An Efficiency Measure for Dynamic Networks with Application to the Internet and Vulnerability Analysis

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Outline

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- The Network Efficiency Measure and Network Component Importance for Dynamic Networks
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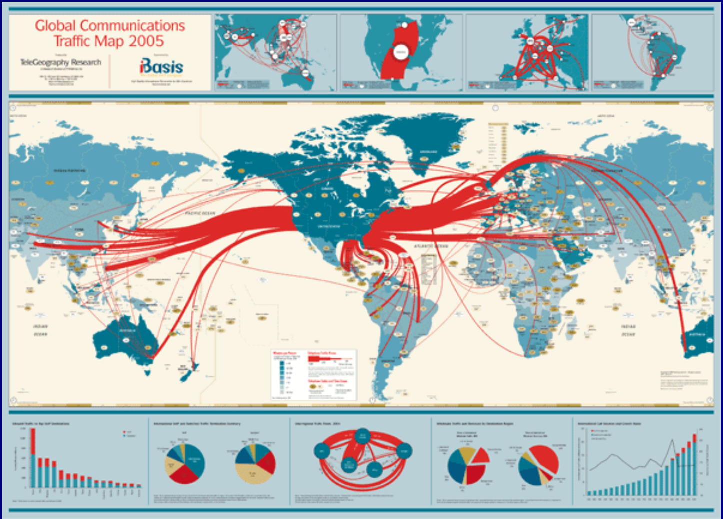
Motivation

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.
- For example:
  - 9/11 Terrorist Attacks, September 11, 2001
  - The biggest blackout in North America, August 14, 2003
  - Two significant power outages during the month of September 2003 one in England and one in Switzerland and Italy
  - Hurricane Katrina, August 23, 2005
  - Minneapolis Bridge Collapse, August 1, 2007

Motivation

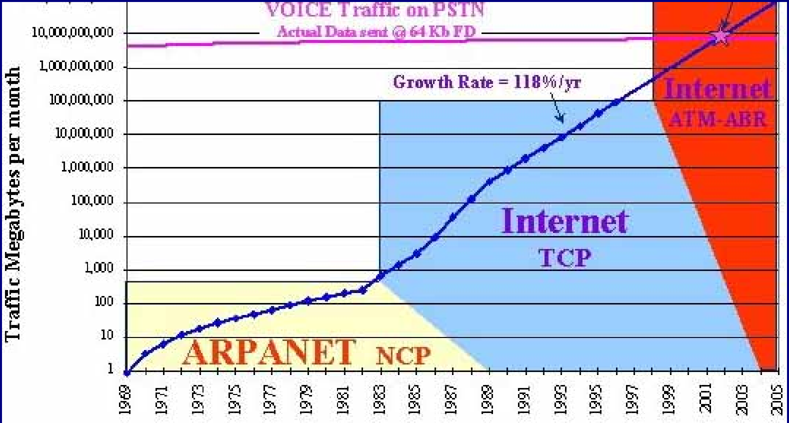
- "A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks." (page 10 of Roughgarden (2005)).
- We can expect that a variety of time-dependent demand structures will occur on the Internet as individuals seek information and news online in response to major events or simply go about their daily activities whether at work or at home. Hence, it is relevant to study the vulnerability of in Internet networks with time-varying traffic.
- "... traffic over the Internet doubling every 100 days" (Frances Hong (1999)).

Global Internet Communication Network



www.telegeography.com

Global Internet Traffic Growth



www.netvalley.com

Examples of Other Dynamic Networks

- Oil and natural gas networks
- Electricity generation and distribution networks
- Supply chain networks
- Transportation networks

Literature on EVIs and the Applications

- Daniele, P., Maugeri, A., Oettli, W. (1999). Time-dependent Traffic Equilibria, *Journal of Optimization Theory and its Applications* 103, 543-555.
- Daniele, P. (2003). Evolutionary Variational Inequalities and Economic Models for Demand Supply Markets, *Mathematical Models and Methods in Applied Sciences* 4, 471-489.
- Daniele, P. (2004). Time-Dependent Spatial Price Equilibrium Problem: Existence and Stability Results for the Quantity Formulation Model, *Journal of Global Optimization* 28, 283-295.
- Daniele, P. (2006). *Dynamic Networks and Evolutionary Variational Inequalities*, Edward Elgar Publishing, Cheltenham, England.

Literature on EVIs and the Applications (Cont'd)

- Nagurney, A. Parkes, D., Daniele, P. (2007). The Internet, Evolutionary Variational Inequalities, and the Time-Dependent Braess Paradox, *Computational Management Science* 4, 355-375.
- Nagurney, A. (2006). *Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits*, Edward Elgar Publishing, Cheltenham, England.
- Nagurney, A., Liu, Z., Cojocaru, M. G., Daniele, P. (2007). Static and Dynamic Transportation Network Equilibrium Reformulations of Electric Power Supply Chain Networks with Known Demands, *Transportation Research E* 43, 624-646.

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Holme, Kim, Yoon and Han (2002)
- Murray-Tuite and Mahmassani (2004)
- Taylor and Deste (2004)
- Barrat, Barthlemy and Vespignani (2005)
- Criado, Flores, Hernández-Bermejo, Pello and Romance (2005)
- Chassin and Posse (2005)
- Sheffi (2005)
- DallAsta, Barrat, Barthlemy and Vespignani (2006)
- Jenelius, Petersen and Mattson (2006)

Our Research on Network Efficiency, Vulnerability, and Robustness

- Nagurney, A., Qiang, Q. (2007a). A Network Efficiency Measure with Application to Critical Infrastructure Networks, *Journal of Global Optimization*, in press.
- Nagurney, A., Qiang, Q. (2007b). A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability, *Proceedings of the POMS 18th Annual Conference*, Dallas, Texas, May 4 to May 7, 2007.
- Nagurney, A., Qiang, Q. (2007c). A Network Efficiency Measure for Congested Networks, *Europhysics Letters*, 38005.
- Nagurney, A., Qiang, Q. (2007d). Robustness of Transportation Networks Subject to Degradable Links, to appear in *Europhysics Letters*.
- Qiang, Q., Nagurney, A. (2007). A Unified Network Performance Measure with Importance Identification and the Ranking of Network Components, *Optimization Letters*, in press.

The Dynamic Model of the Internet (Nagurney, Parkes and Daniele (2007))

The Internet is modeled as a network $G = [N, L]$, consisting of the set of nodes N and the set of directed links L. The set of links L consists of n_L elements. The set of O/D pairs of nodes is denoted by W and consists of n_W elements. We denote the set of routes (with a route consisting of links) joining the O/D pair w by P_w. We assume that the routes are acyclic. Let P with n_P elements denote the set of all routes connecting all the O/D pairs in the Internet. Links are denoted by a, b, etc; routes by r, q, etc., and O/D pairs by w_1, w_2, etc. We assume that the Internet is traversed by a single class of "job" or "task."

Demands, Route Flows, and Link Flows

Let $d_w(t)$ denote the demand, that is, the traffic generated, between O/D pair w at time t. The flow on route r at time t, which is assumed to be nonnegative, is denoted by $x_r(t)$ and the flow on link a at time t by $f_a(t)$.

Conservation of Flow Between Demands and Route Flows

The following conservation of flow equations must be satisfied at each t:

$$d_w(t) = \sum_{r \in P_w} x_r(t), \quad \forall w \in W,$$

Route Capacities

Also, we must have that

$$0 \leq x_r(t) \leq \mu_r(t), \quad \forall r \in P,$$

where $\mu_r(t)$ denotes the capacity on route r at time t.

Conservation of Flow Between Route Flows and Link Flows

$$f_a(t) = \sum_{r \in P} x_r(t)\delta_{ar}, \quad \forall a \in L,$$

where $\delta_{ar} = 1$ if link a is contained in route r, and $\delta_{ar} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on routes that contain that link. All the link flows at time t are grouped into the vector $f(t)$, which is of dimension n_L.

Link Costs and Route Costs

The cost on route r at time t is denoted by $C_r(t)$ and the cost on a link a at time t by $c_a(t)$. We allow the cost on a link, in general, to depend upon the entire vector of link flows at time t, so that

$$c_a(t) = c_a(f(t)), \quad \forall a \in L.$$

The costs on routes are related to costs on links through the following equations:

$$C_r(x(t)) = \sum_{a \in L} c_a(x(t))\delta_{ar}, \quad \forall r \in P,$$

which means that the cost on a route at a time t is equal to the sum of costs on links that make up the route at time t. We group the route costs at time t into the vector $C(t)$, which is of dimension n_P.

Feasible Set

We consider the Hilbert space $\mathcal{L} = L^2([0, T], R^{n_P})$ (where T denotes the time interval under consideration) given by

$$\mathcal{K} = \{x \in L^2([0, T], R^{n_P}) : 0 \leq x(t) \leq \mu(t)$$

$$\text{a.e. in } [0, T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0, T]\}.$$

We assume that the capacities $\mu_r(t)$, for all r, are in $\mathcal{L}$ and that the demands, $d_w \geq 0$, for all w, are also in $\mathcal{L}$. Further, we assume that

$$0 \leq d(t) \leq \Phi\mu(t), \text{a.e. on } [0, T],$$

where Φ is the $n_W \times n_P$-dimensional O/D pair-route incidence matrix, with element (w, r) equal to 1 if route r is contained in P_w, and 0, otherwise. Due to the above assumption, the feasible set $\mathcal{K}$ is nonempty. As noted in Nagurney, Parkes, and Daniele (2007), $\mathcal{K}$ is also convex, closed, and bounded. Note that we are not restricted as to the form that the time-varying demands for the O/D pairs take since convexity is guaranteed even if the demands have a step-wise structure, or are piecewise continuous.

Another Definition

The dual space of $\mathcal{L}$ is denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle \mathcal{G}, x \rangle\rangle := \int_0^T \langle \mathcal{G}(t), x(t) \rangle dt, \quad \mathcal{G} \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \longrightarrow \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

Dynamic Network Equilibrium

A route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle), if, at each time t, only the minimum cost routes not at their capacities are used (that is, have positive flow) for each O/D pair unless the flow on a route is at its upper bound (in which case those routes' costs can be lower than those on the routes not at their capacities). The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every route $r \in P_w$, and a.e. on $[0, T]$:

$$C_r(x^*(t)) - \lambda^*_w(t) \begin{cases} \leq 0, & \text{if} \quad x^*_r(t) = \mu_r(t), \\ = 0, & \text{if} \quad 0 < x^*_r(t) < \mu_r(t), \\ \geq 0, & \text{if} \quad x^*_r(t) = 0. \end{cases}$$

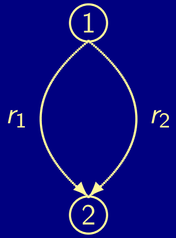
Theorem of Nagurney, Parkes and Daniele (2007)

$x^* \in \mathcal{K}$ is an equilibrium flow according to the definition of dynamic network equilibrium if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

A Simple Numerical Example

Consider a network consisting of two nodes and two links. There is a single O/D pair w=(1,2).

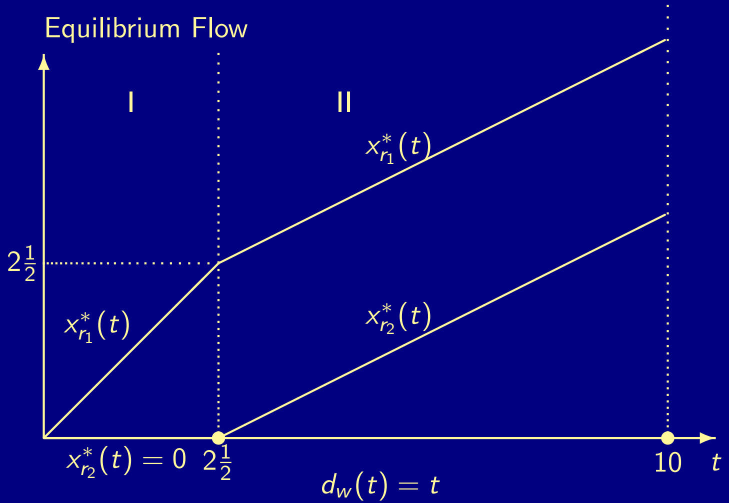


The route costs are

$$C_{r_1}(x(t)) = 2x_{r_1}(t) + 5, \quad C_{r_2}(x(t)) = 2x_{r_2}(t) + 10.$$

Also, we assume that the route capacities are: $\mu_{r_1}(t) = \mu_{r_2}(t) = \infty$. The time horizon is $[0, 10]$ and the time-varying demand is assumed to be $d_w(t) = t$.

Equilibrium Trajectories of the Simple Numerical Example with Time-varying Demands



The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007a, b, c) proposed a network efficiency measure for networks with fixed demands, which captures demand and flow information under the network equilibrium:

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2007a, b, c), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

Network Efficiency Measure for Dynamic Networks - Continuous Time

The network efficiency for the network G with time-varying demand d for $t \in [0, T]$, denoted by $\mathcal{E}(G, d, T)$, is defined as follows:

$$\mathcal{E}(G, d, T) = \frac{\int_0^T [\sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)}]/n_W \, dt}{T}.$$

Note that the above measure is the average network performance over time of the dynamic network.

Network Efficiency Measure for Dynamic Networks - Discrete Time

Let $d_w^1,\, d_w^2, ..., d_w^H$ denote demands for O/D pair w in H discrete time intervals, given, respectively, by:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$. We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair w at the H different time intervals by: $\lambda_w^1,\, \lambda_w^2, ..., \lambda_w^H$. The demand vector d, in this special discrete case, is a vector in $R^{n_W \times H}$. The dynamic network efficiency measure in this case is as follows:

Dynamic Network Efficiency: Discrete Time Version

The network efficiency for the network (G, d) over H discrete time intervals:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, *where $t_H \equiv T$, and with the respective constant demands:*
$d_w^1,\, d_w^2, ..., d_w^H$ *for all $w \in W$ is defined as follows:*

$$\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.$$

Special Case

Assume that $d_w(t) = d_w$*, for all O/D pairs* $w \in W$ *and for* $t \in [0, T]$*. Then, the dynamic network efficiency measure collapses to the Nagurney and Qiang (2007a, b, c) measure:*

$$\mathcal{E} = \frac{1}{n_W} \sum_{w \in W} \frac{d_w}{\lambda_w}.$$

Importance of a Network Component

The importance of network component g of network G with demand d over time horizon T is defined as follows:

$$I(g,d,T) = \frac{\mathcal{E}(G,d,T) - \mathcal{E}(G-g,d,T)}{\mathcal{E}(G,d,T)}$$

where $\mathcal{E}(G-g,d,T)$ is the dynamic network efficiency after component g is removed.

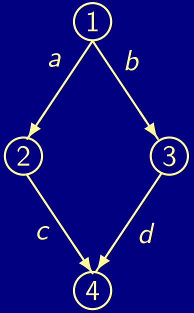
Our Approach to Identifying the Importance and Rankings of Network Components

The elimination of a link is represented in the Nagurney and Qiang measure by the removal of that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Hence, our measure is well-defined even in the case of disconnected networks.

The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $p_1 = (a, c)$ and $p_2 = (b, d)$. For a travel demand of 6, the equilibrium path flows are $x^*_{p_1} = x^*_{p_2} = 3$ and The equilibrium path travel costs are $C_{p_1} = C_{p_2} = 83$.

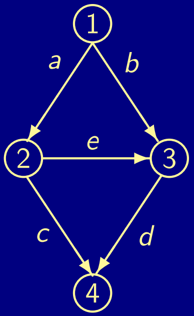


$c_a(f_a) = 10f_a \quad c_b(f_b) = f_b + 50$

$c_c(f_c) = f_c + 50 \quad c_d(f_d) = 10f_d$

Adding a Link Increases Travel Cost for All!

Adding a new link creates a new path $p_3 = (a, e, d)$. The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path p_3, C_{p_3}=70. The new equilibrium path flows are $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$. The equilibrium path travel costs are $C_{p_1} = C_{p_2} = C_{p_3} = 92$.



$c_e(f_e) = f_e + 10$

The Dynamic Braess Network Without Link e

We now construct time-dependent link costs, route costs, and demand for $t \in [0, T]$. It is important to emphasize that the case where time t is discrete, that is, $t = 0, 1, 2, \ldots, T$, is trivially included in the equilibrium conditions and also captured in the EVI formulation.

We consider, to start, the first network, consisting of links: a, b, c, d. We assume that the capacities $\mu_{r_1}(t) = \mu_{r_2}(t) = \infty$ for all $t \in [0, T]$. The link cost functions are assumed to be given and as follows for time $t \in [0, T]$:

$$c_a(f_a(t)) = 10 f_a(t), \quad c_b(f_b(t)) = f_b(t) + 50,$$

$$c_c(f_c(t)) = f_c(t) + 50, \quad c_d(f_d(t)) = 10 f_d(t).$$

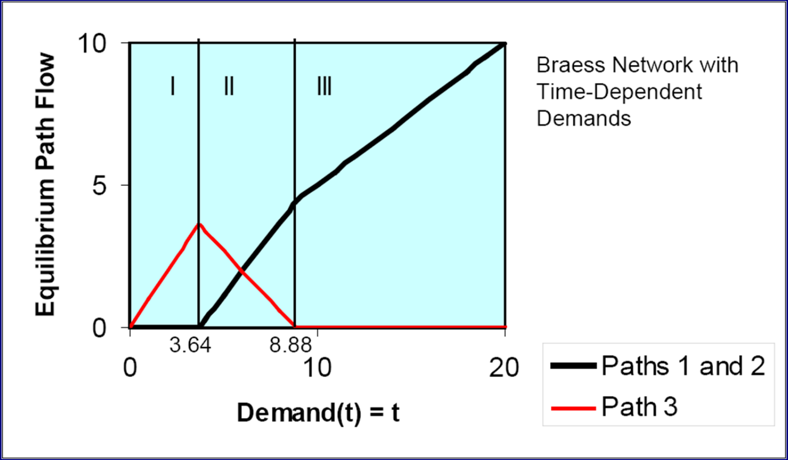
We assume a time-varying demand $d_w(t) = t$ for $t \in [0, T]$.

The Dynamic Braess Network - Solution

Solving the EVI, we have the equilibrium path flows are $x^*_{r_1}(t) = \frac{t}{2}$ and $x^*_{r_2}(t) = \frac{t}{2}$ for $t \in [0, T]$.

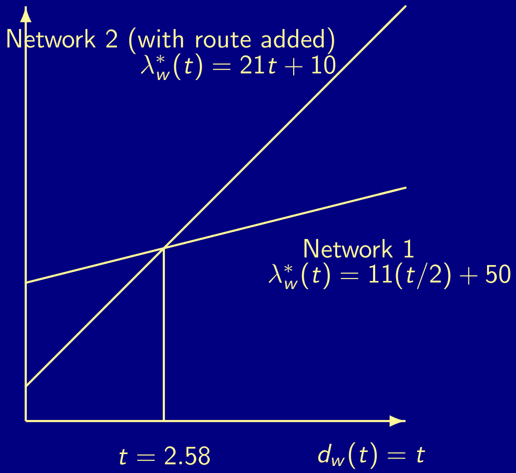
The equilibrium route costs for $t \in [0, T]$ are given by: $C_{r_1}(x^*_{r_1}(t)) = 5\frac{1}{2}t + 50 = C_{r_2}(x^*_{r_2}(t)) = 5\frac{1}{2}t + 50$, and, clearly, equilibrium conditions hold for $\in [0, T]$ a.e..

The Dynamic Braess Network Adding Link e



The Dynamic Braess Network

Minimum Used Route Cost



For demand in the range $2.58 < d_w(t) = t < 8.89$, the addition of the new route will result in everyone being worse off.

Minimum Used Route Costs for Braess Networks 1 and 2.

Importance of Nodes and Links in the Dynamic Braess Network Using the New Measure When $T = 10$

Link	Importance Value	Importance Ranking
a	0.2604	1
b	0.1784	2
c	0.1784	2
d	0.2604	1
e	-0.1341	3

Node	Importance Value	Importance Ranking
1	1.0000	1
2	0.2604	2
3	0.2604	2
4	1.0000	1

Link e is never used after $t = 8.89$ and in the range $t \in [2.58, 8.89]$, it increases the cost, so the fact that link e has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!

Summary and Conclusions

- Our network efficiency measure captures user behavior, flows and costs on networks.
- The proposed measure extends our previous research on the network efficiency measure into the dynamic setting.
- The proposed network efficiency measure is applicable for varying demand in both continuous and discrete time.
- The proposed network efficiency measure can be applied to other critical infrastructure networks.
- The proposed network efficiency measure also has implication for the robustness and vulnerability of networks with partially disrupted network components (Nagurney and Qiang 2007d).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