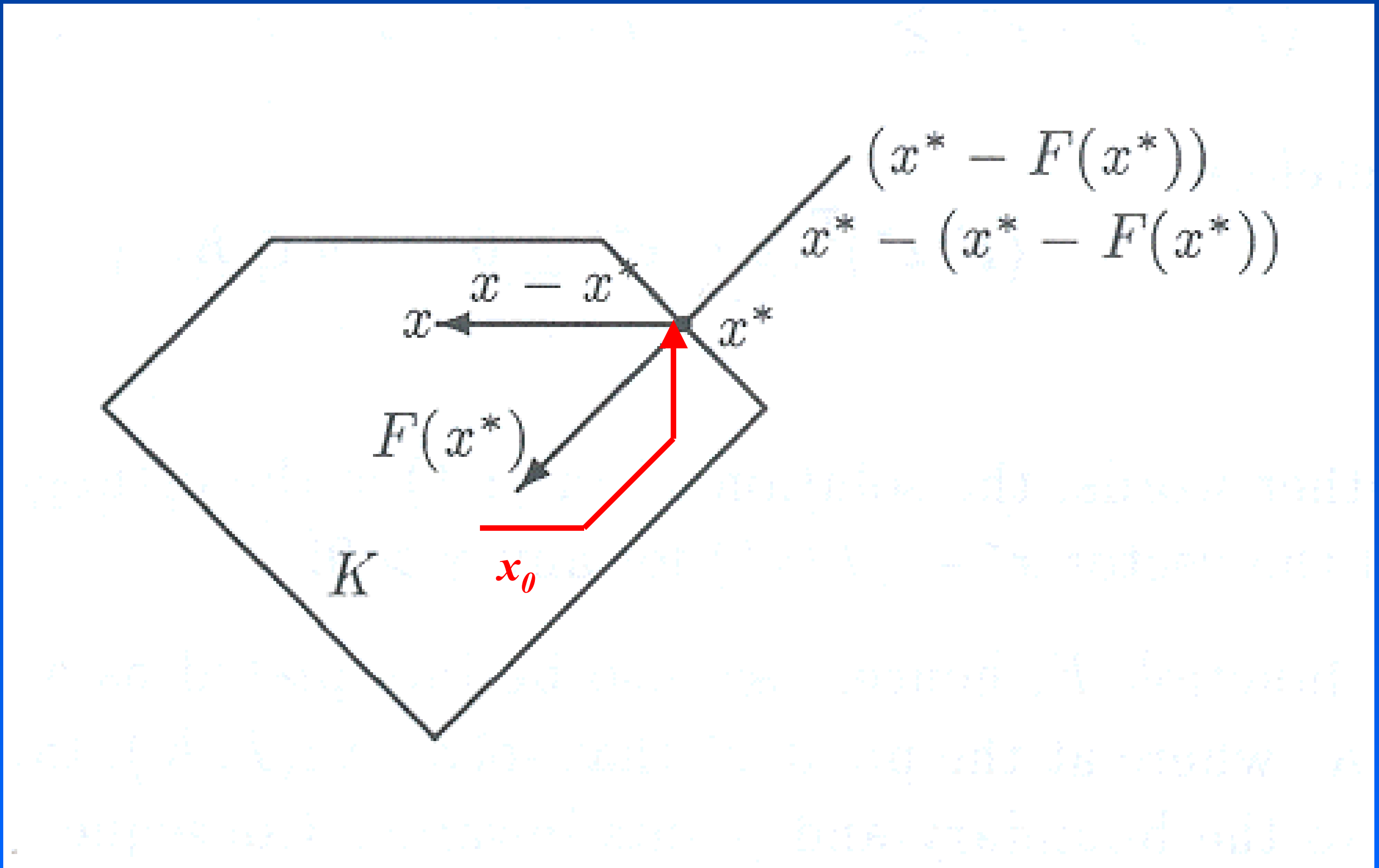
Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

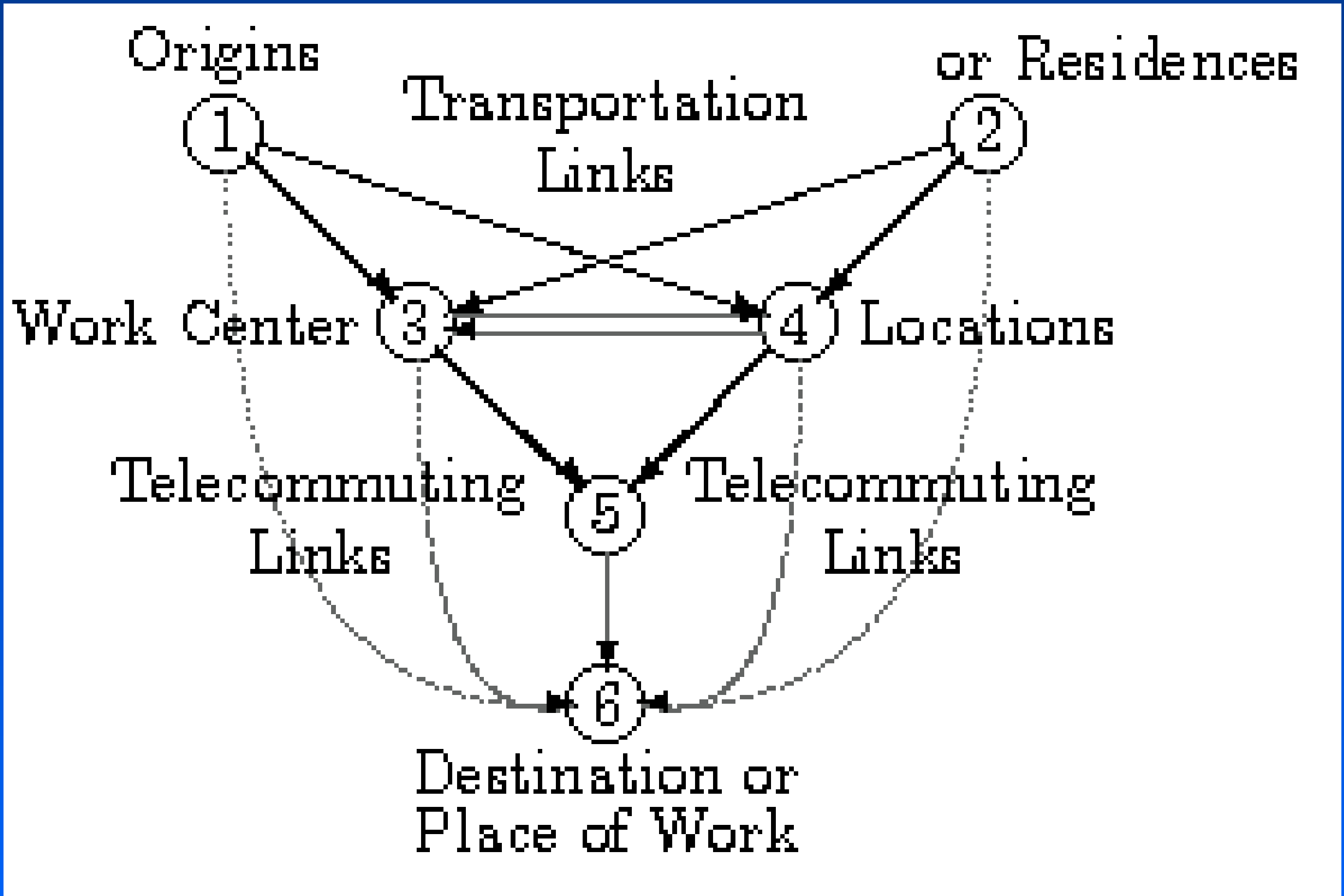
A Geometric Interpretation of a Variational Inequality and a Projected Dynamical System (Dupuis and Nagurney (1993), Nagurney and Zhang (1996))



Some Interesting Applications

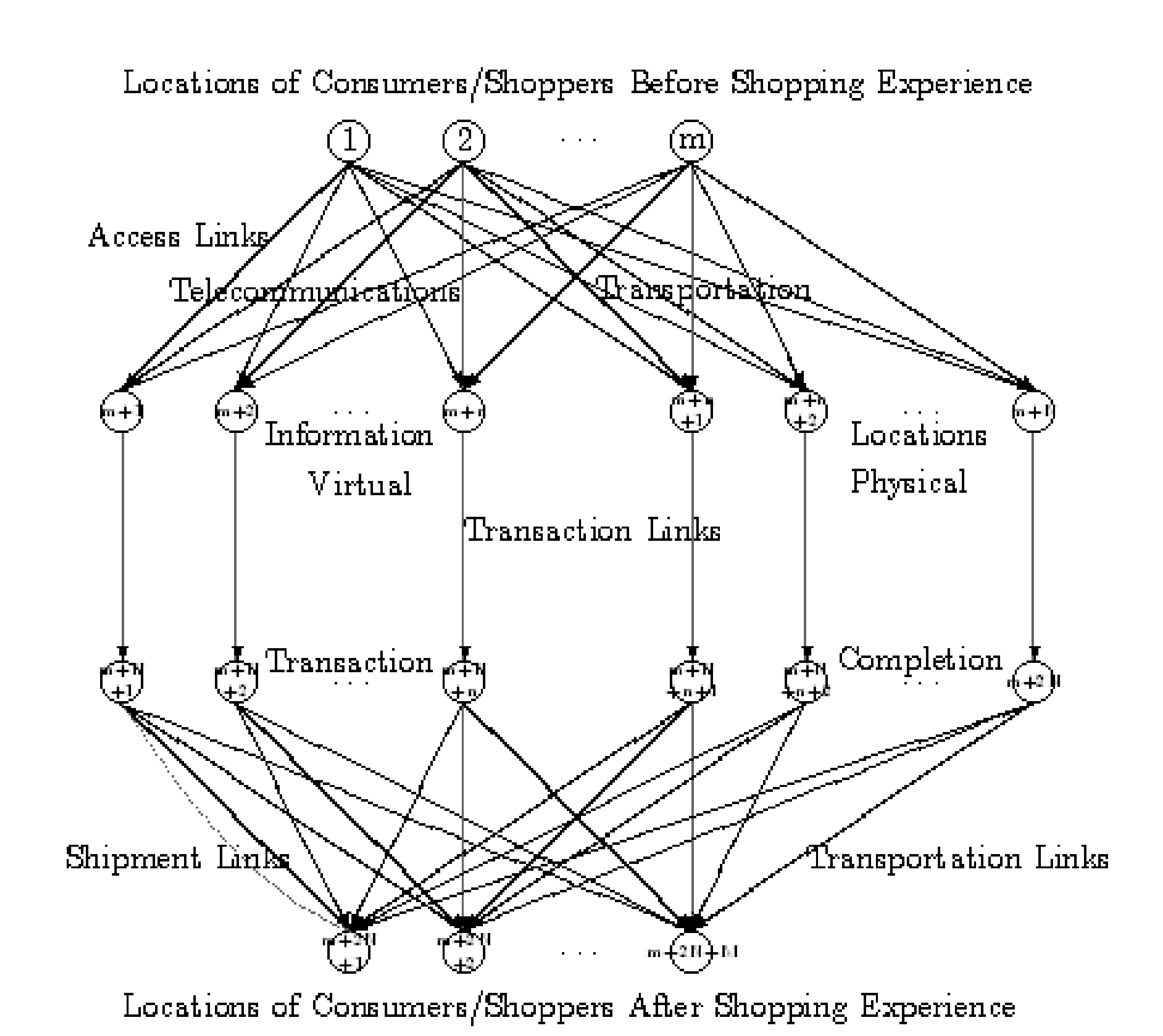
- Telecommuting/Commuting Decision-Making
- Teleshopping/Shopping Decision-Making
- Supply Chain Networks with Electronic Commerce
- Financial Networks with Electronic Transactions
- Reverse Supply Chains with E-Cycling
- Knowledge Networks
- Energy Networks/Power Grids
- Social Networks integrated with Economic Networks

A Conceptualization of Commuting versus Telecommuting



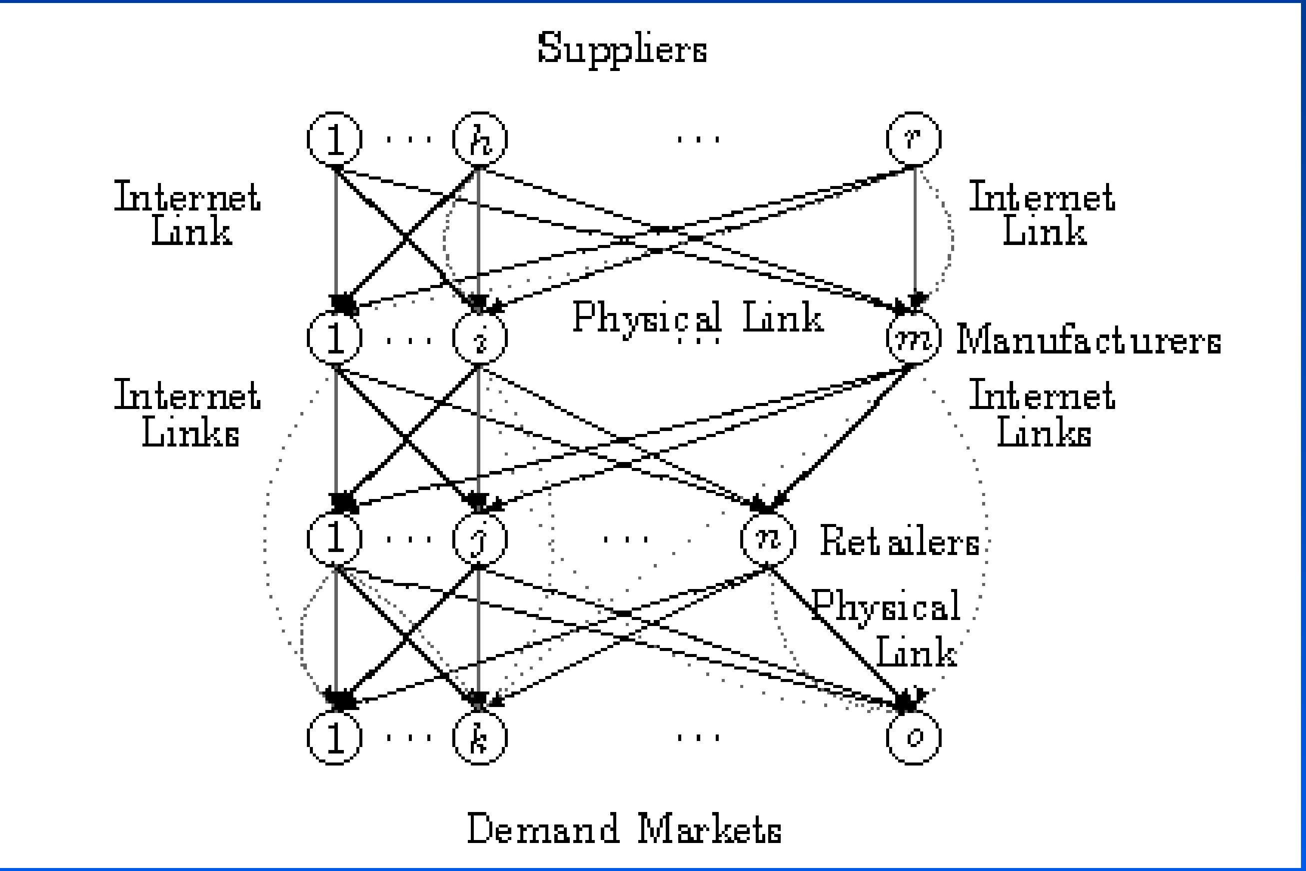
Nagurney, Dong, and Mokhtarian, *JEDC* (2002)

A Framework for Teleshopping versus Shopping



Nagurney, Dong, and Mokhtarian, *JEDC* (2002)

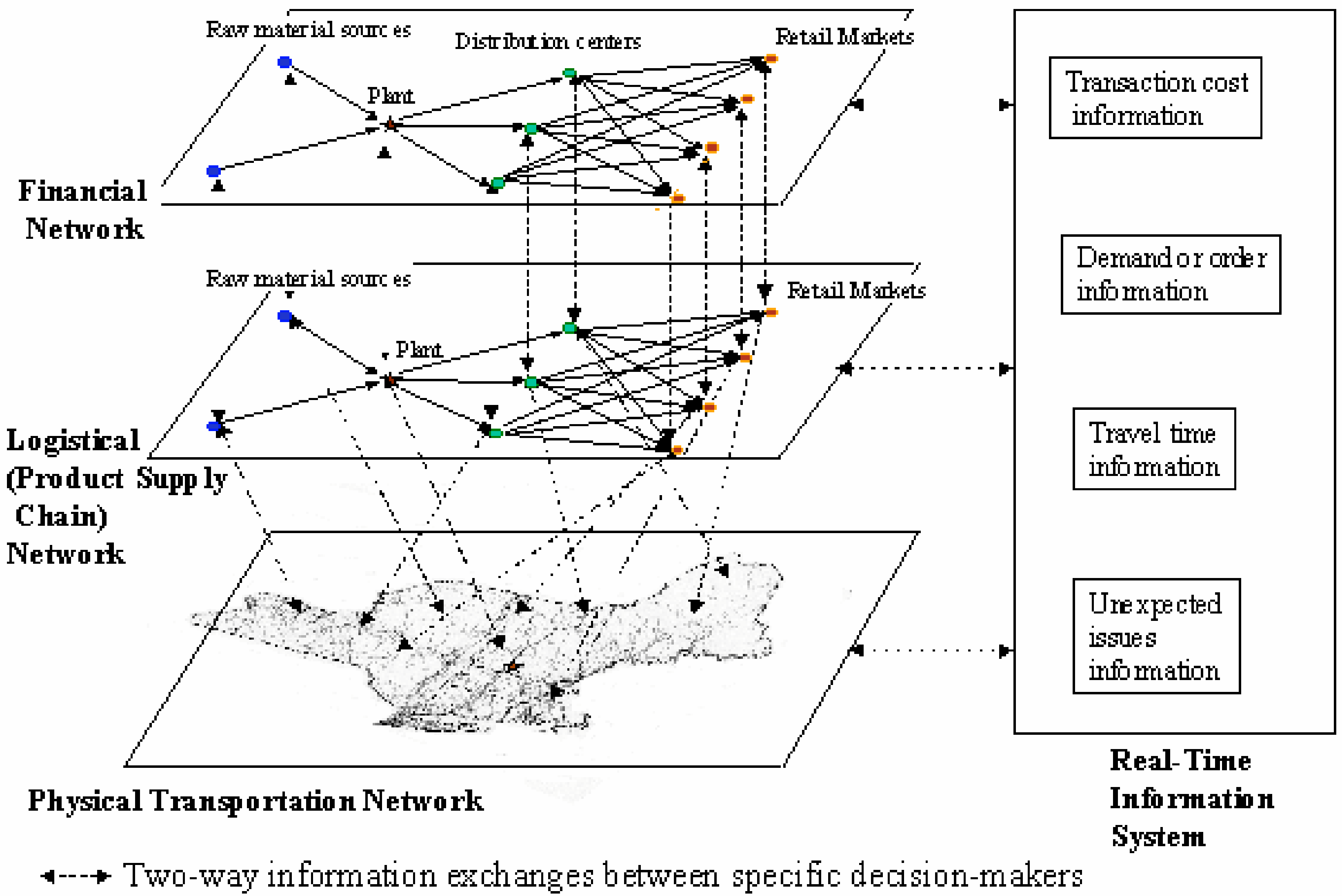
The Structure of a Supply Chain Network



Nagurney, Dong, and Zhang, *Transportation Research E* (2002)

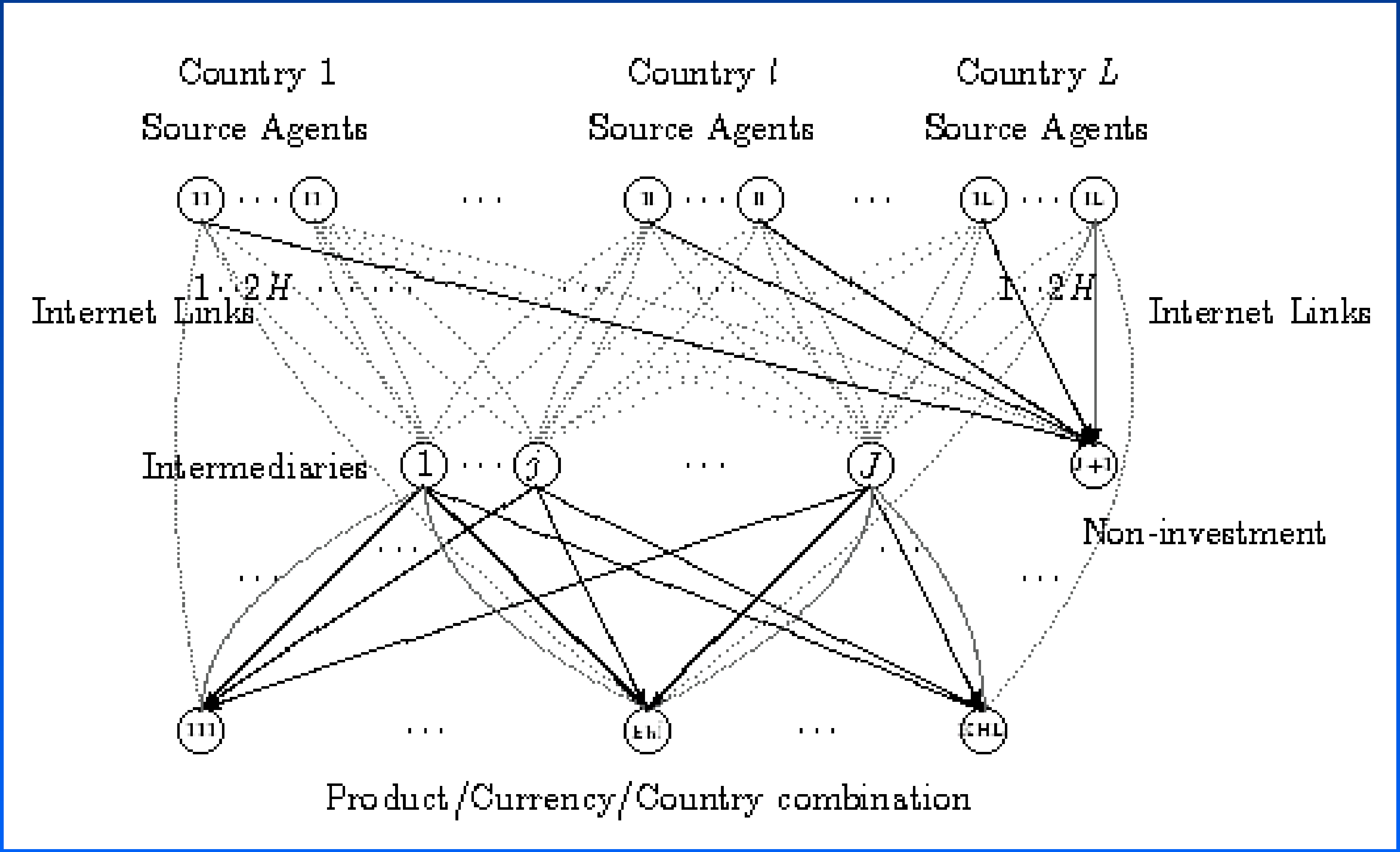
Nagurney, Loo, Dong, and Zhang, *Netnomics* (2002)

Supply Chain -Transportation Supernetwork Representation



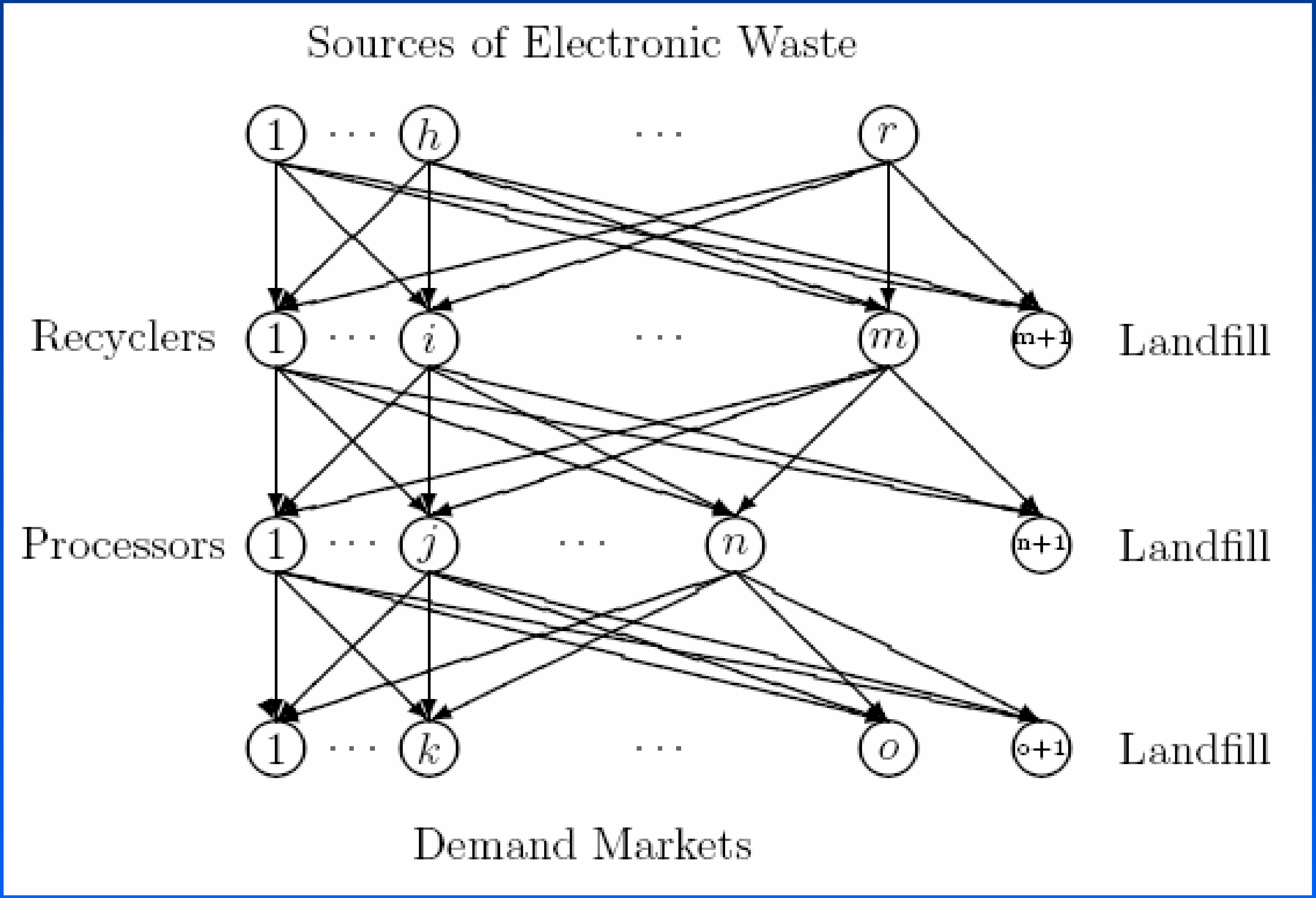
Nagurney, Ke, Cruz, Hancock, Southworth, *Environment and Planning B* (2002)

International Financial Networks with Electronic Transactions



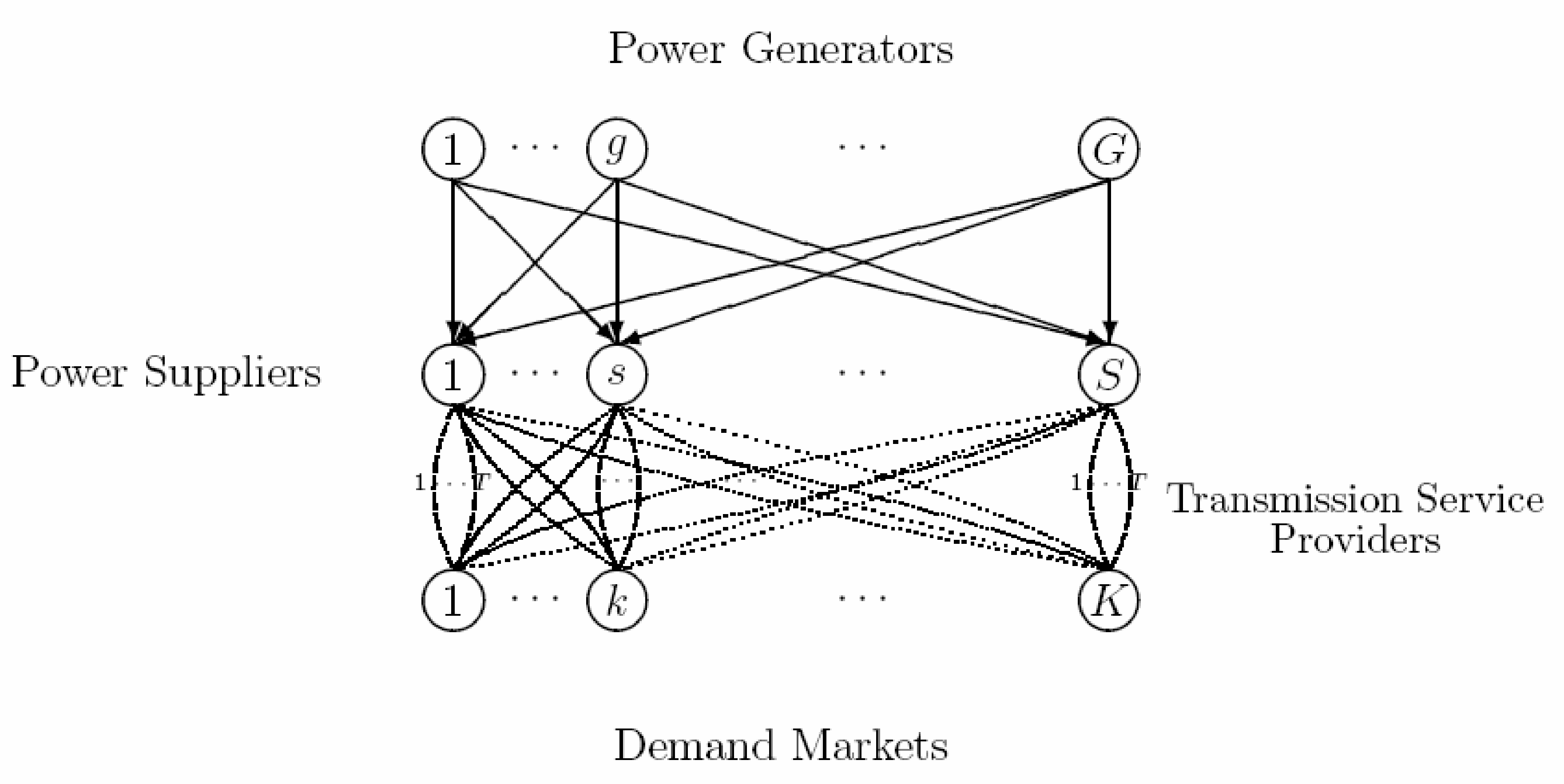
Nagurney and Cruz, in **Innovations in Financial and Economics Networks** (2003)

The 4-Tiered E-Cycling Network



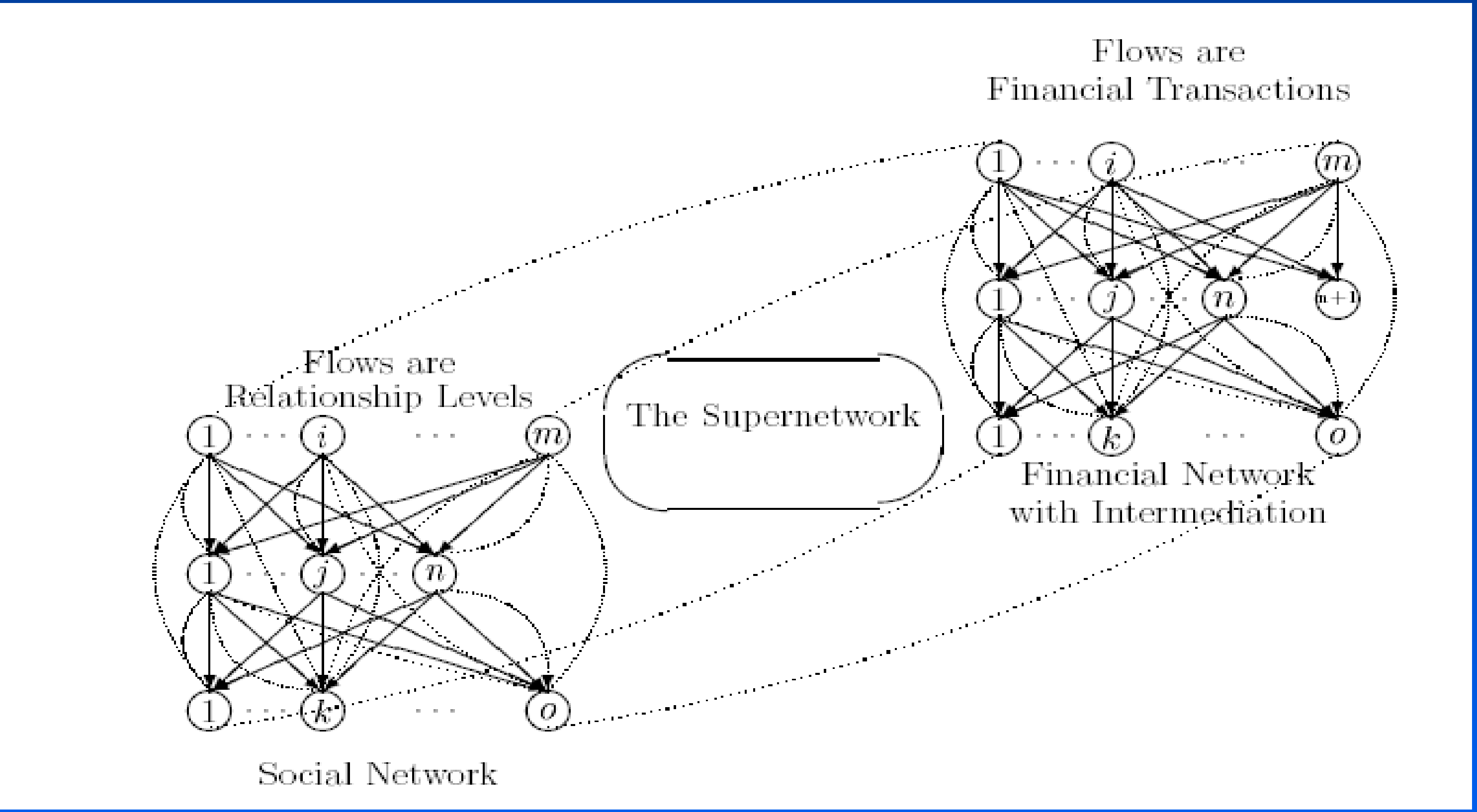
Nagurney and Toyasaki, *Transportation Research E* (2005)

The Electric Power Supply Chain Network



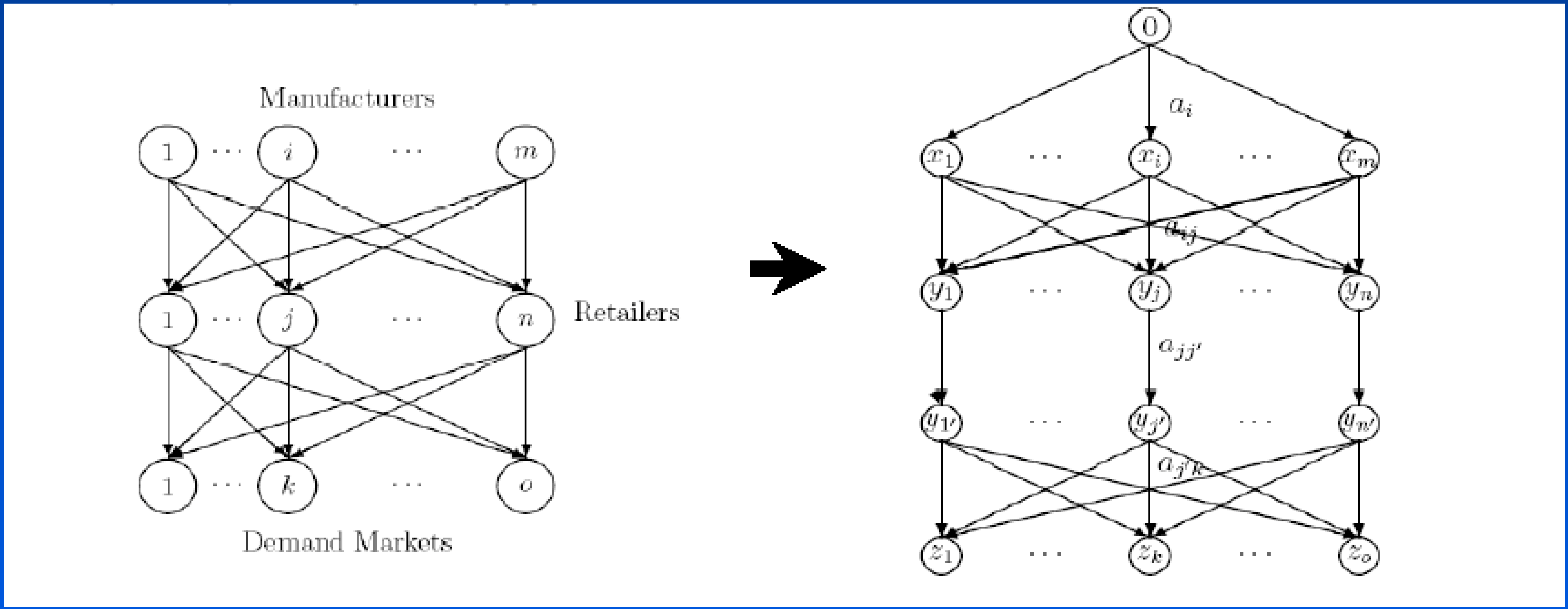
Nagurney and Matsypura, Proceedings of the *CCCT* (2004)

The Integrated Financial/Social Network System



Cruz, Nagurney, and Wakolbinger, *Naval Research Logistics* (2006)

The Equivalence of Supply Chain Networks and Transportation Networks

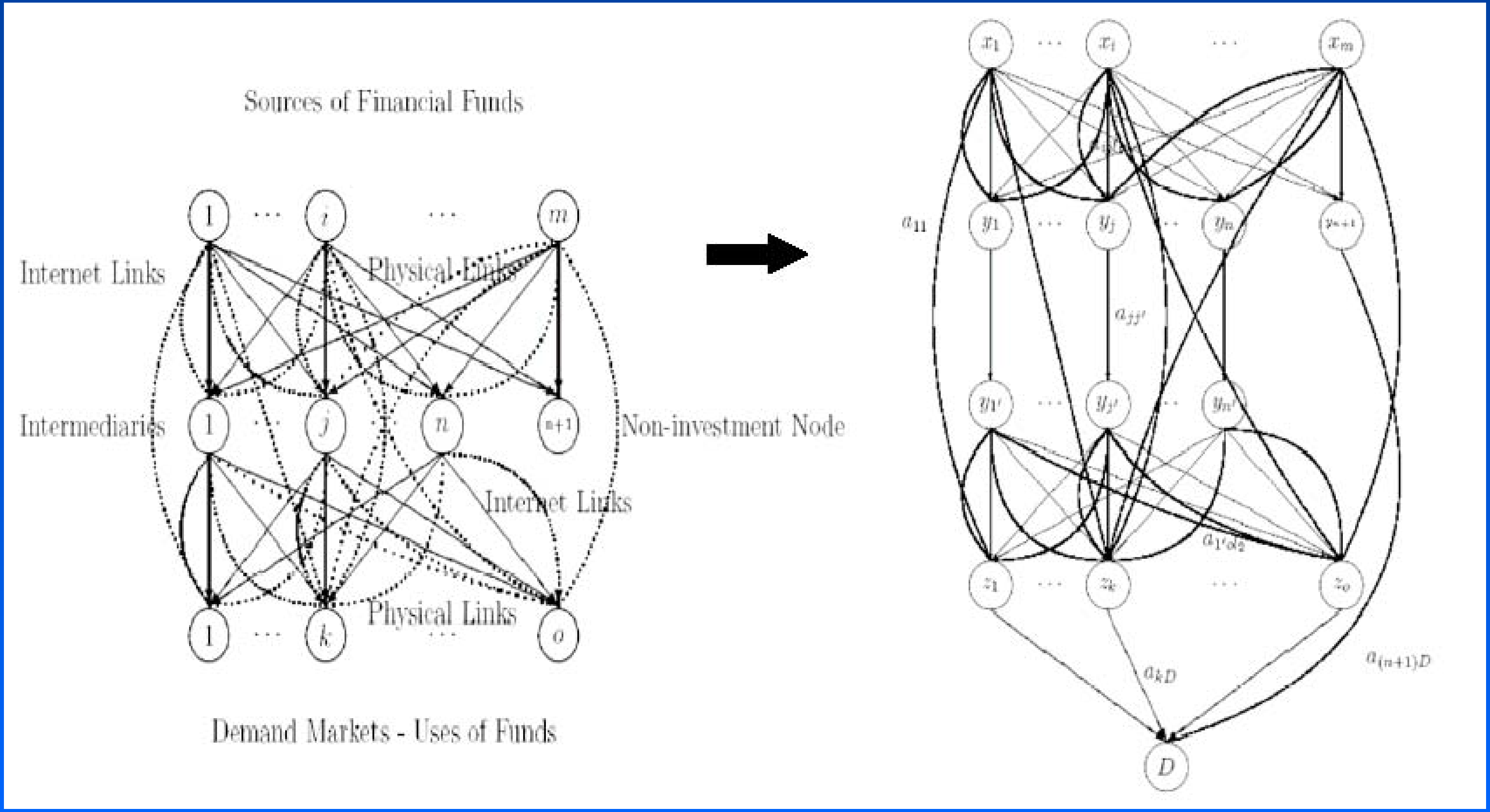


Nagurney, *Transportation Research E* (2006)

Copeland (1952) wondered whether money flows like water or electricity.

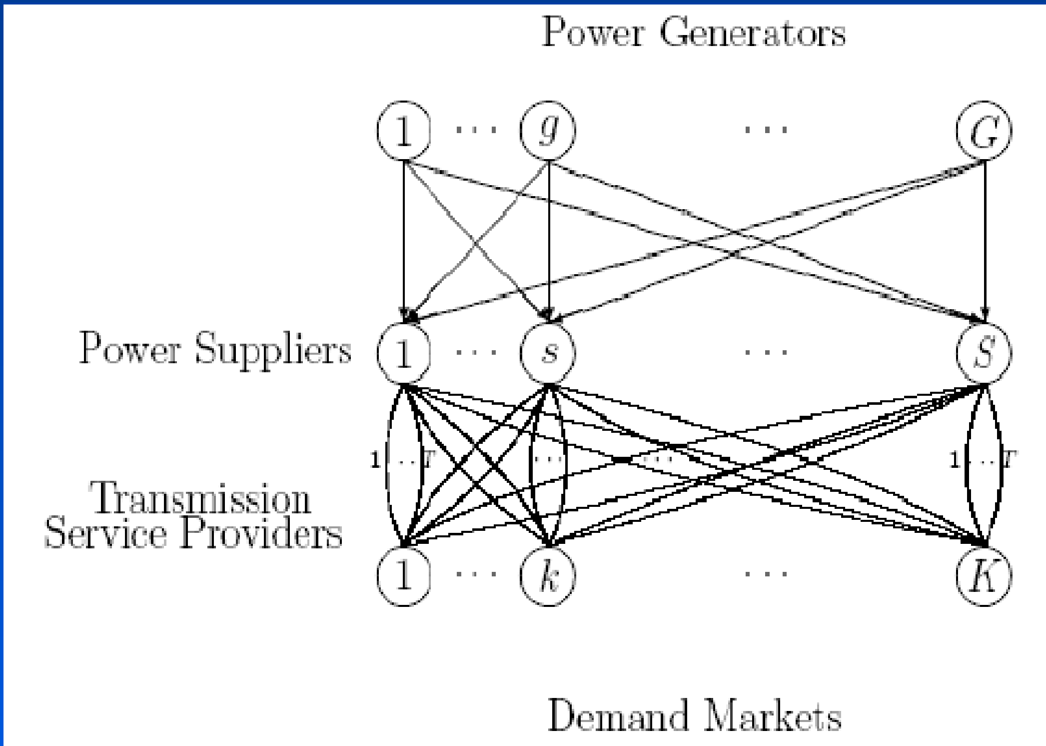
Liu and Nagurney have shown that money and electricity flow like transportation network flows (*Computational Management Science* (2006)).

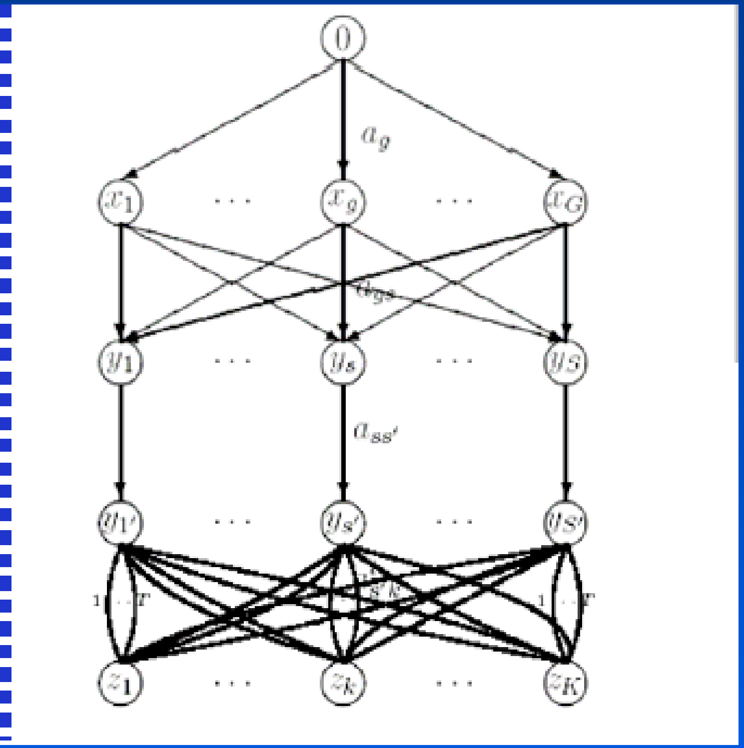
The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation



The fifth chapter of Beckmann, McGuire, and Winsten’s book, **Studies in the Economics of Transportation** (1956) describes some *unsolved problems* including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks





Electric Power Supply Chain Network

Transportation Network

Nagurney et al, to appear in *Transportation Research E*

We have, hence, shown that money as well as electricity flow like *transportation* and have answered questions posed fifty years ago by Copeland and Beckmann, McGuire, and Winsten, respectively.

We are using evolutionary variational inequalities to model dynamic networks with:

- *dynamic (time-dependent)* supplies and demands
- *dynamic (time-dependent)* capacities
- *structural changes* in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).

What happens if the demand is varied in the Braess Network?

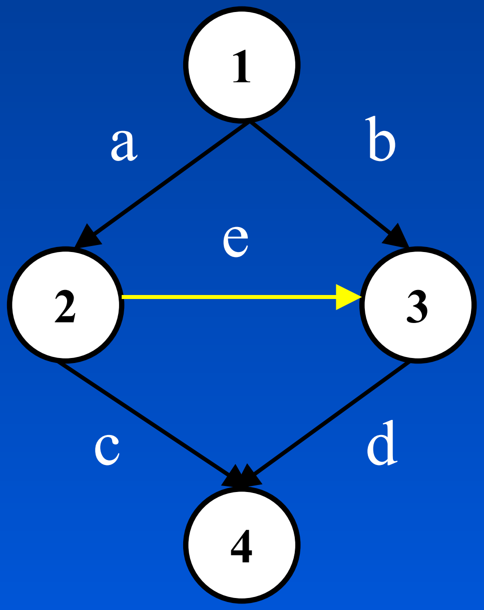
The answer lies in the solution of an Evolutionary (Time-Dependent) Variational Inequality.

Find $x^* \in K$, such that

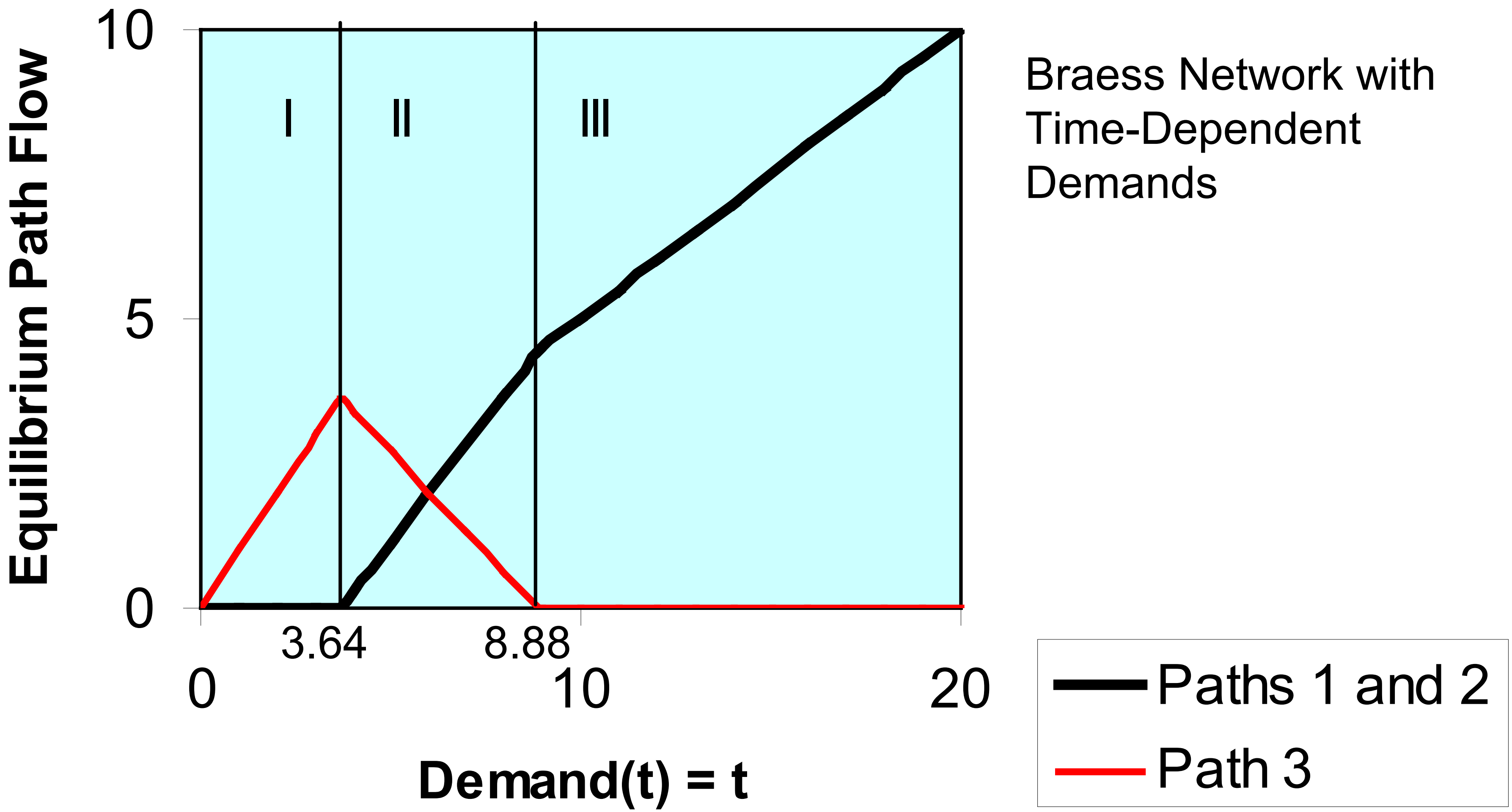
$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle \, dt \geq 0 \qquad \forall x \in K$$

Nagurney, Parkes, and Daniele, *Computational Management Science* (2006)

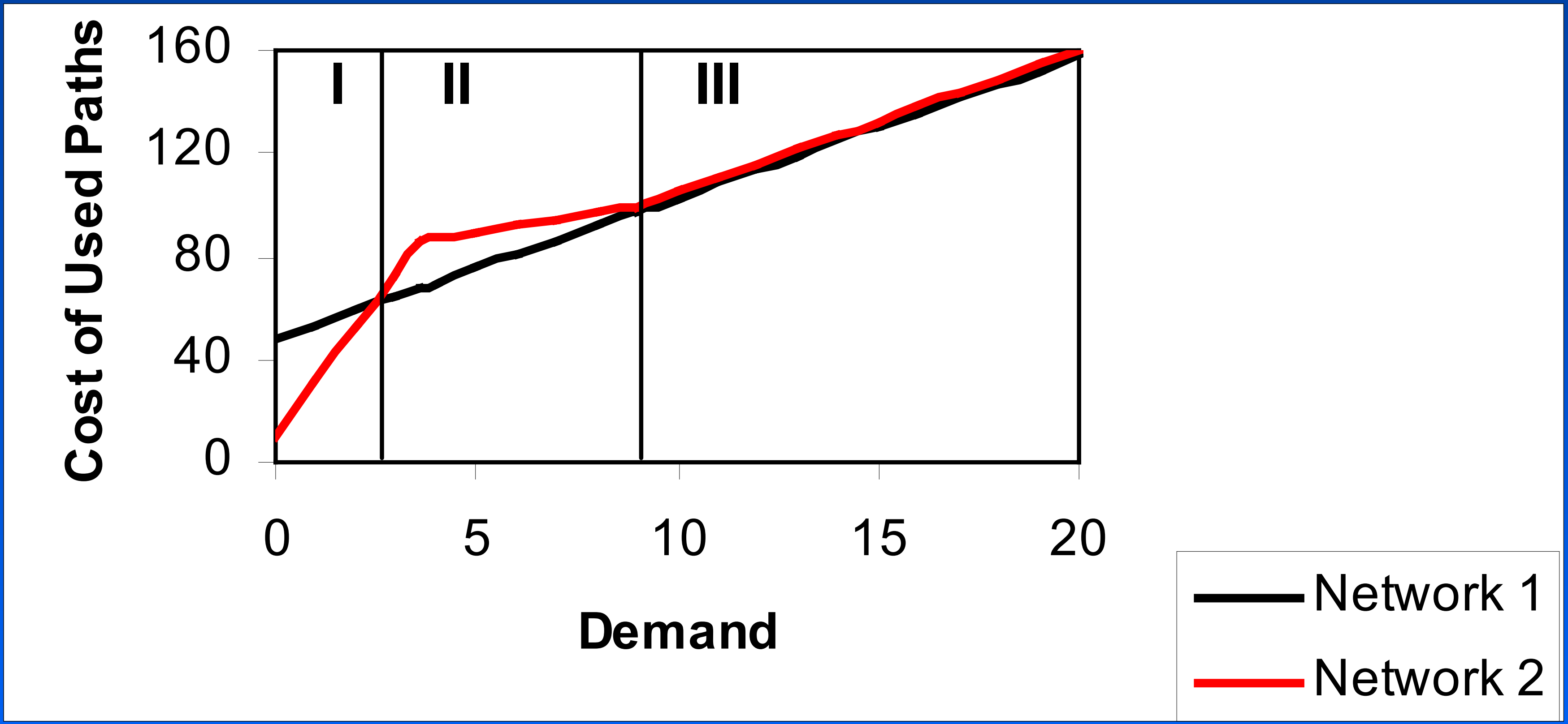
Recall the Braess Network
where we add the link e.



The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)



In Demand Regime I, only the new path is used.
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off!
In Demand Regime III, only the original paths are used.



Network 1 is the Original Braess Network - Network 2 has the added link.

The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!

2005-2006 Radcliffe Institute for Advanced Study Fellowship Year at Harvard Collaboration with Professor David Parkes and Professor Patrizia Daniele (Visiting from Italy)

The Internet -- A Dynamic Network

The Internet has revolutionized the way in which we work, interact, and conduct our daily activities. It has affected the young and the old as they gather information and communicate and has transformed business processes, financial investing and decision-making, and global supply chains. The Internet has evolved into a network that underpins our developed societies and economies.

The motivation for this research comes from several directions:

1. The need to develop a dynamic, that is, time-dependent, model of the Internet, as argued by computer scientists.

Indeed, as noted on page 11 of Roughgarden (2005), *A network like the Internet is volatile. Its traffic patterns can change quicky and dramatically ... The assumption of a static model is therefore particularly suspect in such networks*.

2. Analogues have been identified between transportation networks and telecommunication networks and, in particular, the Internet, in terms of decentralized decision-making, flows and costs, and even the Braess paradox, which allows us to take advantage of such a connection:

Beckmann, McGuire, and Winsten (1956), Beckmann (1967), Braess (1968) Dafermos and Sparrow (1969), Dafermos (1972), Cantor and Gerla (1974), Gallager (1977), Bertsekas and Tsitsiklis (1989), Bertsekas and Gallager (1992), Korilis, Lazar, and Orda (1999), and Boyce, Mahmassani, and Nagurney (2005).

3. The development of a fundamental dynamic model of the Internet will allow for the exploration and development of different incentive mechanisms, including dynamic tolls and pricing mechanisms in order to reduce congestion and also aid in the design of a better Internet, a dynamic network, par excellence.

It has been shown that distributed routing, which is common in computer networks and, in particular, the Internet, and *selfish* (or *source* routing in computer networks) routing, as occurs in the case of user-optimized transportation networks, in which travelers select the minimum cost route between an origin and destination, are one and the same if the cost functions associated with the links that make up the paths/routes coincide with the lengths used to define the shortest paths.

We assume that the costs on the links are congestion-dependent, that is, they depend on the volume of the flow on the link.

Note that the cost on a link may represent travel delay but we utilize *cost* functions since these are more general conceptually than delay functions and they can include, for example, tolls associated with pricing, etc.

It is important to also emphasize that, in the case of transportation networks, it is travelers that make the decisions as to the route selection between origin/destination (O/D) pairs of nodes, whereas in the case of the Internet, it is algorithms, implemented in software, that determine the shortest paths.

We can expect that a variety of time-dependent demand structures will occur on the Internet as individuals seek information and news online in response to major events or simply go about their daily activities whether at work or at home. Hence, the development of this dynamic network model of the Internet is timely.

The costs on routes are related to costs on links through the following equations:

$$C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t))\delta_{ar}, \quad \forall r \in P, \forall k,$$

that is, the cost on a route of class k at a time t is equal to the sum of costs of the class on links that make up the route at time t. We group the path costs at time t into the vector $C(t)$, which is of dimension Kn_P.

We define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0,T], R^{Kn_P})$ (where T denotes the time interval under consideration) given by

$$\mathcal{K} = \left\{ \begin{array}{r} x \in L^2([0,T], R^{Kn_P}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0,T]; \\ \sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k, \text{ a.e. in } [0,T] \end{array} \right\}.$$

We assume that the capacities $\mu_r^k(t)$, for all r and k, are in $\mathcal{L}$ and that the demands, $d_w^k \geq 0$, for all w and k, are also in $\mathcal{L}$. Further, we assume that

$$0 \leq d(t) \leq \Phi \mu(t), \text{a.e. on } [0,T],$$

where Φ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element (kw, kr) equal to 1 if route r is contained in P_w, and 0, otherwise. Hence, the feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded. Note that we are not restricted as to the form that the time-varying demand for the O/D pair takes since convexity is guaranteed even if the demands have a step-wise structure, or are piecewise continuous.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \to \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

Definition: Dynamic Multiclass Network Equilibrium

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle (cf. Wardrop (1952) and Beckmann, McGuire, and Winsten (1956))), if, at each time t, only the minimum cost routes for each class not at their capacities are used (that is, have positive flow) for each O/D pair unless the flow of that class on a route is at its upper bound (in which case those class routes' costs can be lower than those on the routes not at their capacities). The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every path $r \in P_w$, every class k; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if} \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if} \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if} \quad x_r^{k*}(t) = 0. \end{cases}$$

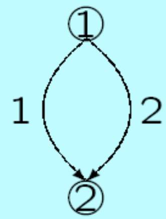
Theorem (Nagurney, Parkes, and Daniele (2006))

$x^* \in \mathcal{K}$ *is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:*

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

A Multiclass Numerical Example

Consider a network (small subnetwork of the Internet) consisting of two nodes and two links. There is a single O/D pair $w = (1, 2)$. Since the routes connecting the O/D pair consist of single links we work with the routes r_1 and r_2 directly:



Network Structure of the Multiclass Numerical Example

There are assumed to be two classes/jobs and the route costs are:

for Class 1:

$$C^1_{r_1}(x(t)) = 2x^1_{r_1}(t) + x^2_{r_1}(t) + 5, \quad C^1_{r_2}(x(t)) = 2x^2_{r_2}(t) + 2x^1_{r_2}(t) + 10,$$

for Class 2:

$$C^2_{r_1}(x(t)) = x^2_{r_1}(t) + x^1_{r_1}(t) + 5, \quad C^2_{r_2}(x(t)) = x^1_{r_2}(t) + 2x^2_{r_2}(t) + 5.$$

The time horizon is $[0, 10]$. The demands for the O/D pair are:

$$d^1_w(t) = 10 - t, \quad d^2_w(t) = t.$$

The upper bounds are: $\mu^1_{r_1} = \mu^1_{r_2} = \mu^1_{r_1} = \mu^2_{r_2} = \infty$.

Equilibrium Route Flows for the Multiclass Numerical Example

Equilibrium Multiclass Route Flows at time t					
Flow	$t=0$	$t=2.5$	$t=5$	$t=7.5$	$t=10$
$x^{1*}_{r_1}(t)$	6.25	6.25	5.00	2.50	0.00
$x^{1*}_{r_2}(t)$	3.75	1.25	0.00	0.00	0.00
$x^{2*}_{r_1}(t)$	0.00	0.00	$1.6\bar{6}$	$4.16\bar{6}$	$6.6\bar{6}$
$x^{2*}_{r_2}(t)$	0.00	2.50	$3.3\bar{3}$	$3.3\bar{3}$	$3.3\bar{3}$

For completeness, we also provide the following class O/D pair minimum costs at times $t = 0, 2.5, 5, 7.5$ and 10:

$$\lambda^{1*}_w(0) = 17.50, \quad \lambda^{1*}_w(2.5) = 17.50,$$

$$\lambda^{1*}_w(5) = 16.6\bar{6}, \quad \lambda^{1*}_w(7.5) = 14.16\bar{6}, \quad \lambda^{1*}_w(10) = 11.6\bar{6}$$

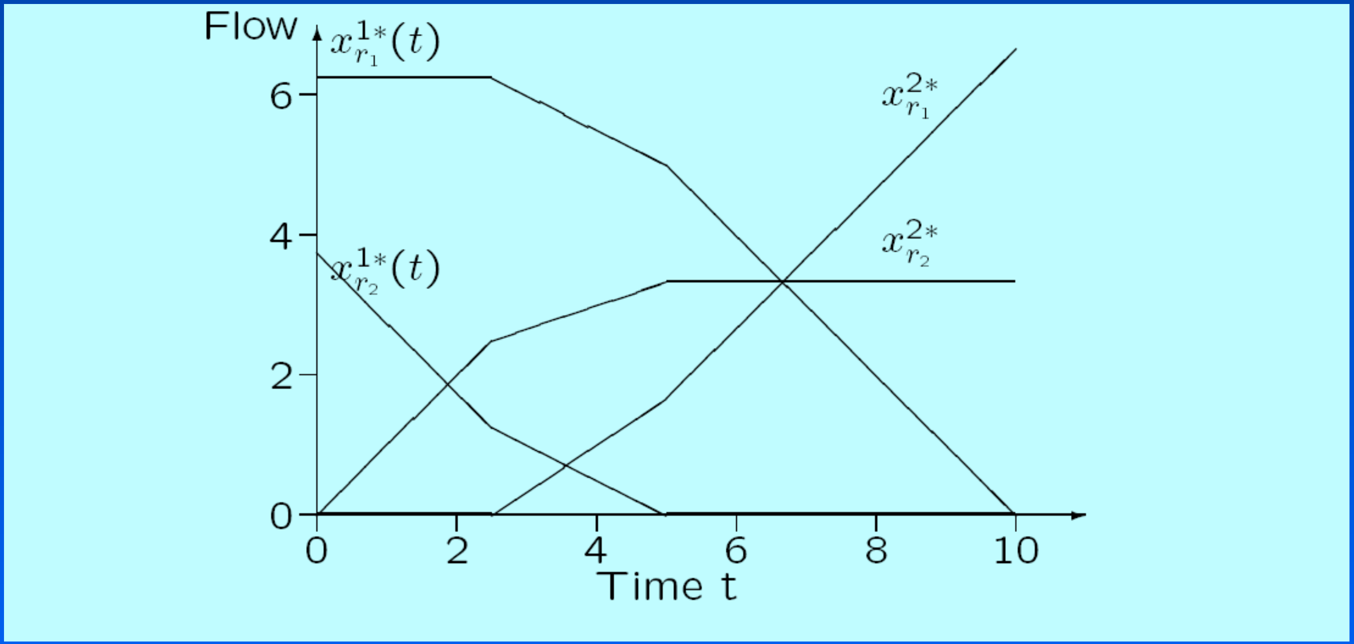
and

$$\lambda^{2*}_w(0) = 8.75, \quad \lambda^{2*}_w(2.5) = 11.25,$$

$$\lambda^{2*}_w(5) = 11.6\bar{6}, \quad \lambda^{2*}_w(7.5) = 11.6\bar{6}, \quad \lambda^{2*}_w(10) = 11.6\bar{6}.$$

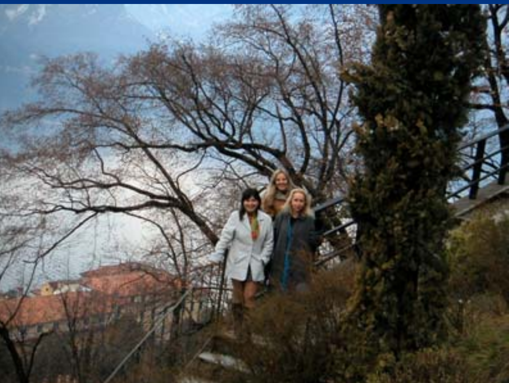
We provide a graph of the equilibrium route trajectories, where we display also the interpolations between the discrete solutions. Since the route cost functions are strictly monotone over the time horizon [0, 10] we know that the equilibrium trajectories are unique.

As the theory predicts, the trajectories are also continuous for this example. It is interesting to see that after time t = 5 route r_2 is never used by class 1, whereas route r_1 is not utilized for class 2 traffic until after t = 2.

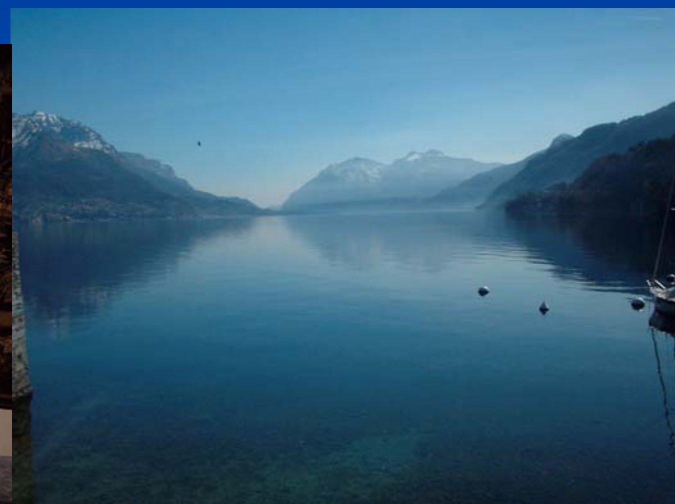


Equilibrium Trajectories for the Multiclass Numerical Example

New Challenges and Opportunities: The Unification of EVIs and PDSs



Bellagio Research Team Residency March 2004



Double-Layered Dynamics

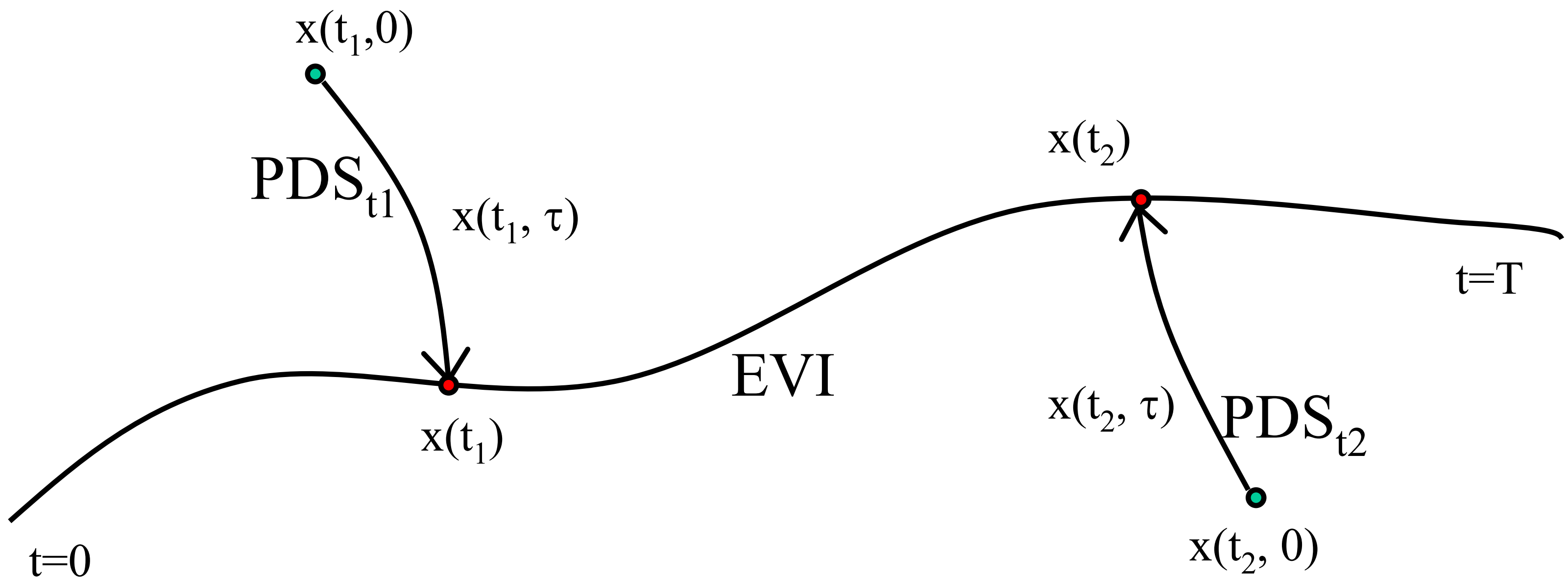
The unification of EVIs and PDSs allows the modeling of dynamic networks over *different time scales*.

Papers:

Projected Dynamical Systems and Evolutionary Variational Inequalities via Hilbert Spaces with Applications (Cojocaru, Daniele, and Nagurney), *Journal of Optimization Theory and Applications*, vol. 127, no. 3, pp. 1-15, December 2005.

Double-Layered Dynamics: A Unified Theory of Projected Dynamical Systems and Evolutionary Variational Inequalities (Cojocaru, Daniele, and Nagurney), *European Journal of Operational Research,* in press.

A Pictorial of the Double-Layered Dynamics



There are new exciting questions, both theoretical and computational, arising from this multiple time structure.

In the course of answering these questions, a new theory is taking shape from the synthesis of PDS and EVI, and, as such, it deserves a name of its own; we call it **double-layered dynamics.**

STANDARD EVI FORM (TIME-DEPENDENT) VARIATIONAL INEQUALITIES

Recall that $\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt$ is the duality mapping on $L^p([0,T], R^q)$, where $\phi \in (L^p([0,T], R^q))^*$ and $u \in L^p([0,T], R^q)$. Let $F : K \to (L^p([0,T], R^q))^*$.

The standard form of the evolutionary variational inequality (EVI) that we work with is:

find $x^* \in K$ such that $\ll F(x^*), x - x^* \gg \geq 0, \; \forall x \in K,$

or, equivalently, find $x^* \in K$ such that

$$\int_0^T \langle F(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in K.$$

EQUILIBRIA of PDS and VARIATIONAL INEQUALITIES

An important feature of any PDS is that it is intimately related to a variational inequality problem (VI).

Theorem

The equilibria of a PDS:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_K(x(t), -F(x(t)))$$

$$= \lim_{\delta \to 0} \frac{P_K(x(t) - \delta F(x(t))) - x(t)}{\delta}, \quad x(0) = x_0,$$

that is, $x^* \in K$ *such that*

$$\Pi_K(x^*, -F(x^*)) = 0$$

are solutions to the $\mathrm{VI}(F, K)$*: find* $x^* \in K$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

and vice-versa, where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* X*, where* X *is a Hilbert space.*

Theorem (Cojocaru, Daniele, and Nagurney (2005))

The solutions to the EVI problem are the same as the critical points of the PDS and vice versa, that is, the critical points of the PDS are the solutions to the EVI.

Hence, by choosing the Hilbert space to be $L^2([0,T], R^q)$*, we find that the solutions to the evolutionary variational inequality: find* $x^* \in K$ *such that*

$$\int_0^T \langle F(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in K$$

are the same as the critical points of the equation:

$$\frac{\partial x(t,\tau)}{\partial \tau} = \Pi_K(x(t,\tau), -F(x(t,\tau))),$$

that is, the points such that

$$\Pi_K(x(t,\tau), -F(x(t,\tau))) \equiv 0 \quad a.e. in \ [0,T],$$

which are obviously stationary with respect to τ*.*

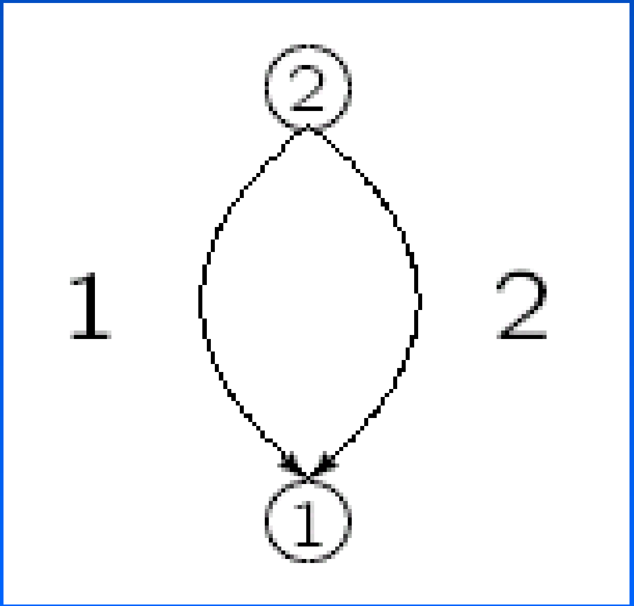
To solve the associated evolutionary variational inequality, we discretize the time horizon T and the corresponding variational inequality (or, equivalently, projected dynamical system) at each discrete point in time is then solved.

Obviously, this procedure is correct if the continuity of the solution is guaranteed.

Continuity results for solutions to evolutionary variational inequalities, in the case where $F(x(t)) = A(t)x(t) + B(t)$ is a linear operator, $A(t)$ is a continuous and positive definite matrix in $[0, T]$, and $B(t)$ is a continuous vector can be found in Barbagallo (2005).

A Dynamic Network Example with Time-Varying Demand and Capacities

We consider a network consisting of a single origin/destination pair of nodes and two paths connecting these nodes.



Let cost on path **1** be: $\mathbf{2x_1(t)-1.5}$ and cost on path **2** be: $\mathbf{x_2(t)-1}$.

The demand is **t** in the interval **[0,2].**

Suppose that we also have capacities**:**

$\mathbf{(0,0) \leq (x_1(t), x_2(t)) \leq (t, 3/2\, t).}$

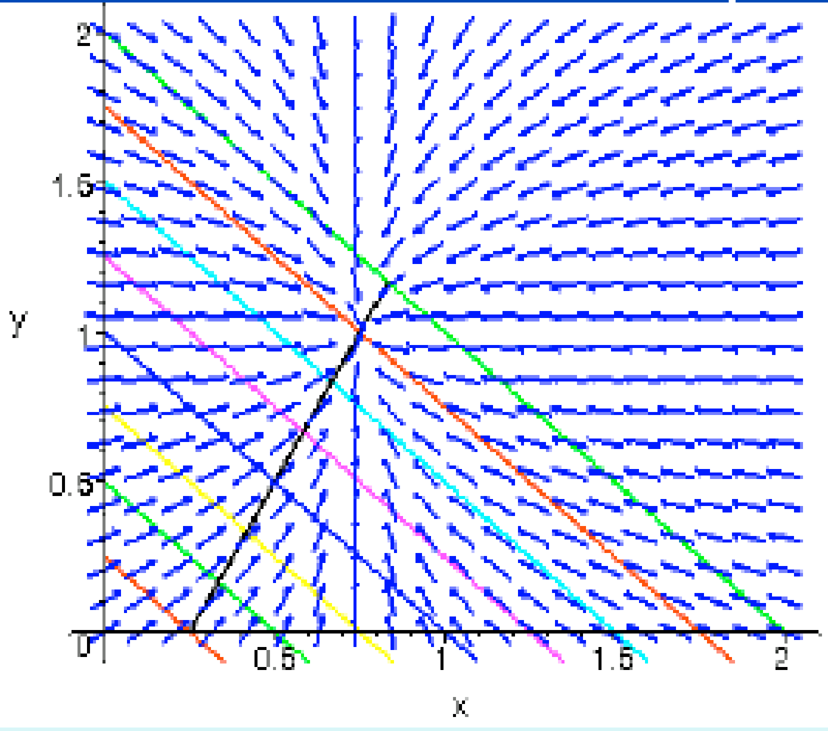
With the help of PDS theory, we can compute an approximate curve of equilibrium by choosing

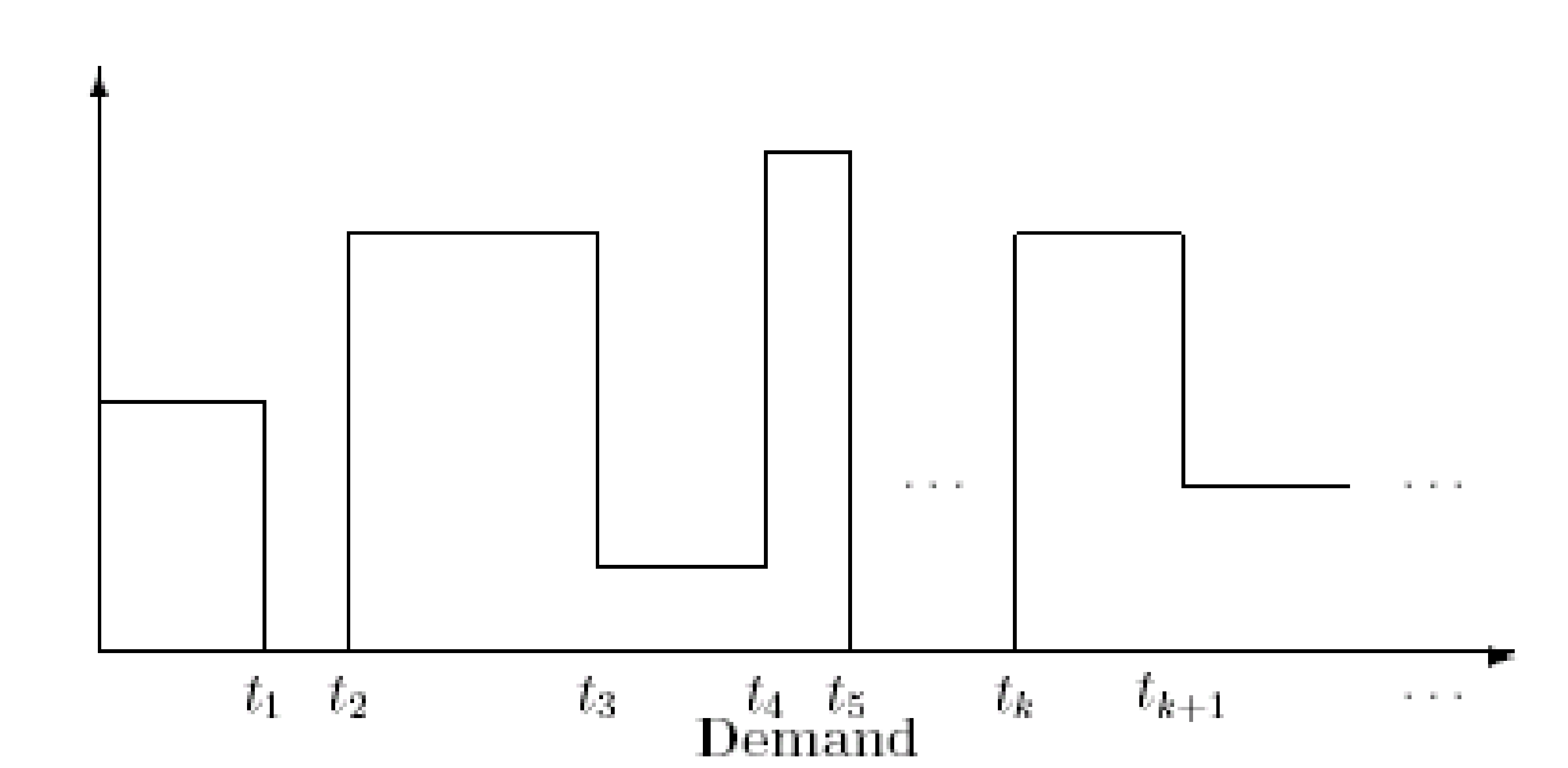
$$t_0 \in \left\{ \frac{k}{4} | k \in \{0, \ldots, 8\} \right\}.$$

Using a simple MAPLE computation, we obtain that the equilibria are the points:

$$\left\{(0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{3}, \frac{1}{6}\right), \left(\frac{5}{12}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{7}{12}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{5}{6}\right), \left(\frac{3}{4}, 1\right), \left(\frac{5}{6}, \frac{7}{6}\right)\right\}.$$

Interpolating these points, we obtain the approximate curve of network equilibria:



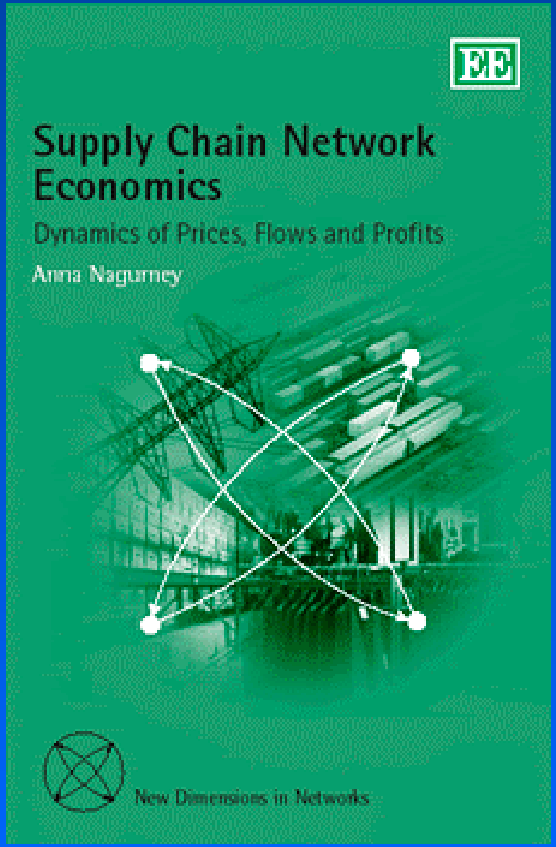


If the demand is a step function, the solution to the EVI has the structure:

$$x^*(t) = \begin{cases} x_1^* & \text{if } 0 \le t \le t_1 \\ x_2^* & \text{if } t_1 < t \le t_2 \\ \vdots & \vdots \\ x_{k+1}^* & \text{if } t_k < t \le t_{k+1} \\ \vdots & \vdots \end{cases}$$

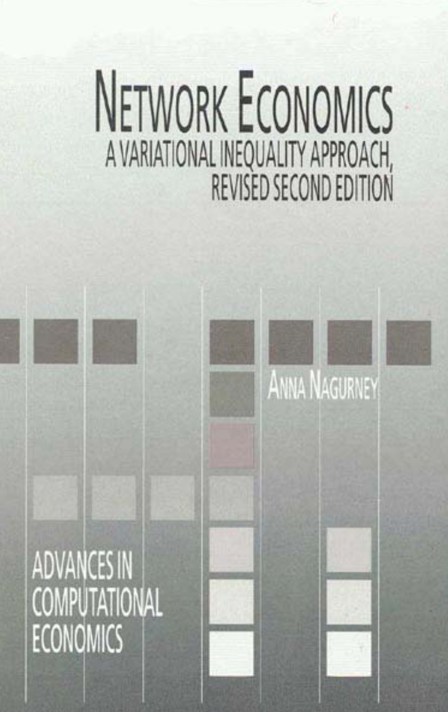
For additional background and new applications see:

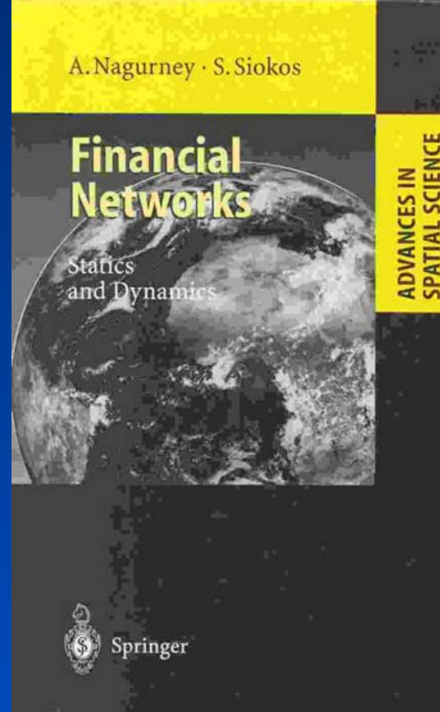
Supply Chain Network Economics

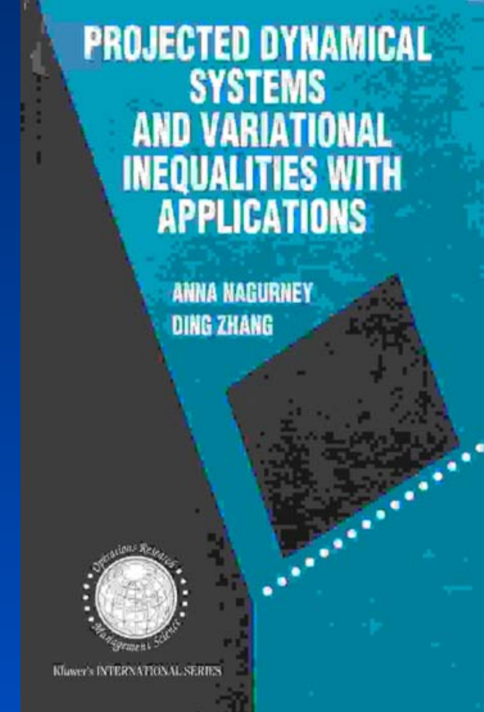


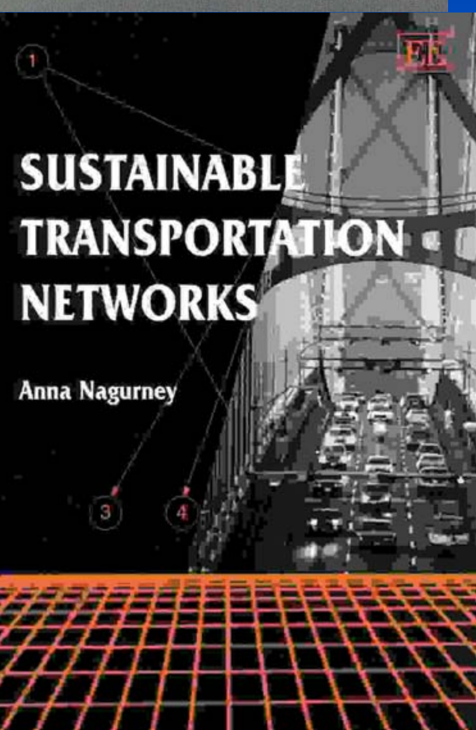
Edward Elgar Publishing

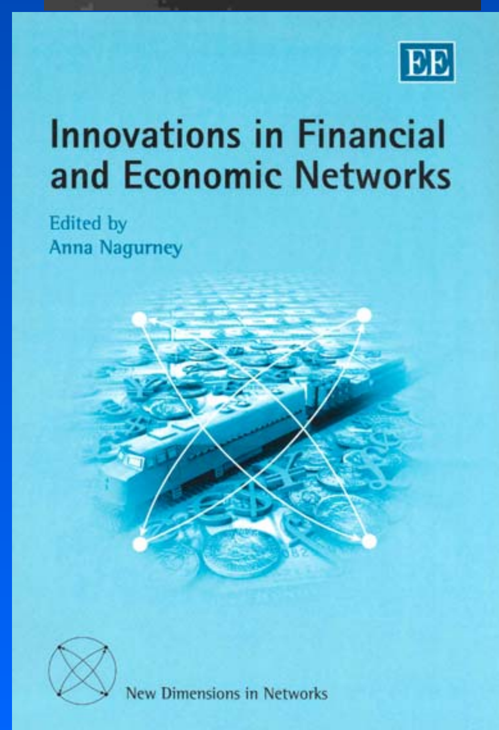
Published 2006

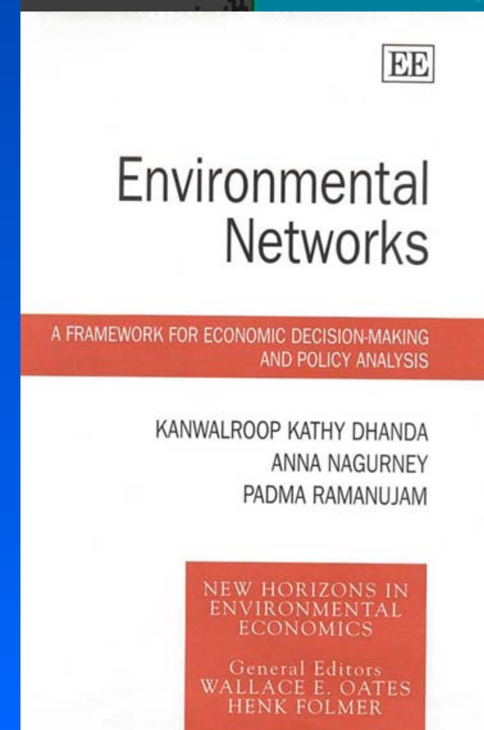

NETWORK ECONOMICS
A VARIATIONAL INEQUALITY APPROACH,
REVISED SECOND EDITION
ANNA NAGURNEY
ADVANCES IN
COMPUTATIONAL
ECONOMICS


A. Nagurney · S. Siokos
Financial
Networks
Statics
and Dynamics
ADVANCES IN
SPATIAL SCIENCE
Springer

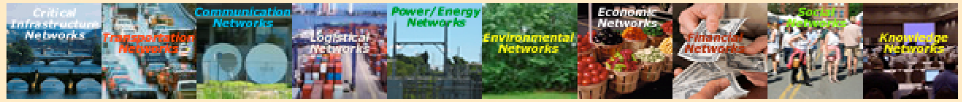

PROJECTED DYNAMICAL
SYSTEMS
AND VARIATIONAL
INEQUALITIES WITH
APPLICATIONS
ANNA NAGURNEY
DING ZHANG
Kluwer's INTERNATIONAL SERIES


EE
1
SUSTAINABLE
TRANSPORTATION
NETWORKS
Anna Nagurney
3
4


EE
Innovations in Financial
and Economic Networks
Edited by
Anna Nagurney
New Dimensions in Networks


EE
Environmental
Networks
A FRAMEWORK FOR ECONOMIC DECISION-MAKING
AND POLICY ANALYSIS
KANWALROOP KATHY DHANDA
ANNA NAGURNEY
PADMA RAMANUJAM
NEW HORIZONS IN
ENVIRONMENTAL
ECONOMICS
General Editors
WALLACE E. OATES
HENK FOLMER

The Virtual Center for Supernetworks



Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

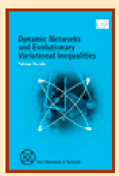
The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

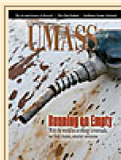
Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, logistical, telecommunication, and power networks to economic, environmental, financial, knowledge and social networks.

The applications of Supernetworks include: transportation, logistics, critical infrastructure, telecommunications, power and energy, electronic commerce, supply chain management, environment, economics, finance, knowledge and social networks, and decision-making.

NEW!

Books



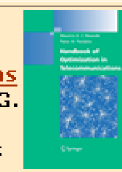


Getting There from Here
The route to sustainable transportation is paved with knowledge and creativity

NEW!

Professor Dietrich Braess visits the Virtual Center for Supernetworks
April 7, 2006

Handbook of Optimization in Telecommunications
edited by Mauricio G. C. Resende and Panos M. Pardalos

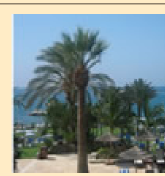


NEW!

New Papers on
The Internet, Supply Chains, and Electric Power Networks

NEW!

Recent Conference Photos



RADCLIFFE INSTITUTE FOR ADVANCED STUDY
HARVARD UNIVERSITY

A Year as a Science Fellow at the Radcliffe Institute for Advanced Study at Harvard
by Anna Nagurney

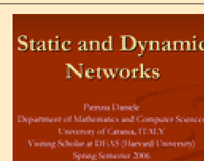
Modeling Generator Power Plant Portfolios and Pollution Taxes in Electric Power Supply Chain Networks: A Transportation Network Transformation
is the Lead Article in May 2006 *Transportation Research D*

You are visitor number

32,224

to the Virtual Center for Supernetworks.

Visuals



Center Associate
Dr. Stavros Siokos of Citigroup is interviewed for
Financial Times Mandate
May 2006
and featured in
FTSE Global Markets
May/June 2005

Google

Google Search

Thank you!

For more information, see
http://supernet.som.umass.edu

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks