Double-Layered Dynamics: A Unified Theory of Projected Dynamical Systems and Evolutionary Variational Inequalities

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OUTLINE

I will present here brief introductions to projected dynamical systems and evolutionary variational inequalities and highlight the way the two theories intertwine in a new theory of double-layered dynamics. This provides us with novel ways of interpreting applied problems. I shall proceed as follows:

- Background and application contexts;

- Projected differential equations (PrDE) and projected dynamical systems (PDS);

- Evolutionary variational inequalities (EVI);

- A unified framework for EVI constraint sets and how the PDS and EVI mesh;

- Qualitative analysis including stability analysis;

- Answers to three critical questions regarding the double-layered dynamics;

- Application of double-layered dynamics to traffic networks.

BACKGROUND

Numerous problems in engineering, in operations research and the management sciences, as well as in economics and finance involve interactions among decision-makers and the competition for resources.

In such problems, the *concept of equilibrium* plays a *central role* and provides a valuable benchmark against which an existing state of such complex systems can be compared.

Examples, par excellence, of such equilibrium problems include:

- congested urban transportation networks,
- spatial price equilibrium problems,
- the Internet and supernetworks,
- financial equilibrium problems, and
- decentralized supply chain networks with various related applications.

Transportation science has historically been the discipline that has **pushed the frontiers** in terms of methodological developments for such problems (which are often large-scale) beginning with the work of Beckmann, McGuire, and Winsten (1956).

Dafermos (1980) later showed that the traffic network equilibrium conditions as formulated by Smith (1979) were a finite-dimensional variational inequality and then utilized the theory to establish both existence and uniqueness results of the equilibrium traffic flow pattern as well as to propose an algorithm with convergence results.

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above, as well as to **game theoretic problems**, such as oligopolistic market equilibrium problems, and to general economic equilibrium problems (see, e.g., Nagurney (1993) and the references therein).

PROJECTED DYNAMICAL SYSTEMS and FINITE-DIMENSIONAL VARIATIONAL INEQUALITIES

As important as the study of the equilibrium state is that of the **study of the underlying dynamics or disequilibrium behavior of such systems**.

Since such problems typically involve more constraints (such as, for example, budgetary, conservation of flow, nonnegativity assumptions on the variables, among others) classical dynamical systems theory is no longer sufficient for the formulation and solution of such problems.

In 1993, Dupuis and Nagurney introduced a new class of dynamical system with a discontinuous right-hand side and provided the foundational theory for such projected dynamical systems.

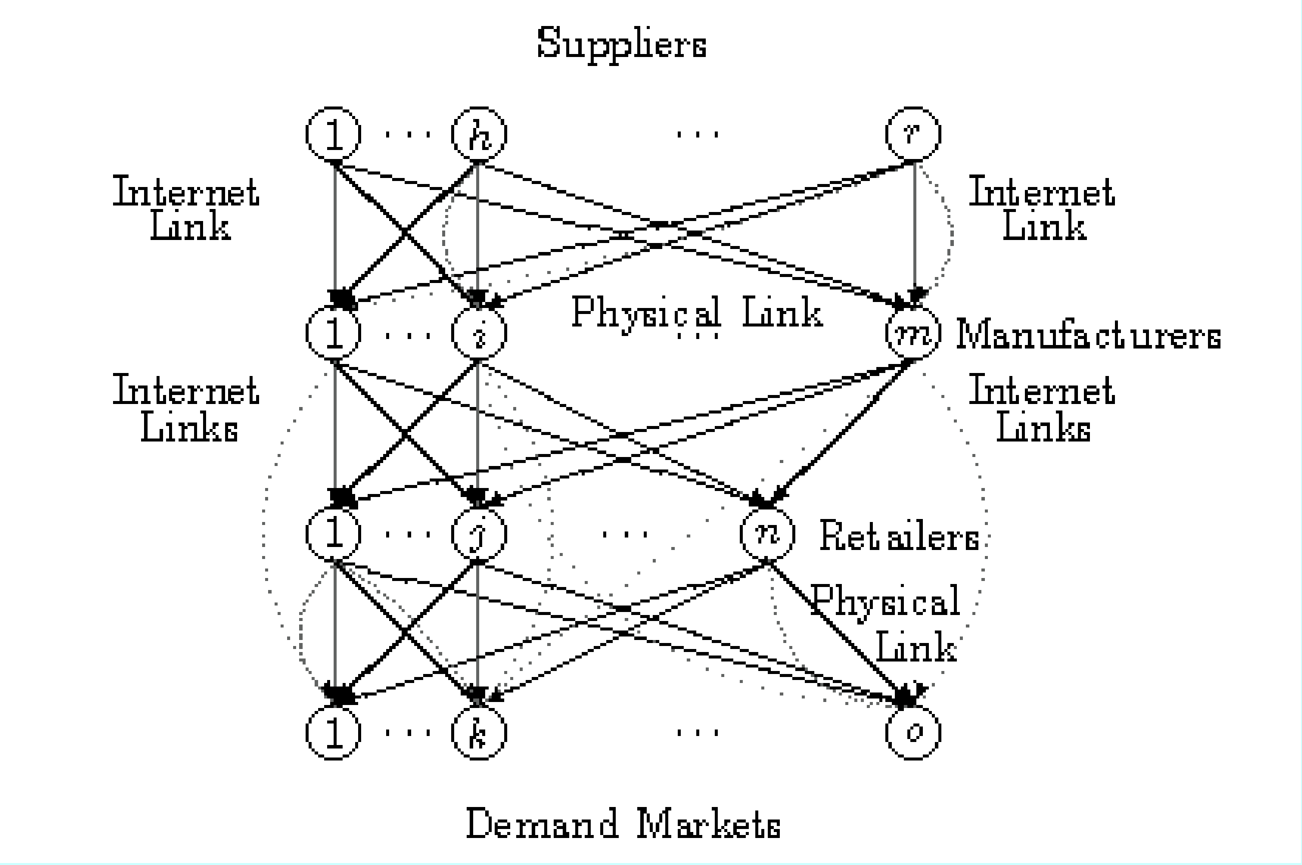
Moreover, they established, under suitable conditions, that the **set of stationary points of a projected dynamical system** coincided with the **set of solutions of the associated finite-dimensional variational inequality (VI).**

This connection allowed for the investigation of the disequilibrium behavior preceding the attainment of the equilibrium.

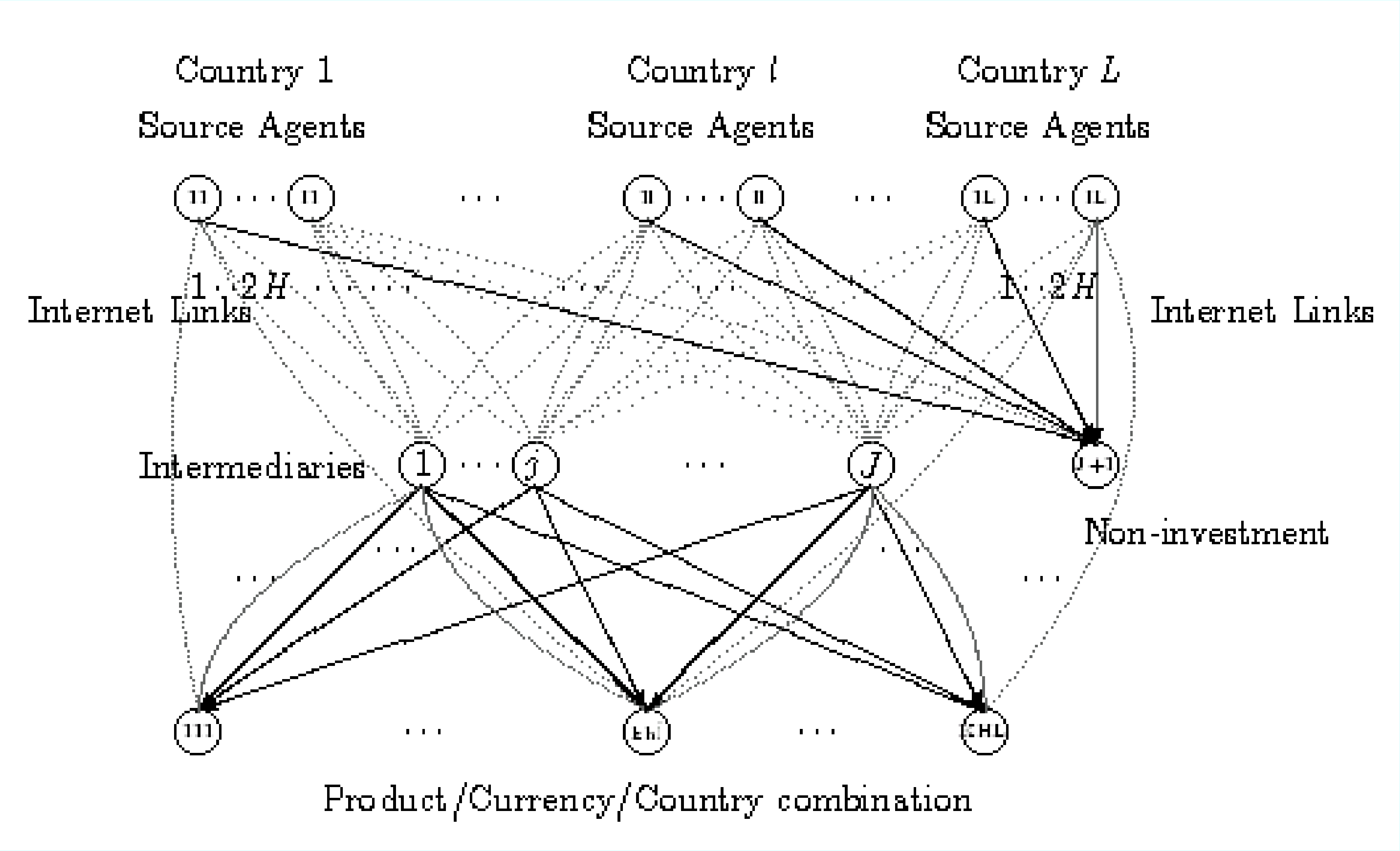
In 1995, Zhang and Nagurney (1995) developed the stability theory for finite-dimensional projected dynamical systems.

To-date, PDS theory and finite-dimensional VI theory has been used to formulate and solve a plethora of applications ranging from congested urban transportation networks to supply chains (with and without electronic commerce) and a variety of financial networks (with and without intermediation) with the incorporation even of financial transactions.

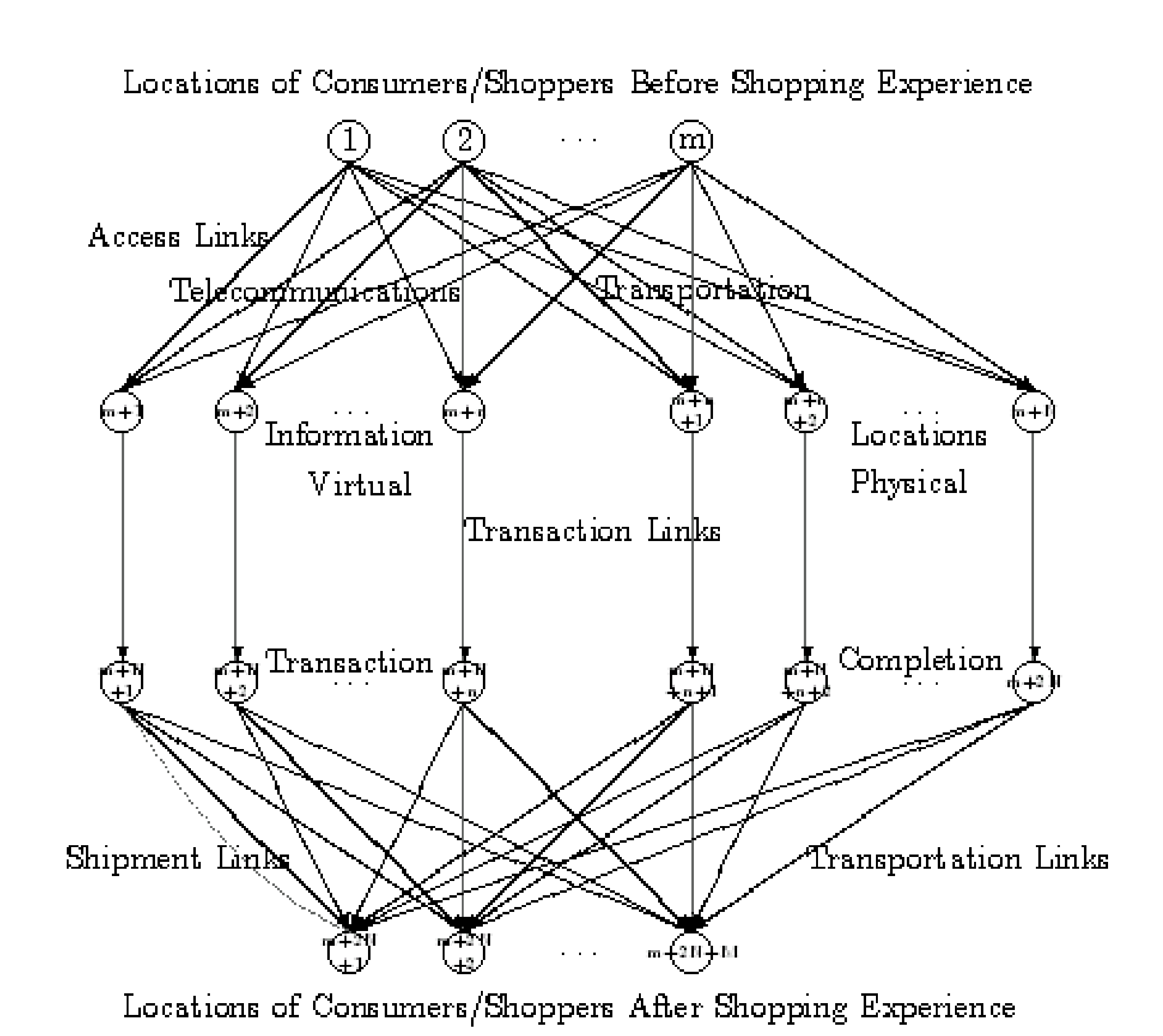
Supply Chain Network



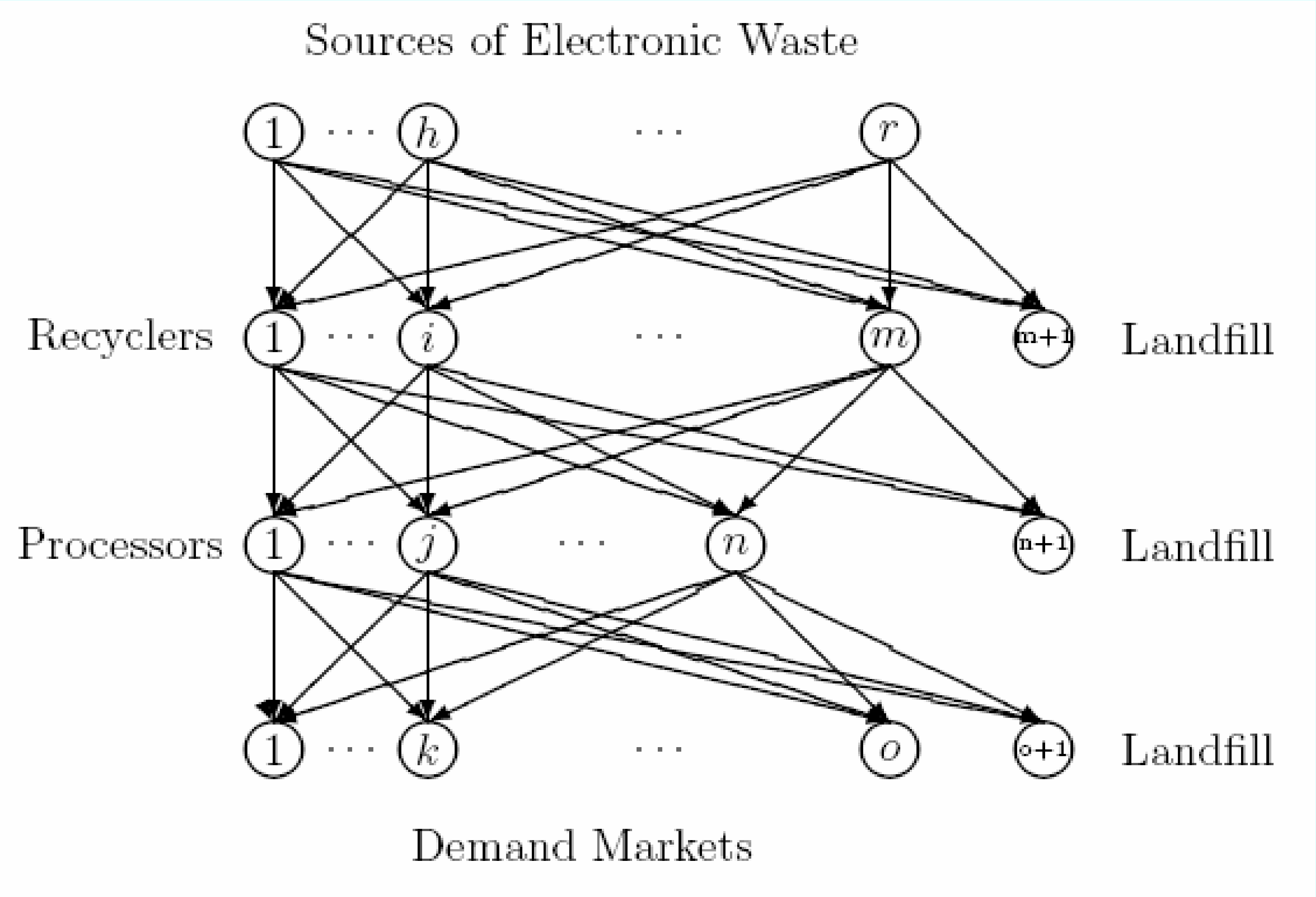
International Financial Network with Electronic Transactions



Teleshopping versus Shopping



The 4-Tiered E-Cycling Network



PROJECTED DYNAMICAL SYSTEMS ON INFINITE-DIMENSIONAL HILBERT SPACES

Isac and Cojocaru (2002, 2004) initiated the systematic study of projected dynamical systems on infinite-dimensional Hilbert spaces with the fundamental issue of existence of solutions to such problems answered by Cojocaru (2002) in her thesis (see also Cojocaru and Jonker (2004)).

EVOLUTIONARY VARIATIONAL INEQUALITIES

Evolutionary variational inequalities, which are also infinite-dimensional, were originally introduced by Lions and Stampacchia (1967) and by Brezis (1967) in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems.

Steinbach (1998) studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity.

Daniele, Maugeri, and Oettli (1998, 1999), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others. See also Ran and Boyce(1996).

As noted by Cojocaru, Daniele, and Nagurney (2004), the theory and application of evolutionary variational inequalities was developing in parallel to that of projected dynamical systems.

Cojocaru, Daniele, and Nagurney (2004) built the basis for merging the theory of projected dynamical systems (PDS) and that of evolutionary variational inequalities (EVI), in order to further develop the theoretical analysis and computation of solutions to applied problems in which dynamics plays a central role.

The intriguing feature of the merger is that it allows for the modeling of problems that present two (theoretically) distinct timeframes, most simply put, a *big scale time* and a *small scale time*.

The existing literature has focused on understanding human decision-making for a specific timescale rather than viewing decision-making over multiple timescales. The ability to capture multiple timescales can also further support combined strategic and operational decision-making and planning.

There are new exciting questions, both theoretical and computational, arising from this "multiple time structure."

In the course of answering these questions, a new theory is taking shape from the synthesis of PDS and EVI, and, as such, it deserves a name of its own; we call it **double-layered dynamics**.

PrDE and PDS - I

The most general mathematical context to date in which we can define a projected differential equation (PrDE) and, consequently, a projected dynamical system (PDS), is that of a Hilbert space X of arbitrary (finite or infinite) dimension.

Suppose that we have $K \subset X$, a nonempty, closed, convex subset in a Hilbert space X. Let $F : K \to X$ be a Lipschitz continuous mapping. It is well-known that the ODE:

$$\frac{\partial x(t)}{\partial t} = -F(x(t)), \quad x(0) \in K$$

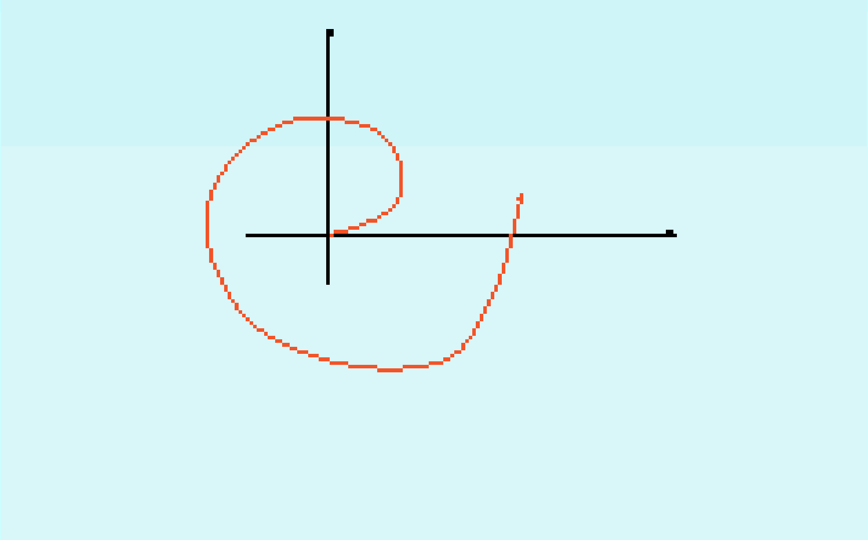
has solutions in a suitable class of functions; here that class will be that of absolutely continuous functions AC$([0, \infty), X)$.

Let us define a PrDE on an example, "with drawings:"

Suppose $X := R^2$, $K := R^2_+$, and suppose that the image below represents a trajectory of the equation

$$\frac{\partial x(t)}{\partial t} = -F(x(t)),$$

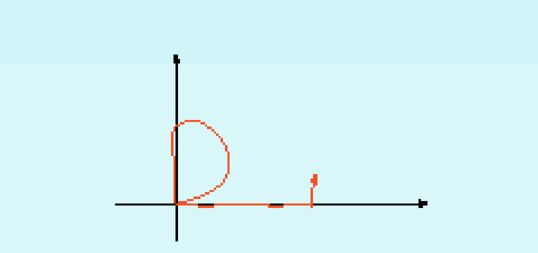
starting in R^2_+.



A PrDE describes the control problem:

$$\frac{\partial}{\partial t}(x(t)) = -F(x(t)), \quad x(0) \in R^2_+$$

such that $x(t) \in R^2_+$, as shown in the figure below:



In other words, a trajectory of a projected differential equation is always "trapped" in the constraint set $K = R^2_+$ and the velocity field along any such trajectory is not continuous.

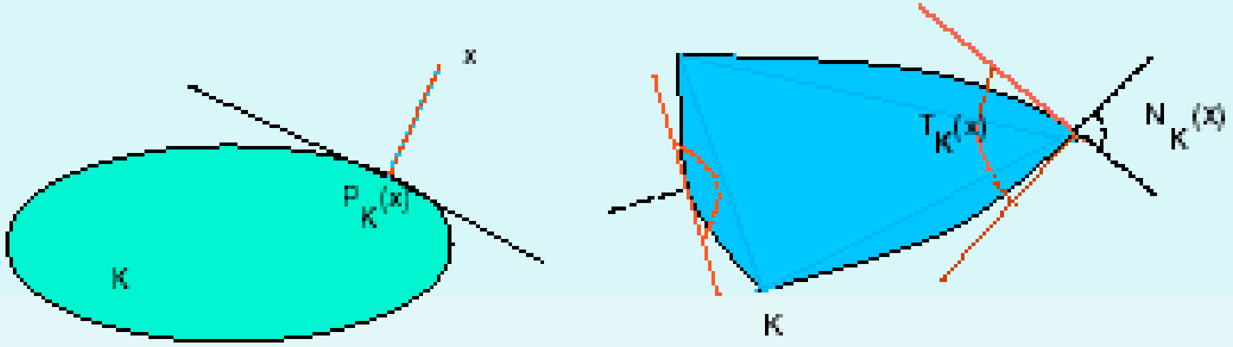
PrDE and PDS - II

To rigorously define the two notions, we recall the following:

1). the projection of X onto K by $P_K : X \to K$, with

$$\|P_K(x) - x\| = \inf_{z \in K} \|z - x\|, \quad \forall z \in X,$$

2. the tangent cone $T_K(x) = \overline{\cup_{h>0} \frac{1}{h}(K - x)}$.



PrDE and PDS - III

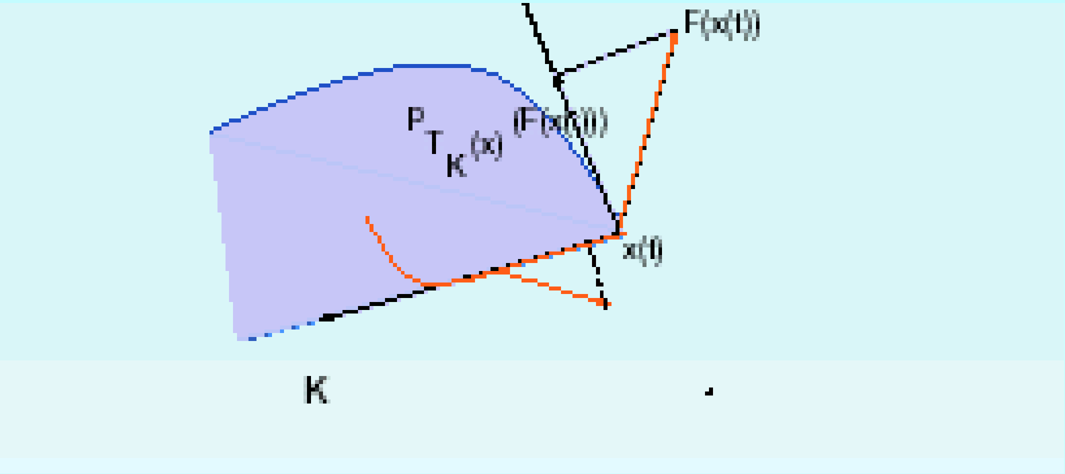
Let X, $K \subset X$, and $F : K \to X$ as before. Then a PrDE is defined by:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_K(x(t), -F(x(t))), \quad x(0) = x_0,$$

where

$$\Pi_K(x, -F(x)) = \lim_{\delta \to 0+} \frac{P_K(x - \delta F(x)) - x}{\delta} =: P_{T_K(x)}(-F(x)),$$

where $T_K(x)$ is the tangent cone to the set K at x and $N_K(x)$ is the normal cone to K at the same point x.



The right-hand side of any PrDE is nonlinear and discontinuous.

An existence result for such equations was obtained by Dupuis and Nagurney (1993) for $X := R^n$, and by Cojocaru (2002) for general Hilbert spaces.

Theorem 1

Let X be a Hilbert space of arbitrary dimension and let $K \subset X$ be a non-empty, closed, and convex subset. Let $F : K \to X$ be a Lipschitz continuous vector field on K with $x_0 \in K$. Then the initial value problem

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))), x(0) = x_0$$

has a unique solution in $AC([0, \infty), K)$.

A projected dynamical system (PDS) is the dynamical system given by the set of trajectories of a PrDE.

EQUILIBRIA of PDS and VARIATIONAL INEQUALITIES

An important feature of any PDS is that it is intimately related to a variational inequality problem (VI).

The starting point of VI theory: 1966 (Hartman and Stampacchia); 1967 (Lions and Stampacchia); it is now part of the calculus of variations; it has been used to show existence of equilibrium in a plethora of equilibrium problems and free boundary problems.

The following relation between a PDS and a VI was shown by Dupuis and Nagurney (1993) for $X := R^n$ and by Cojocaru (2002) for any Hilbert space. Here $F : K \to X$.

Theorem 2

The equilibria of a PDS:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_K(x(t), -F(x(t))),$$

that is, $x^* \in K$ *such that*

$$\Pi_K(x^*, -F(x^*)) = 0$$

are solutions to the VI(F, K)*: find* $x^* \in K$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

and vice-versa, where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* X.

We are interested in studying an evolutionary variational inequality in the form proposed by Daniele, Maugeri, and Oettli (1998, 1999). They modeled and studied the traffic network problem with feasible path flows which have to satisfy time–dependent capacity constraints and demands.

They proved that the equilibrium conditions (in the form of generalized Wardrop (1952) conditions) can be expressed by means of an EVI, for which existence theorems and computational procedures were given. The algorithm proposed was based on the subgradient method.

In addition, the EVI for spatial price equilibrium problems (see Daniele and Maugeri (2001) and Daniele (2004)) and for financial equilibria (2003) have been derived.

STANDARD EVI FORM (TIME-DEPENDENT) VARIATIONAL INEQUALITIES

Recall that $\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt$ is the duality mapping on $L^p([0,T], R^q)$, where $\phi \in (L^p([0,T], R^q))^*$ and $u \in L^p([0,T], R^q)$. Let $F : K \to (L^p([0,T], R^q))^*$.

The standard form of the evolutionary variational inequality (EVI) that we work with is:

find $u \in K$ such that $\ll F(u), v - u \gg \geq 0, \ \forall v \in K,$

or, equivalently, find $u \in K$ such that

$$\int_0^T \langle F(u(t)), v(t) - u(t) \rangle dt \geq 0, \quad \forall v \in K.$$

A UNIFIED FEASIBLE SET and EVI FORMULATON (Cojocaru, Daniele, and Nagurney (2004))

We consider a nonempty, convex, closed, bounded subset of the reflexive Banach space $L^p([0,T],R^q)$ given by:

$$K = \bigcup_{t\in[0,T]} \{u \in L^p([0,T],R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\},$$

$$j \in \{1,\ldots,l\}\}.$$

Let $\lambda, \mu \in L^p([0,T],R^q)$, $\rho \in L^p([0,T],R^l)$ be convex functions in the above definition. For chosen values of the scalars ξ_{ji}, of the dimensions q and l, and of the boundaries λ, μ, we obtain each of the previous above-cited model constraint set formulations as follows:

- for the traffic network problem (see Daniele, Maugeri, and Oettli (1998, 1999)) we let $\xi_{ji} \in \{0,1\}$, $i \in \{1,..,q\}$, $j \in \{1,\ldots,l\}$, and $\lambda(t) \geq 0$ for all $t \in [0,T]$;

- for the quantity formulation of spatial price equilibrium (see Daniele (2004)) we let $q = n + m + nm$, $l = n + m$, $\xi_{ji} \in \{0,1\}$, $i \in \{1,..,q\}$, $j \in \{1,\ldots,l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0,T]$;

- for the price formulation of spatial price equilibrium (see Daniele (2003)) we let $q = n + m + mn$, $l = 1$, $\xi_{ji} = 0$, $i \in \{1, .., q\}$, $j \in \{1, \ldots, l\}$, and $\lambda(t) \geq 0$ for all $t \in [0, T]$;

- for the financial equilibrium problem (cf. Daniele (2003)) we let $q = 2mn + n$, $l = 2m$, $\xi_{ji} = \{0, 1\}$ for $i \in \{1, .., n\}$, $j \in \{1, \ldots, l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0, T]$.

SOME PRELIMINARIES AND DEFINITIONS

In the general theory of variational inequalities, of which EVI are a part, as well as in Nonlinear Analysis and Optimization, the concept of monotone mappings and its extensions have been extensively used in existence / uniqueness-type results.

From among the extensions of monotonicity, we recall here definitions of pseudomonotonicity, which are used throughout the analysis.

Definition

Let E be a reflexive Banach space with dual E^*, $\ll \cdot, \cdot \gg$ the duality map between E^* and E, K a non-empty closed, convex subset of E and $F : K \to E^*$. Then:
(1) A map F is called **pseudo-monotone** *on K if, for every pair of points $x, y \in K$, we have*

$$\langle F(x), y - x \rangle \geq 0 \to \langle F(y), y - x \rangle \geq 0.$$

(2) A map F is **strictly pseudo-monotone** *on K if, for every pair of distinct points x, y, we have*

$$\langle F(x), y - x \rangle \geq 0 \to \langle F(y), y - x \rangle > 0.$$

(3) A map F is **strongly pseudo-monotone** *on K if, there exists $\eta > 0$ such that, for every pair of distinct points x, y, we have*

$$\langle F(x), y - x \rangle \geq 0 \to \langle F(y), y - x \rangle \geq \eta ||y - x||^2.$$

Daniele, Maugeri, and Oettli (1998) gave an existence result for an EVI as above:

Theorem 3

If F satisfies either of the following conditions:

1. *F is hemicontinuous with respect to the strong topology on K, and there exist $A \subseteq K$ nonempty, compact, and $B \subseteq K$ compact such that, for every $v \in K \setminus A$, there exists $v \in B$ with $\ll F(u), v - u \gg \geq 0$;*

2. *F is hemicontinuous with respect to the weak topology on K;*

3. *F is pseudomonotone and hemicontinuous along line segments,*

then the EVI problem above admits a solution over the constraint set K.

DOUBLE-LAYERED DYNAMICS: MERGING PDS and EVI

The theory of EVI and that of PDS can be intertwined for the purpose of deepening the analysis of many dynamic applied problems arising in different disciplines. The fundamental theoretical ideas, together with an example of such problems, specifically, a dynamic traffic network problem, were given in Cojocaru, Daniele, and Nagurney (2004). However, the implications of one theory over the other have to be further studied.

Here we continue to develop and consolidate the mathematical formalism of this new emerging theory which we call **double-layered dynamics**, thus opening up new questions as topics for future work.

First and foremost, we have seen that the EVI considered involves a constraint set of a Banach space, but to be used in conjunction with PDS theory, we need to limit ourselves to Hilbert spaces; therefore, we set $p := 2$ and consider only constraint sets $K \in L^2([0,T], R^q)$, as given.

By definition, such sets are closed and convex.

Also note that the elements in the set K vary with time, but K is fixed in the space of functions $L^2([0,T], R^q)$, $T > 0$ given.

DOUBLE-LAYERED DYNAMICS:

Consider the above (EVI), where F is pseudomonotone and Lipschitz continuous and $K \in L^2([0,T], R^q)$ is given as above.

Lipschitz continuity implies hemicontinuity, which, in turn, implies hemicontinuity on line segments, so according to Theorem 3, the EVI problem has solutions.

We are also in the scope of Theorem 1, and, therefore, we can consider the PDS defined on the closed and convex set K by the PrDE:

$$\frac{du(\cdot,\tau)}{d\tau} = \Pi_K(u(\cdot,\tau), -F(u(\cdot,\tau))),$$

$$u(\cdot,0) = u(\cdot) \in K,$$

where time τ is different than time t in the EVI. In general, the PDS has solutions in the set of absolutely continuous functions in the τ variable, $AC([0,\infty), K)$. However, we will limit ourselves to finite intervals for τ, i.e., with $\tau \in [0,l]$, $l > 0$, given.

The meaning of the "two times" used here needs to be well understood. Intuitively, at each moment $t \in [0, T]$, the solution of the EVI represents a static state of the underlying system.

As t varies over the interval $[0, T]$, the static states describe one (or more) curve(s) of equilibria. In contrast, τ is the time that describes the dynamics of the system until it reaches one of the equilibria on the curve(s). This structure motivates the name of the new theory, which has many interesting features. Our intuitive explanation is rigorously confirmed by the following result.

Theorem 4 (Cojocaru, Daniele, and Nagurney (2004))

The solutions to the EVI problem are the same as the critical points of the PDS and vice versa, that is, the critical points of the PDS are the solutions to the EVI.

Hence, by choosing the Hilbert space to be $L^2([0,T], R^q)$, *we find that the solutions to the evolutionary variational inequality: find* $u \in K$ *such that*

$$\int_0^T \langle F(u(t)), v(t) - u(t) \rangle dt \geq 0, \quad \forall v \in K$$

are the same as the critical points of the equation:

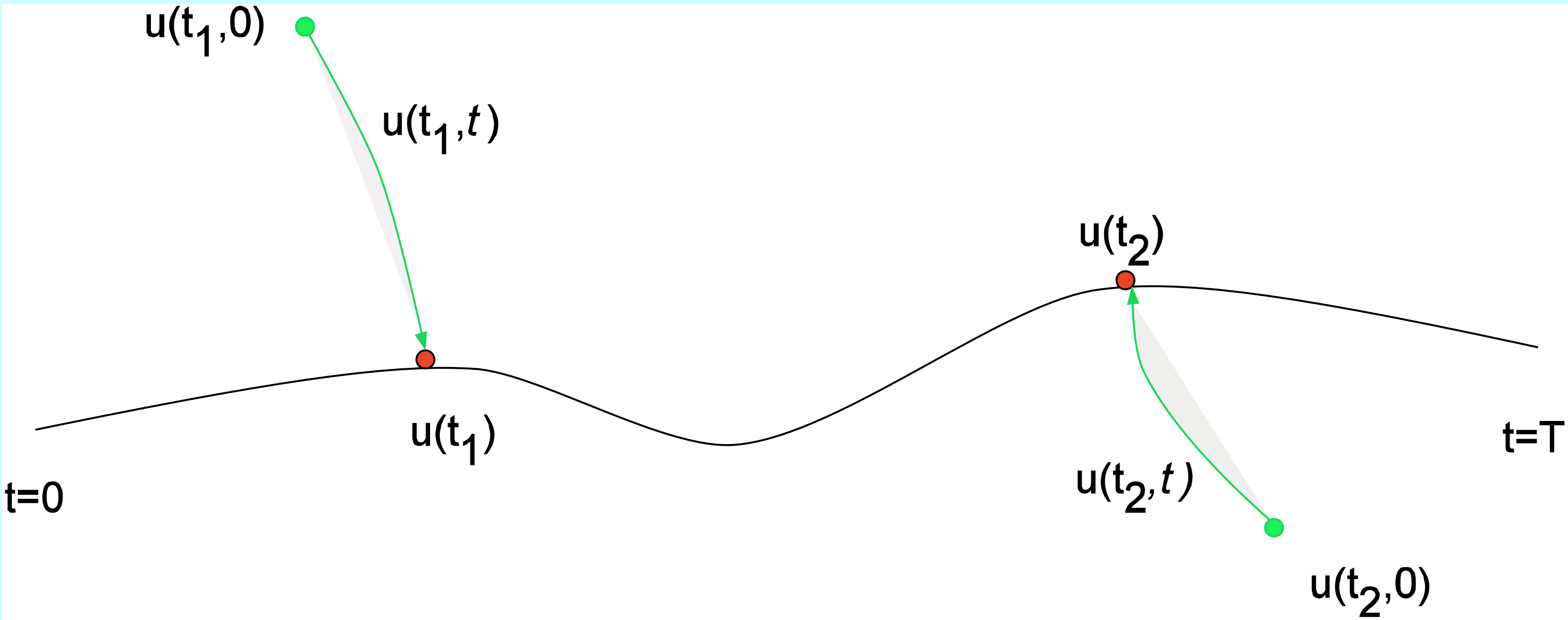
$$\frac{\partial u(t,\tau)}{\partial \tau} = \Pi_K(u(t,\tau), -F(u(t,\tau))),$$

that is, the points such that

$$\Pi_K(u(t,\tau), -F(u(t,\tau))) \equiv 0 \quad a.e.in \; [0,T],$$

which are obviously stationary with respect to τ.

A Pictorial of the Double-Layered Dynamics



This result is the most important feature in merging the two theories and in computing and interpreting problems ranging from spatial price (quantity and price formulations), traffic network equilibrium problems, and general financial equilibrium problems.

Now we are ready to answer the question of uniqueness of solutions to the EVI. It is known that, in general, strict monotonicity implies uniqueness of solutions for a variational inequality (Stampacchia (1968)) and, hence, if F is strictly monotone, then it is pseudomonotone and the solution to the EVI is unique.

But, generally, pseudomonotonicity or strict pseudomonotonicity alone cannot guarantee uniqueness of such solutions for the PDS.

This is not so in the PDS theory, where it is easy to show that if F is only strictly pseudomonotone, but not strictly monotone, the PDS still has a unique equilibrium.

Proposition 1 (Cojocaru, Daniele, and Nagurney (2004b))

Assume that F is strictly pseudomonotone and Lipschitz on K. Then the PDS has at most one equilibrium point.

Here is a direct, important consequence of the new theory of double-layered dynamics:

Proposition 2

Assume either one of the hypotheses (2) or (3) of Theorem 3, where F is strictly pseudomonotone on K and assume DLDH. Then the EVI has at most one solution.

STABILITY PROPERTIES of the CURVE of EQUILIBRIA; the RELATION BETWEEN the TWO TIMEFRAMES

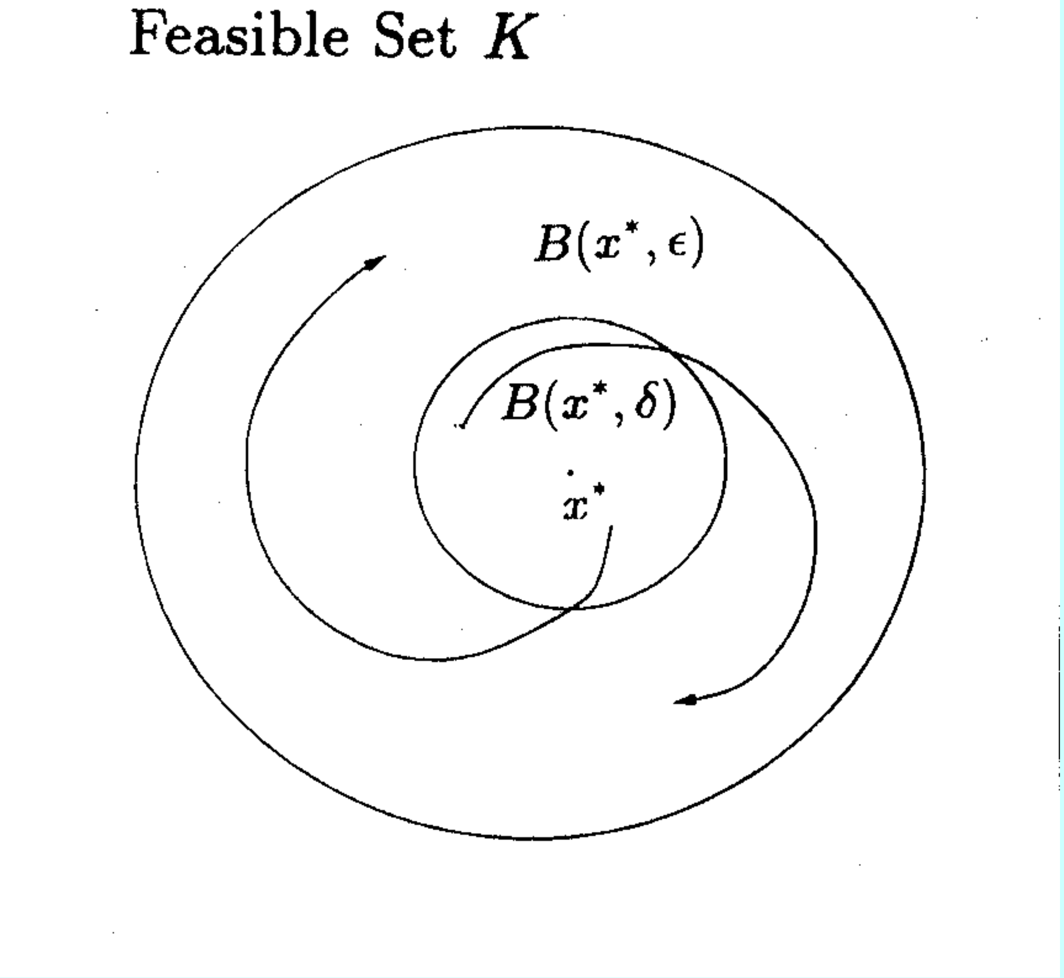
Here we address the stability properties of solution(s) to the EVI, viewed as curves of equilibria for PDS. We also make precise the relation between PDS time and EVI time, together with its meaning in applications.

In Cojocaru, Daniele, and Nagurney (2004) we remarked that, intuitively, time t describes the curve of equilibria, while time τ describes the evolution of the projected dynamics in the presence of this curve. In our first paper, we assumed that the projected dynamics should describe how the underlying problem "approaches" this curve, but we did not give a proof of why and how this happens.

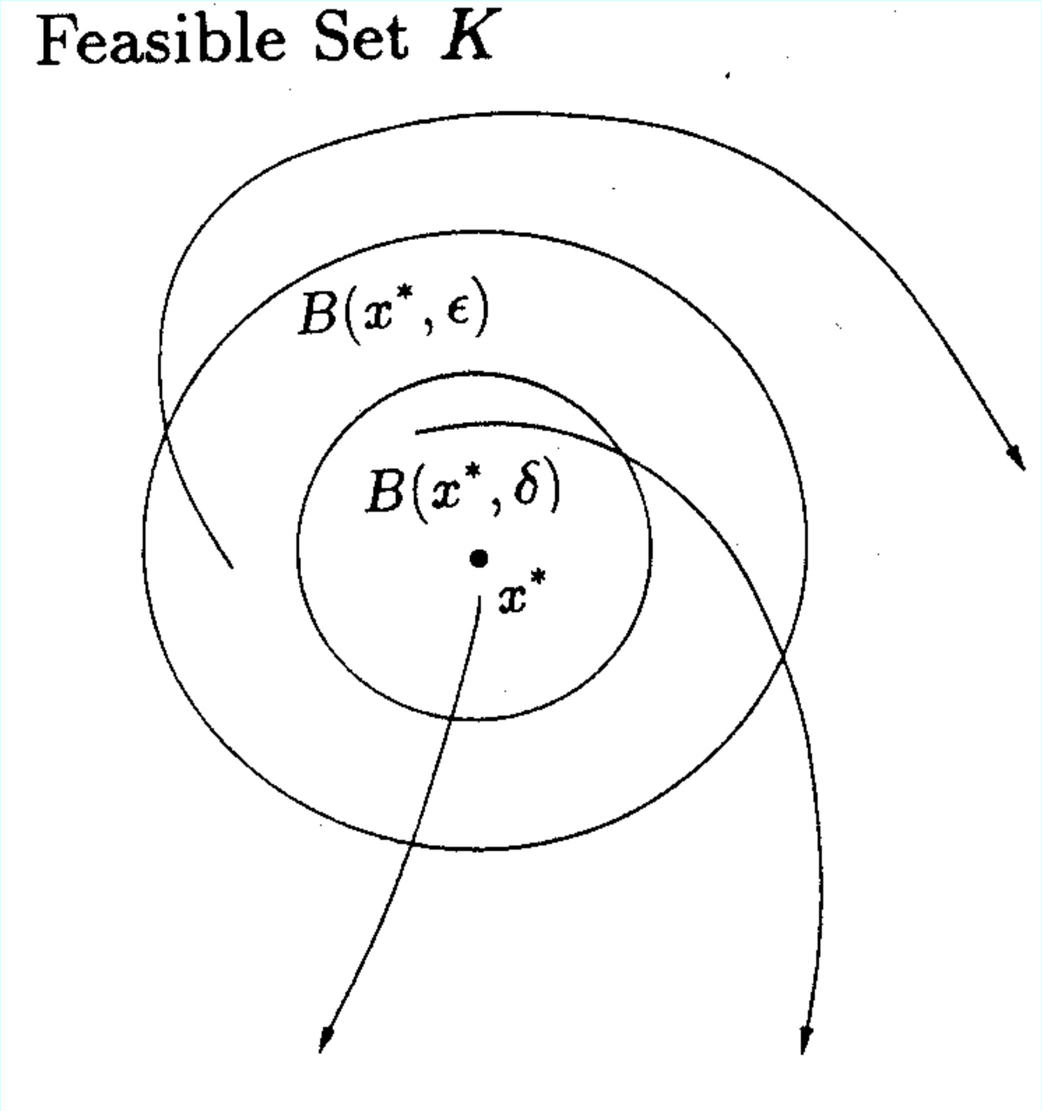
The assumption of pseudomonotonicity is vital to the existence of EVI solutions, but not so for solutions to PDS.

However, it plays a very important role in the stability study of perturbed equilibria of PDS, more precisely, in the study of the local/global properties of the projected systems around these equilibria.

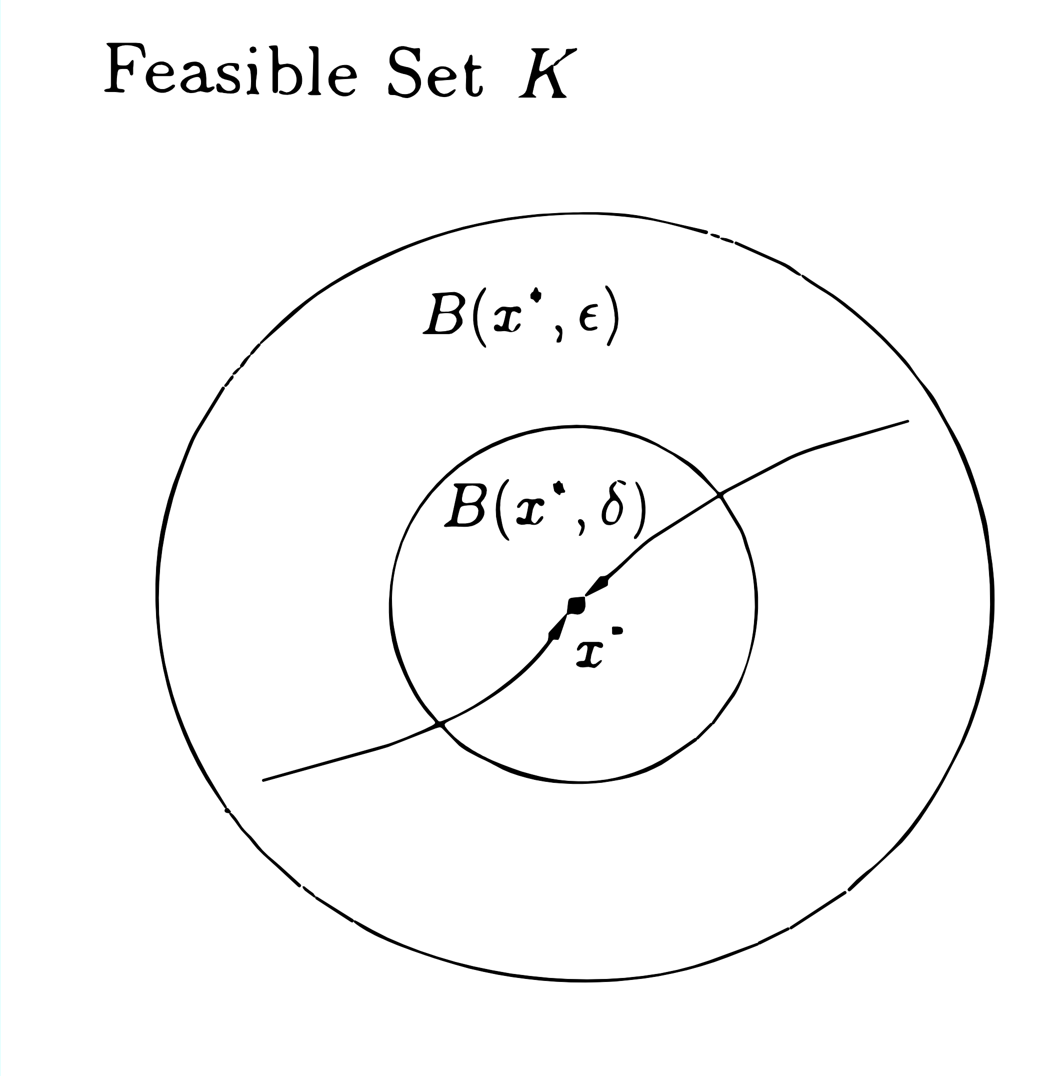
This stability question remains meaningful in the double-layered dynamics theory, where we seek to unravel the behavior of perturbations of the curve(s) of equilibria.



A Stable Equilibrium Point



An Unstable Equilibrium Point



A Finite Time Attractor

THREE IMPORTANT QUESTIONS

We see next that pseudomonotonicity-type conditions fully answer three important questions along the lines of our remarks above:

1. Is it accurate to expect that for almost all $t \in [0, T]$ given, the trajectories of the PDS at t (which we denote by PDS_t) evolve towards the curve of equilibria?

2. What is the relation between an arbitrarily chosen $t \in [0, T]$ and the time it takes for solutions to PDS_t to actually reach the curve of equilibria?

3. What is the interpretation of the double-layered dynamics for applications?

Answer to question (1). The first question is answered positively, and is a consequence of the stability study of perturbed equilibria for PDS on Hilbert spaces (see Isac and Cojocaru (2002)). Before stating the main results, we need to recall the notion of monotone attractor. While the classical notion of an attractor for a dynamical system is well-known, that of a **monotone attractor** is different and was initially introduced to study the properties of equilibrium points of projected dynamical systems (Zhang and Nagurney (1995)).

Definition

Let X be a Hilbert space, $K \subset X$ closed, convex subset.
(1) A point $x^* \in K$ is called a **local monotone attractor for the PDS** *if there exists a neighborhood V of x^* such that the function $d(t) := ||x(t) - x^*||$ is a non-increasing function of t, for any solution $x(t)$ of the PDS, starting in the neighborhood V.*
(2) A point $x^* \in K$ is a **local strict monotone attractor** *if the function $d(t)$ is decreasing.*

A point $x^* \in K$ is a **global** *monotone attractor (respectively a* **global** *strict monotone attractor) if conditions (1) and (2) are satisfied for solutions starting at any point of K.*

It is not difficult to see that the notion of monotone attractor and that of an attractor are different. For example, a monotone attractor is not necessarily an attractor, if say $d(x,t)$ decreases for $t \in [0, t_1]$ and remains constant in time for $t \geq t_1$, for some $t_1 \in R_+$. In the same way, an attractor is not necessarily a monotone attractor, unless $d(x,t)$ is monotonically decreasing to zero.

We now give stability results for the perturbed curve of equilibria, based on pseudo-monotonicity-type.

Theorem 5 (Cojocaru, Daniele, and Nagurney (2004b))

Assume $F : K \to L^2([0,T], R^q)$ is Lipschitz continuous on K and consider the EVI and the PDS. Then the following hold:

1. *if F is (locally) pseudomonotone on K, then the curve(s) of equilibria (solution(s) of EVI) is(are) a (local) monotone attractor;*

2. *if F is (locally) strictly pseudomonotone on K, then the unique curve of equilibria is a (local) strict monotone attractor;*

3. *if F is (locally) strongly pseudomonotone on K, then the unique curve of equilibria is exponentially stable and a (local) attractor.*

Answer to question (2). The stability properties of the curve of equilibria as a whole, given by Theorem 5, show that the curve is attracting solutions of almost all PDS_t and that it is possible for the curve to be reached for some of the moments $t \in [0, T]$.

To answer question (2), we start first by noticing that for almost all $t \in [0, T]$, arbitrarily fixed, we can identify a closed and convex subset $K_t \in R^q$, given by

$$K_t := \{u(t) \in R^q \,|\, \lambda(t) \leq u(t) \leq \mu(t);\ \lambda(t), \mu(t) \text{ given};$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t),\ \xi_{ji} \in \{0, 1\}, i \in \{1, .., q\}, j \in \{1, \ldots, l\}\}.$$

Evidently, to each such fixed t, we have a PDS_t given by

$$\frac{du(t,\tau)}{d\tau} = \Pi_{K_t}(u(t,\tau), -F(u(t,\tau))),\ u(t,0) = u_0^t \in K_t.$$

We recall the following definition (Zhang and Nagurney (1995)).

Definition

A map F is called **strongly pseudo-monotone with degree** α *on* K *if, there exists* $\eta > 0$ *such that, for every pair of distinct points* x, y*, we have*

$$\langle F(x), y - x \rangle \geq 0 \rightarrow \langle F(y), y - x \rangle \geq \eta ||y - x||^{\alpha}.$$

Evidently, if F is strongly pseudo-monotone with degree α, then it is strictly pseudomonotone. Hence, the EVI gives a unique curve of equilibria.

We answer question (2) by the following:

Theorem 6 (Cojocaru, Daniele, and Nagurney (2004b))

Consider the above EVI with F Lipschitz continuous and strongly pseudo-monotone with degree $\alpha < 2$ on K, for almost all fixed $t \in [0, T]$, there exists $l_t > 0$, finite, such that the unique equilibrium $u^* := u^*(t)$ of the PDS_t is reached by the (unique) solution $u(t, \tau)$ of the PDS_t, starting at the initial point $u_0^t \in K_t$. The time l_t depends upon η, α and $||u_0^t - u^*||$.

We have proved that for each $u_0^* \in K_t$, there exists $l_t < \infty$, depending on $\eta, \alpha, ||u_0^t - u^*||$, given by

$$l_t := \frac{||u_0^t - u^*||^{2-\alpha}}{(2-\alpha)\eta},$$

such that whenever $\alpha < 2$,

$$D(\tau) > 0 \text{ when } \tau < l_t \text{ and } D(\tau) = 0 \text{ when } \tau \geq l_t.$$

In other words, u^* is a globally finite-time attractor for the unique solution of PDS_t starting at u_0^t and it will be reached in l_t units of time.

Answer to question (3). In real life, there is only one concept of time in terms of a timeline. Therefore, in applications it is important to have a clear, easy way to estimate if, under what conditions, and, when, the curve of equilibria is reached. Theorem 6 provides exactly the desired answer: for almost any $t \in [0, T]$, we can estimate that the equilibrium on the curve corresponding to t will be reached in the time l_t if and only if

$$t \geq l_t := \frac{||u_0^t - u^*||^{2-\alpha}}{(2-\alpha)\eta}.$$

Otherwise, although the equilibrium can be computed, the solution to PDS_t does not have enough time to reach the curve.

But l_t depends intrinsically upon three parameters,

$$\eta, \quad \alpha, \quad ||u_0^t - u^*||,$$

two of which are given by F. Hence, we have, in fact, only one that we can manipulate, and that is $||u_0^t - u^*||$, i.e., the distance between the initial point of the trajectory and the equilibrium u^* at t.

Naturally, if we want to find/compute those solutions that will be arriving on the curve of equilibria at a fixed moment t, all we have to do is to make sure that we choose a trajectory of the PDS_t starting at a distance $||u_0^t - u^*||$ from the curve, so that the above is satisfied.

Let t be arbitrarily fixed in $[0,T]$ and consider the PDS given by:

$$\frac{\partial u(t,\tau)}{\partial \tau} = P_{T_K(u(t,\tau))}(-F(u(t,\tau))), \quad u(t,0) = t(t) \in K,$$

where τ is the evolution time of the PDS and t is the evolution time of the EVI. Note that:

- at each fixed $t \in [0,T]$, the solution(s) of the EVI represent one or more equilibria of the PDS;

- as t varies over $[0,T]$, these equilibria describe one (or more) curve(s);

- also, for each fixed t; $\tau \in [0,l]$ is the time it takes the system to reach one of the equilibria on the curve(s).

HOW THESE CONCEPTS CAN BE APPLIED TO TRAFFIC NETWORKS

To use these concepts/results we apply the following steps:

- we discretize the evolution time interval of the EVI;

- we obtain a finite collections of PDS's, defined on distinct closed, convex set K_t;

- we compute the equilibria of each PDS, i.e., we find the equilibria at the discrete chosen moments $t \in [0, T]$;

- we interpolate the sequence of equilibria and obtain an approximation of the curve(s) of equilibria.

A DYNAMIC NETWORK EXAMPLE

Consider a network consisting of a single origin/destination pair of nodes and two paths connecting these nodes of a single link each. The feasible set is given before where $u(t)$ denotes the vector of path flows at t. The cost functions on the paths are defined as: $2u_1(t) - 1.5$ for the first path and $u_2(t) - 1$ for the second path. We consider a vector field F given by

$$F : K \to L^2([0,1], R^2),$$

$$(F_1(u(t)), F_2(u(t))) = (2u_1(t) - 1.5, u_2(t) - 1).$$

The theory of EVI states that the system has a unique equilibrium, since F is strictly monotone, for any arbitrarily fixed point $t \in [0,2]$. One can easily see that

$$\langle F(u_1,u_2) - F(v_1,v_2), (u_1-v_1, u_2-v_2)\rangle = 2(u_1-v_1)^2 + (u_2-v_2)^2 > 0,$$

for any

$$u \neq v \in L^2([0,2], R^2).$$

With the help of PDS theory, we can compute an approximate curve of equilibria, by choosing $t_0 \in \left\{\frac{k}{4} | k \in \{0, \ldots, 8\}\right\}$. Therefore, we obtain a sequence of PDS defined by the vector field

$$-F(u_1(t_0), u_2(t_0)) = (-2u_1(t_0) + 1.5, -u_2(t_0) + 1)$$

on nonempty, closed, convex, 1-dimensional subsets

$$K_{t_0} := \left\{\left\{[0, t_0] \times \left[0, \frac{3}{2}t_0\right]\right\} \cap \{x + y = t_0\}\right\}.$$

For each we can compute the unique equilibrium of the system at t_0, i.e., the point

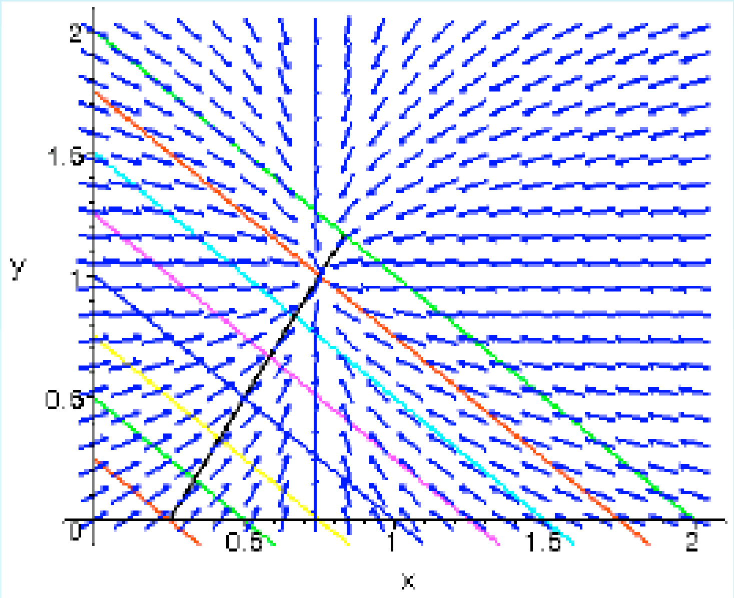
$$(u_1(t_0), u_2(t_0)) \in R^2$$

such that $-F(u_1(t_0), u_2(t_0)) \in N_{K_{t_0}}(u_1(t_0), u_2(t_0))$.

Using a simple MAPLE computation, we obtain that the equilibria are the points:

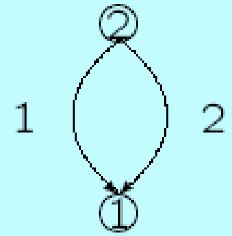
$$\left\{(0,0), \left(\frac{1}{4},0\right), \left(\frac{1}{3},\frac{1}{6}\right), \left(\frac{5}{12},\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{7}{12},\frac{2}{3}\right), \left(\frac{2}{3},\frac{5}{6}\right), \left(\frac{3}{4},1\right), \left(\frac{5}{6},\frac{7}{6}\right)\right\}.$$

Interpolating these points we obtain the approximate curve of network equilibria as displayed in the Figure.



ANOTHER DYNAMIC TRAFFIC NETWORK EXAMPLE

We again consider a transportation network consisting of a single origin/destination pair of nodes and two paths connecting these nodes of a single link each.



Network Structure of the Numerical Examples

The feasible set K is as before, where we take $p := 2$. We also have that $q := 2$, $j := 1$, $T := 2$, $\rho(t) := t$, and $\xi_{ji} := 1$ for $i \in \{1, 2\}$:

$$K = \bigcup_{t \in [0,2]} \Big\{ u \in L^2([0,2],^2) |$$

$$(0,0) \le (u_1(t), u_2(t)) \le \left(t, \frac{3}{2}t\right) \text{ a.e. in } [0,2];$$

$$\sum_{i=1}^{2} u_i(t) = t \text{ a.e. in } [0,2] \Big\} .$$

In this application $u(t)$ denotes the vector of path flows at t. The cost functions on the paths are defined as: $u_1(t) + 1$ for the first path and $u_2(t) + 2$ for the second path. We consider a vector field F defined by

$$F : L^2([0,2],^2\,) \to L^2([0,2],^2\,);$$

$$(F_1(u(t), F_2(u(t))) = (u_1(t) + 1, u_2(t) + 2).$$

The theory of EVI (as described above) states that the system has a unique equilibrium, since F is strictly monotone, for any arbitrarily fixed point $t \in [0,2]$. Indeed, one can easily see that $\langle F(u_1, u_2) - F(v_1, v_2), (u_1 - v_1, u_2 - v_2)\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2 > 0$, for any $u \neq v \in L^2([0,2],^2\,)$. With the help of PDS theory, we can compute an approximate curve of equilibria, by selecting $t_0 \in \left\{\frac{k}{4} | k \in \{0, \ldots, 8\}\right\}$. Hence, we obtain a sequence of PDS defined by the vector field $-F(u_1(t_0), u_2(t_0)) = (-u_1(t_0) + 1, -u_2(t_0) + 2)$ on nonempty, closed, convex, 1–dimensional subsets:

$${}_{t_o} := \left\{ \left\{ [0, t_0] \times \left[0, \frac{3}{2} t_0\right] \right\} \cap \{x + y = t_0\} \right\}.$$

For each, we can compute the unique equilibrium of the system at the point t_0, that is, the point:

$$(u_1(t_0), u_2(t_0)) \in R^2 \text{ such that}$$

$$-F(u_1(t_0), u_2(t_0)) \in N_{K_{t_0}}(u_1(t_0), u_2(t_0)).$$

Proceeding in this manner, we obtain the equilibria consisting of the points:

$$\left\{(0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), (1,0), \left(\frac{9}{8}, \frac{1}{8}\right), \right.$$

$$\left. \left(\frac{5}{4}, \frac{1}{4}\right), \left(\frac{11}{8}, \frac{3}{8}\right), \left(\frac{3}{2}, \frac{1}{2}\right)\right\}.$$

The interpolation of these points yields the curve of equilibria.

We note that due to the simplicity of the network topology and the linearity (and separability of the cost functions in this example) we can also obtain explicit formulae for the path flows over time as given below:

$$\begin{cases} u_1(t) = t, \\ \\ u_2(t) = 0 \end{cases} \quad \text{if } \; 0 \le t \le 1$$

and

$$\begin{cases} u_1(t) = \frac{t+1}{2}, \\ \\ u_2(t) = \frac{t-1}{2}. \end{cases} \quad \text{if } \; 1 \le t \le 2.$$

The above results demonstrate how the two theories of projected dynamical systems and evolutionary variational inequalities that have been developed in parallel can be connected to enhance the modeling, analysis, and computation of solutions to a plethora of time–dependent equilibrium problems that arise in such disciplines as engineering, operations research/management science, economics, and finance.

FINAL COMMENTS

- The PDS theory gives the natural environment in which an EVI can be applied;

- The PDS-EVI mesh opens up more questions to study, some of them theoretical and some regarding the future PDS-EVI applications;

- An EVI can be defined on Banach spaces, a PDS only on Hilbert spaces so far. This motivates the theoretical extensions of the PDS to B-spaces;

- as far as applications, we believe that the PDS-EVI has perhaps the strongest potential to model applied problems involving double-layered timeframes.

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Copies of this presentation and related work can be found on the Virtual Center for Supernetworks

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Thank you!