

Projected Dynamical Systems and Applications

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Projected Dynamical System (PDS)

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))) \quad (PDS)$$

$K \subseteq \mathbf{R}^n$: convex polyhedral set;

$F : K \rightarrow \mathbf{R}^n$: Lipschitz continuous function with linear growth;

$\Pi_K : \mathbf{R} \times K \rightarrow \mathbf{R}^n$: Gateaux directional derivative

$$\Pi_K(x, -F(x)) = \lim_{\delta \rightarrow 0^+} \frac{P_K(x - \delta F(x)) - x}{\delta};$$

$P_K : \mathbf{R}^n \rightarrow K$: projection operator $\|P_K(z) - z\| = \inf_{y \in K} \|y - z\|$;

Theor. [Dupuis-Nagurney (1993)]

Critical points of (PDS) \Leftrightarrow solutions to

$$\text{Find } x \in K : \langle F(x), y - x \rangle \geq 0, \quad \forall y \in K.$$

Isac-Cojocaru (2002, 2004): study of
PDS in infinite-dimensional Hilbert spaces

Lions-Stampacchia (1967),

Brezis (1967): introduction of Evolutionary Variational Inequalities (EVI)

Steinbach (1998): study of obstacle problems by means of VI

Daniele-Maugeri-Oettli (1998): applications of EVI to network problems

Cojocaru-Daniele-Nagurney (2004): connection between PDS and EVI and introduction of Double-Layered Dynamics

General Formulation of the set K for traffic network problems, spatial price equilibrium problems, financial equilibrium problems

$$\mathbf{K} = \bigcup_{t \in [0, T]} \left\{ u \in L^2([0, T], \mathbf{R}^q) : \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0, T]; \right. \\ \left. \sum_{i=1}^q \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0, T], \xi_{ji} \in \{0, 1\}, i = 1, \dots, q, j = 1, \dots, l \right\}$$

$\lambda, \mu, \rho \in L^p([0, T], \mathbf{R}^q)$: convex functions

Preliminary Definitions

➤ $M \subset H \Rightarrow M^0 = \{ \xi \in H : \langle \xi, x \rangle \leq 1, \forall x \in M \}$

➤ $C \text{ cone} \Rightarrow C^0 = C^- = \{ \xi \in H : \langle \xi, x \rangle \leq 0, \forall x \in C \}$

➤ $K \subset H$ nonempty, closed, convex \Rightarrow

$$T_K(x) = \overline{\bigcup_{\lambda > 0} \lambda(K - x)} = \text{support cone}$$

➤ $N_K(x) = \{ \xi \in H : \langle \xi, z - x \rangle \leq 0, \forall z \in K \} =$
normal cone to K at x

➤ **Prop. 1:** $(T_K(x))^0 = N_K(x) = (T_K(x))^-$

➤ **Theor. 1:** $P_K(x + \lambda h) = x + \lambda P_{T_K(x)} h + o(\lambda)$
 $\forall x, h, \lambda > 0$

➤ **Cor. 1:** If $\Pi_K(x, h) = \lim_{\lambda \rightarrow 0^+} \frac{P_K(x + \lambda h) - x}{\lambda}$,
then $\Pi_K(x, h) = P_{T_K(x)} h$.

➤ $n_K(x) = \{v : \|v\| = 1 \text{ and } \langle v, x - y \rangle \leq 0, \forall y \in K\} =$
set of unit inward normals to K at x

➤ **Prop. 2:** $n_K(x) = \partial B(0,1) \cap -(T_K(x))^0$

➤ $\text{qi } K = \{x \in K : T_K(x) = H\}$

➤ $\text{qbdry } K = K \setminus \text{qi } K$

➤ **Prop. 3:** $x \in \text{qbdry } K \Leftrightarrow n_K(x) \neq \emptyset$

➤ **Theor. 2:**

$$x \in \text{qi } \mathbf{K} \Rightarrow \Pi_{\mathbf{K}}(x, h) = h, \quad \forall h \in H;$$

$$x \in \text{qbdry } \mathbf{K} \Rightarrow \forall v \in H \setminus T_{\mathbf{K}}(x), \exists n^*(x) \in n_{\mathbf{K}}(x):$$

$$\beta(x) = -\langle v, n^*(x) \rangle > 0,$$

$$\Pi_{\mathbf{K}}(x, v) = v + \beta(x) n^*(x).$$

➤ **Cor. 2:**

$$\Pi_{\mathbf{K}}(x, v) = P_{v - N_{\mathbf{K}}(x)}(0) = (v - N_{\mathbf{K}}(x))^{\#} \quad \forall v \in H.$$

$$\text{Cor. 2} \Rightarrow \exists ! n_x : \Pi_{\mathbf{K}}(x, -F(x)) = -F(x) - n_x$$

where $n_x = 0$ if $x \in \text{qi } \mathbf{K}$

➤ Remark 1: Cor. 2 implies

$$\frac{d \dot{x}(t)}{d t} = \Pi_K(x, -F(x)) = P_{-F(x) - N_K(x)}(\mathbf{0}) = \left\{ \bar{v} \in -(F(x) + N_K(x)) : \|\bar{v}\| = \min_{y \in -(F(x) + N_K(x))} \|y\| \right\}$$

$$\left\{ \begin{array}{l} \frac{d \dot{x}(t)}{d t} = \Pi_K(x, -F(x)) \\ x(0) = x_0 \in K \end{array} \right. \Leftrightarrow \begin{array}{l} \text{slow solution to} \\ \dot{x}(t) \in -(N_K(x(t)) + F(x(t))) \\ x(0) = x_0 \end{array}$$

$$\Pi_K(x(t), -F(x(t))) = P_{T_K(x(t))}(-F(x(t)))$$



$$\begin{cases} \frac{d \dot{x}(t)}{d t} = \Pi_K(x, -F(x)) \\ x(0) = x_0 \in K \end{cases} \Leftrightarrow \begin{cases} \dot{x}(t) \in P_{T_K(x)}(-F(x(t))) \\ x(0) = x_0 \end{cases} \quad \text{slow solution to}$$

➤ Remark 2:

Find $u \in \mathbf{K}$: $\langle\langle F(u), v-u \rangle\rangle \geq 0, \quad \forall v \in \mathbf{K}$

where $\langle\langle \Phi, u \rangle\rangle = \int_0^T \langle \Phi(t), u(t) \rangle dt,$

$\Phi \in L^2([0, T], \mathbf{R}^q)^*, \quad u \in L^2([0, T], \mathbf{R}^q)$



Find $u \in \mathbf{K}$: $\langle F(u(t)), v(t) - u(t) \rangle \geq 0,$

$\forall v \in \mathbf{K}, \text{ a.e. in } [0, T]$

Computational Procedure

F strictly monotone \Rightarrow

$u(t) \in C^0([0, T], \mathbf{R}^q)$ unique solution to EVI:

$$\langle F(u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall t \in [0, T]$$

Partitions of $[0, T]$:

$$\pi_n = (t_n^0, t_n^1, \dots, t_n^{N_n}), \quad 0 = t_n^0 < t_n^1 < \dots < t_n^{N_n} = T$$

$$k_n = \max \{ t_n^j - t_n^{j-1} : j = 1, \dots, N_n \}, \quad \lim_n k_n = 0.$$

Finite-dimensional Variational Inequality:

$$\left\langle F\left(u\left(t_n^{j-1}\right)\right), v-u\left(t_n^{j-1}\right)\right\rangle \geq 0, \quad \forall v \in \mathbf{K}\left(t_n^{j-1}\right) \quad (FDVI)$$

where

$$\mathbf{K}\left(t_n^{j-1}\right)=\left\{v \in \mathbf{R}^q: \lambda\left(t_n^{j-1}\right) \leq v \leq \mu\left(t_n^{j-1}\right), \sum_{i=1}^q \xi_{ji} v_i=\rho_j\left(t_n^{j-1}\right)\right\}$$

Unique solution to (FDVI) \Leftrightarrow critical point of

$$\Pi_{\mathbf{K}}\left(u\left(t_n^{j-1}, \tau\right),-F\left(u\left(t_n^{j-1}, \tau\right)\right)\right)=0$$

Interpolation function $u_n(t)$:

$$\lim_n \left\|u_n(t)-u(t)\right\|_{L^\infty([0, T], \mathbf{R}^q)}=0$$

$$P_n([0, T], \mathbf{R}^m) = \left\{ v \in L^\infty([0, T], \mathbf{R}^m) : v_{(t_n^{j-1}, t_n^j]} = v_j \in \mathbf{R}^m, j = 1, \dots, N_n \right\}$$

Mean value operators $\mu_n : L^1([0, T], \mathbf{R}^m) \rightarrow P_n([0, T], \mathbf{R}^m)$

$$\mu_n v_{(t_n^{j-1}, t_n^j]} = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s) ds$$

$$\mathbf{K} = \left\{ F(t) \in L^2([0, T], \mathbf{R}^m) : \lambda \leq F(t) \leq \nu, \text{ a.e. in } [0, T], \right. \\ \left. \Phi F(t) = \rho(t), \lambda, \nu \geq 0 \right\}$$

$C : [0, T] \times \mathbf{K} \rightarrow L^2([0, T], \mathbf{R}^m)$ linear mapping

$$C[t, F(t)] = A(t)F(t) + B(t), \quad A(t) \in L^\infty, B(t) \in L^2$$

Find $H(t) \in \mathbf{K}$:

$$\int_0^T \langle A(t)H(t) + B(t), F(t) - H(t) \rangle dt =$$

$$\sum_{j=1}^{N_n} \int_{t_n^{j-1}}^{t_n^j} \langle A(t)H(t) + B(t), F(t) - H(t) \rangle dt \geq 0, \quad \forall F(t) \in \mathbf{K}$$

Find $u_j^n(t) \in \mathbf{K}$:

$$\int_{t_n^{j-1}}^{t_n^j} \langle A(t)H_j^n(t) + B(t), F_j^n(t) - H_j^n(t) \rangle dt \geq 0,$$

$$\forall F_j^n(t) \in \mathbf{K}, \quad \forall t \in [t_n^{j-1}, t_n^j]$$

Finite-dimensional problem:

Find $H_j^n \in \mathbf{K}_m \subset \mathbf{R}^m$:

$$\left\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \right\rangle \geq 0, \quad \forall F_j^n \in \mathbf{K}_m$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t) dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t) dt.$$

$$H_n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1}, t_n^j) H_j^n \quad \forall n \in N$$

piecewise constant approximations to solutions to

$$\int_0^T \langle A(t)H(t) + B(t), F(t) - H(t) \rangle dt \geq 0, \quad \forall F(t) \in K? \quad (a)$$

➤ Theor. 3:

$A(t)$ positive definite a.e. in $[0, T] \Rightarrow U = \{H_n\}_{n \in N}$

weakly compact; its cluster points are feasible.

If \overline{H} weak cluster point for $U \Rightarrow \overline{H}$ solution to (a)

Time-dependent convex set:

$$\mathbf{K} = \left\{ F(t) \in L^2([0, T], \mathbf{R}^m) : \lambda(t) \leq F(t) \leq \nu(t), \text{ a.e. in } [0, T], \right. \\ \left. \lambda(t), \nu(t) \geq 0, \Phi F(t) = \rho(t), \text{ a.e. in } [0, T] \right\}$$

$$\mathbf{K}_j^n = \left\{ F(t) \in L^2([0, T], \mathbf{R}^m), \text{ piecewise constant:} \right. \\ \left. \bar{\lambda}_{j,n} \leq F_j(t) \leq \bar{\nu}_{j,n}, \text{ a.e. in } (t_{j-1}, t_j), \right. \\ \left. \Phi F(t) = \bar{\rho}_{j,n}, \text{ a.e. in } (t_{j-1}, t_j) \right\}$$

where

$$\bar{\lambda}_{j,n} = \mu_{j,n} \lambda(t), \quad \bar{\nu}_{j,n} = \mu_{j,n} \nu(t), \quad \bar{\rho}_{j,n} = \mu_{j,n} \rho(t)$$

➤ **Lemma 1:**

The set sequence $K^n = \bigcap K_j^n$ converges to K

➤ **Theor. 4:**

$A(t)$ positive definite a.e. in $[0, T] \Rightarrow$ the sequence

$H^n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1}, t_n^j) H_j^n$ admits weak cluster points.

Each weak cluster point is feasible and solves (a).

➤ Application to Dynamic Traffic Network



Cost functions:

$$C_1(H(t)) = H_1(t) + 1$$

$$C_2(H(t)) = H_2(t) + 2$$

$$\mathbf{K} = \bigcup_{t \in [0,2]} \left\{ F(t) \in L^2([0,2], \mathbf{R}^2) : 0 \leq F_1(t) \leq t, 0 \leq F_2(t) \leq \frac{3}{2}t \text{ a.e. in } [0,2]; \right. \\ \left. F_1(t) + F_2(t) = t \text{ a.e. in } [0,2] \right\}$$

Vector field:

$$F : L^2([0, 2], \mathbf{R}^2) \rightarrow L^2([0, 2], \mathbf{R}^2)$$

$$(F_1(H(t)), F_2(H(t))) = (H_1(t) + 1, H_2(t) + 2).$$

$t_0 \in \left\{ \frac{k}{4} : k = 0, \dots, 8 \right\} \Rightarrow$ sequence of PDS defined by:

$$-F(H_1(t_0), H_2(t_0)) = (-H_1(t) - 1, -H_2(t) - 2) \text{ on}$$

$$\mathbf{K}_{t_0} = \left\{ \left\{ [0, t_0] \times \left[0, \frac{3}{2} t_0 \right] \right\} \cap \{x + y = t_0\} \right\}$$

Unique equilibrium at t_0 :

$$\left(H_1(t_0), H_2(t_0)\right) \in \mathbf{R}^2 : -F\left(H_1(t_0), H_2(t_0)\right) \in N_{\mathbf{K}_{t_0}}\left(H_1(t_0), H_2(t_0)\right)$$

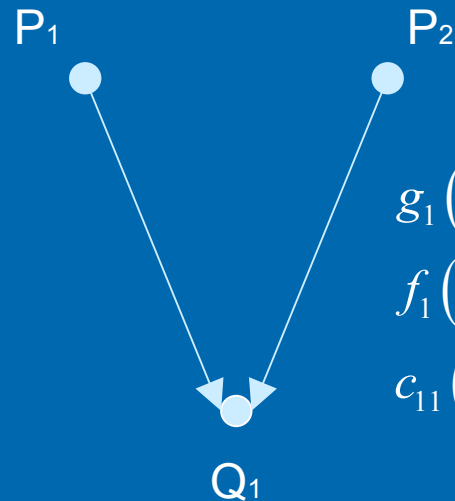
Equilibria:

$$\left\{ (0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), (1,0), \left(\frac{9}{8}, \frac{1}{8}\right), \left(\frac{5}{4}, \frac{1}{4}\right), \left(\frac{11}{8}, \frac{3}{8}\right), \left(\frac{3}{2}, \frac{1}{2}\right) \right\}$$

Explicit formulae:

$$\begin{cases} H_1(t) = t \\ H_2(t) = 0 \end{cases} \quad \text{if } 0 \leq t \leq 1; \quad \begin{cases} H_1(t) = \frac{t+1}{2} \\ H_2(t) = \frac{t-1}{2} \end{cases} \quad \text{if } 1 \leq t \leq 2.$$

➤ Application to Spatial Price Economic Markets



$$g_1(p(t)) = p_1(t) + h(t); g_2(p(t)) = p_2(t) + k(t);$$

$$f_1(q(t)) = k(t) - q_1(t);$$

$$c_{11}(x(t)) = x_{11}(t); c_{21}(x(t)) = x_{21}(t) + 1;$$

$$\mathbf{K} = \left\{ u(t) = (p(t), q(t), x(t)) \in L^2 \left(\left[0, \frac{1}{2} \right], \mathbf{R}^2 \right) \times L^2 \left(\left[0, \frac{1}{2} \right], \mathbf{R} \right) \times L^2 \left(\left[0, \frac{1}{2} \right], \mathbf{R}^2 \right) : \right.$$

$$0 \leq p_1(t) \leq h(t) + k(t) + 1, 0 \leq p_2(t) \leq h(t) + k(t) + 1, 0 \leq q_1(t) \leq h(t) + k(t) + 1,$$

$$\left. 0 \leq x_{11}(t) \leq h(t) + k(t) + 1, 0 \leq x_{21}(t) \leq h(t) + k(t) + 1 \right\}$$

➤ Direct method

$$\begin{cases} v(u(t))=0 \\ u(t) \in \mathbf{K} \end{cases} \Rightarrow \begin{cases} p_1(t) - x_{11}(t) + h(t) = 0 \\ q_1(t) + x_{11}(t) + x_{21}(t) - k(t) = 0 \\ p_1(t) + x_{11}(t) - q_1(t) = 0 \\ x_{21}(t) - q_1(t) + 1 = 0 \\ u(t) \in \mathbf{K} \end{cases} \Rightarrow \emptyset$$

$$\mathbf{K} \cap \{p_2(t)=0\} : \begin{cases} -3h(t) + k(t) + 1 \geq 0 \\ -h(t) + 2k(t) - 3 \geq 0 \end{cases}$$

$$p_1(t) = \frac{-3h(t) + k(t) + 1}{5}; \quad p_2(t) = 0; \quad q_1(t) = \frac{-h(t) + 2k(t) + 2}{5};$$

$$x_{11}(t) = \frac{2h(t) + k(t) + 1}{5}; \quad x_{21}(t) = \frac{-h(t) + 2k(t) - 3}{5};$$

$$g_1(p(t)) = \frac{2h(t) + k(t) + 1}{5}; \quad g_2(p(t)) = k(t); \quad f_1(q(t)) = \frac{h(t) + 3k(t) - 2}{5};$$

$$c_{11}(x(t)) = \frac{2h(t) + k(t) + 1}{5}; \quad c_{21}(x(t)) = \frac{-h(t) + 2k(t) + 2}{5}; \quad s_2(t) = \frac{h(t) + 3k(t) + 3}{5}.$$

➤ Discretization method: $h(t) = \frac{t}{2}, k(t) = t + \frac{8}{7}$

$$\int_0^{\frac{1}{2}} \langle A(t)u^*(t) + b(t), u(t) - u^*(t) \rangle dt \geq 0, \quad \forall u(t) \in \mathbf{K}$$

$$A(t) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}, \quad b(t) = \begin{bmatrix} h(t) \\ k(t) \\ -k(t) \\ 0 \\ 1 \end{bmatrix}$$

$$\left[0, \frac{1}{2}\right]: 0 < \frac{1}{2n} < \dots < \frac{j-1}{2n} < \frac{j}{2n} < \dots < \frac{1}{2}$$

$\mathbf{K}_j^n = \{u_j^n(t) = (p_{j1}^n(t), p_{j2}^n(t), q_{j1}^n(t), x_{j11}^n(t), x_{j21}^n(t)) \text{ piecewise constant:}$

$$0 \leq p_{j1}^n(t) \leq \frac{6j-3}{8n} + \frac{15}{7}, 0 \leq p_{j2}^n(t) \leq \frac{6j-3}{8n} + \frac{15}{7}, 0 \leq q_{j1}^n(t) \leq \frac{6j-3}{8n} + \frac{15}{7},$$

$$\left. 0 \leq x_{j11}^n(t) \leq \frac{6j-3}{8n} + \frac{15}{7}, 0 \leq x_{j21}^n(t) \leq \frac{6j-3}{8n} + \frac{15}{7} \right\}$$

Variational Inequality:

$$\left\langle A(t)(u_j^n(t))^* + b_j^n(t), u_j^n(t) - (u_j^n(t))^* \right\rangle \geq 0 \quad \forall u_j^n(t) \in \mathbf{K}_j^n(t)$$

where $b_j^n(t) = \begin{bmatrix} \frac{2j-1}{8n} \\ \frac{2j-1}{4n} + \frac{8}{7} \\ -\frac{2j-1}{4n} - \frac{8}{7} \\ 0 \\ 1 \end{bmatrix}$

$$\begin{cases} A(t)u_j^n(t) + b_j^n(t) = 0 \\ u_j^n(t) \in \mathbf{K}_j \end{cases} \Rightarrow \emptyset$$

$$\mathbf{K}_j^n \cap \{p_{j2}^n(t) = 0\} : \begin{cases} A(t)u_j^n(t) + b_j^n(t) = 0 \\ p_{j2}^n(t) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} p_{j1}^n(t) = \frac{-2j+1}{40n} + \frac{3}{7}; & p_{j2}^n(t) = 0; & q_{j1}^n(t) = \frac{6j-3}{40n} + \frac{6}{7}; \\ x_{j11}^n(t) = \frac{2j-1}{10n} + \frac{3}{7}; & x_{j21}^n(t) = \frac{6j-3}{40n} - \frac{1}{7}; \\ g_{j1}^n(p(t)) = \frac{2j-1}{10n} + \frac{3}{7}; & g_{j2}^n(p(t)) = \frac{2j-1}{4n} + \frac{8}{7}; & f_{j1}^n(q(t)) = \frac{14j-7}{40n} + \frac{2}{7}; \\ c_{j11}^n(x(t)) = \frac{2j-1}{10n} + \frac{3}{7}; & c_{j21}^n(x(t)) = \frac{6j-3}{40n} + \frac{6}{7}; \end{cases}$$

$$\text{Excess: } s_{j2}^n(t) = \frac{14j-7}{40n} + \frac{6}{7};$$

Solution on $\mathbf{K}^n = \bigcap \mathbf{K}_j^n$:

$$u_n(t) = \sum_{j=1}^n \chi\left(\frac{j-1}{2n}, \frac{j}{2n}\right) u_j^n(t).$$