Equilibrium Modeling and Vulnerability Analysis of Complex Network Systems:

Which Nodes and Links Really Matter?

Anna Nagurney
John F. Smith Memorial Professor
Department of Finance and Operations Management
University of Massachusetts
Amherst, MA 01003

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Center for Women in Mathematics at Smith College

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Outline of Presentation

- Background
- The Transportation Network Equilibrium Problem and Methodological Tools
- The Braess Paradox
- Transportation and Complex Networks
- A New Network Performance/Efficiency Measure with Applications to Network Systems
- What About Dynamic Networks?
- Evolutionary Variational Inequalities, the Internet, an the Time-Dependent (Demand-Varying) Braess Paradox
- Extension of the Efficiency Measure to Dynamic Networks
- Where Are We Now? An Empirical Case Study to Real-World Electric Power Supply Chains

Background

Interdisciplinary Impact of Networks

Mathematics

Networks

Economics

Interregional Trade

General Equilibrium

Industrial Organization

Portfolio Optimization

Flow of Funds Accounting

Engineering

Energy

Manufacturing

Telecommunications

Transportation

Biology

DNA Sequencing

Targeted Cancer Therapy

Sociology

Social Networks

Organizational Theory

Computer Science

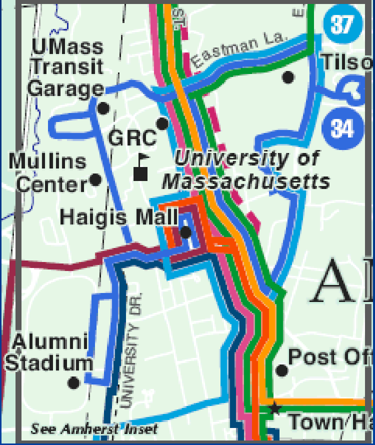
Routing Algorithms

We Are in a New Era of Decision-Making Characterized by:

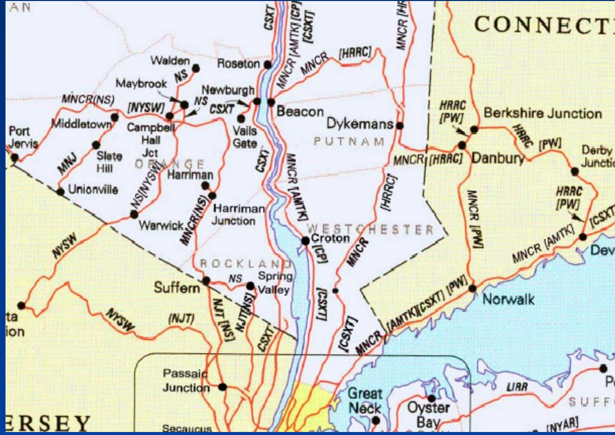
- *complex interactions* among decision-makers in organizations;
- alternative and at times *conflicting criteria* used in decision-making;
- *constraints on resources*: natural, human, financial, time, etc.;
- *global reach* of many decisions;
- *high impact* of many decisions;
- increasing *risk and uncertainty*, and
- the *importance of dynamics* and realizing a fast and sound response to evolving events.

Network problems are their own class of problems and they come in various forms and formulations, i.e., as optimization (linear or nonlinear) problems or as equilibrium problems and even dynamic network problems.

Complex network problems, with a focus on on transportation, will be the focus of this talk.



Transportation, Communication, and Energy Networks



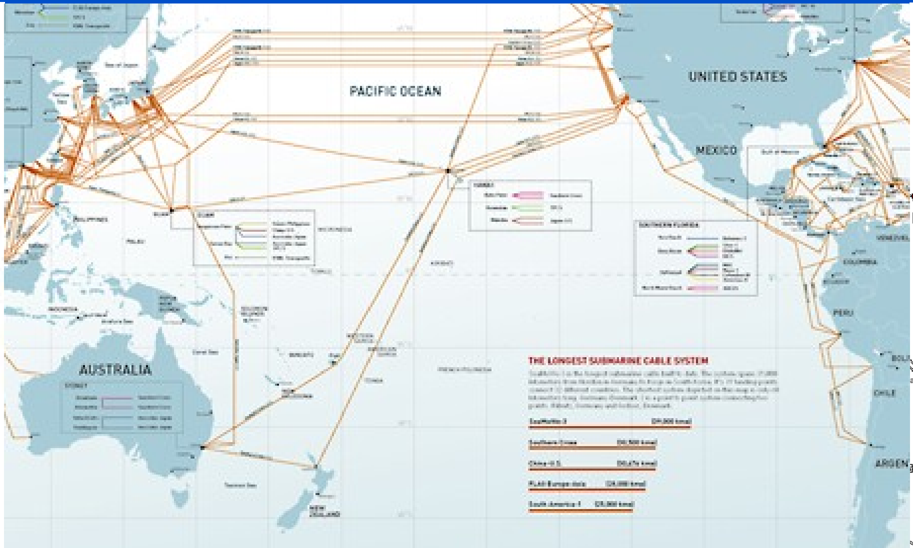
Bus Network

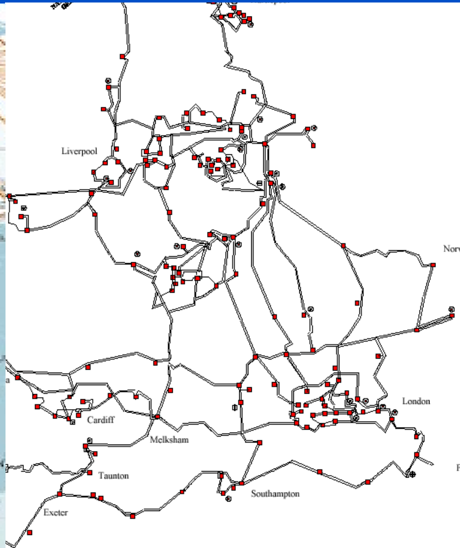
Rail Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks

British Electricity Grid



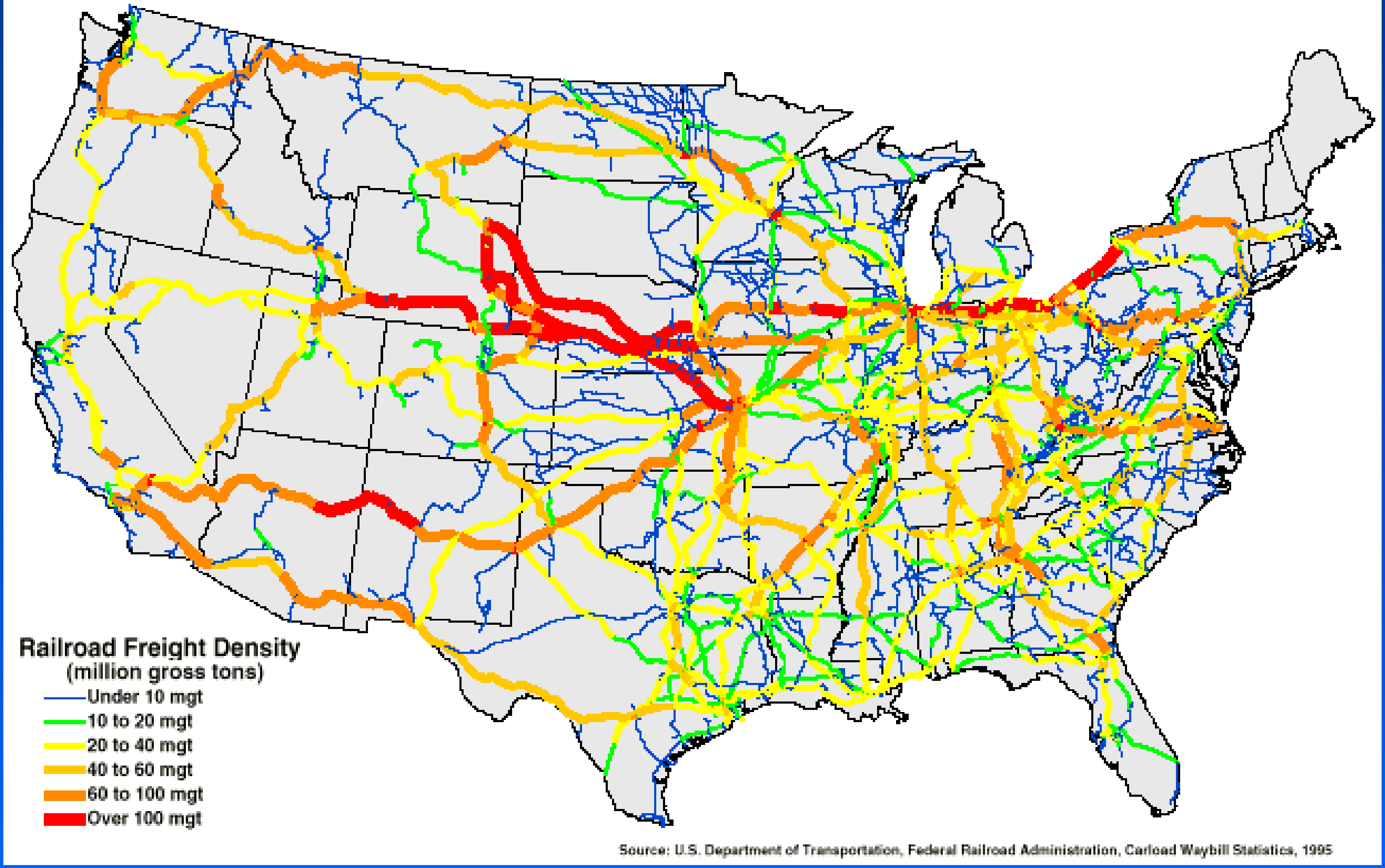


Components of Common Physical Networks

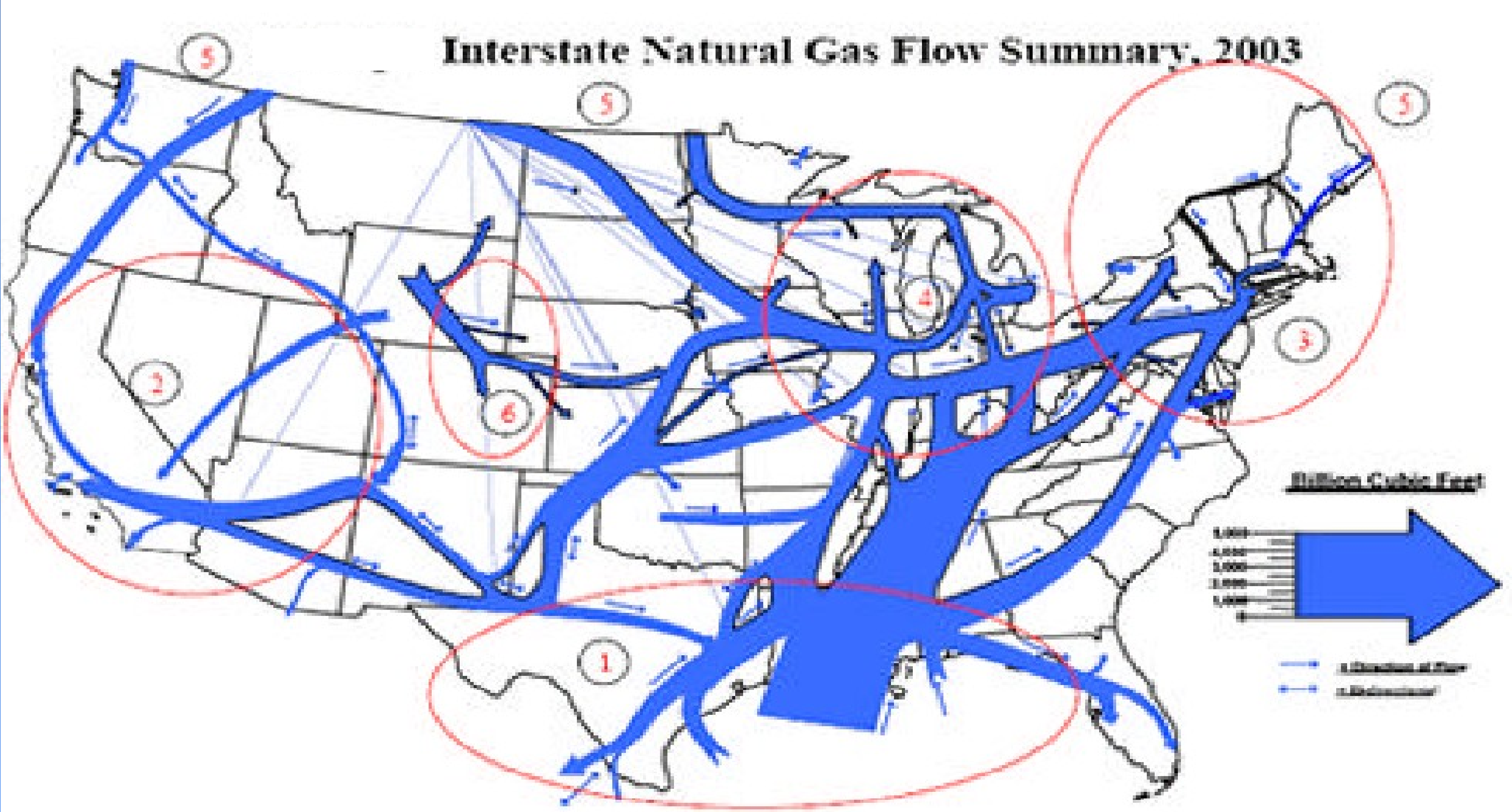
Network System	Nodes	Links	Flows
Transportation	Intersections, Homes, Workplaces, Airports, Railyards	Roads, Airline Routes, Railroad Track	Automobiles, Trains, and Planes,
Manufacturing and logistics	Workstations, Distribution Points	Processing, Shipment	Components, Finished Goods
Communication	Computers, Satellites, Telephone Exchanges	Fiber Optic Cables Radio Links	Voice, Data, Video
Energy	Pumping Stations, Plants	Pipelines, Transmission Lines	Water, Gas, Oil, Electricity

Interstate Highway System

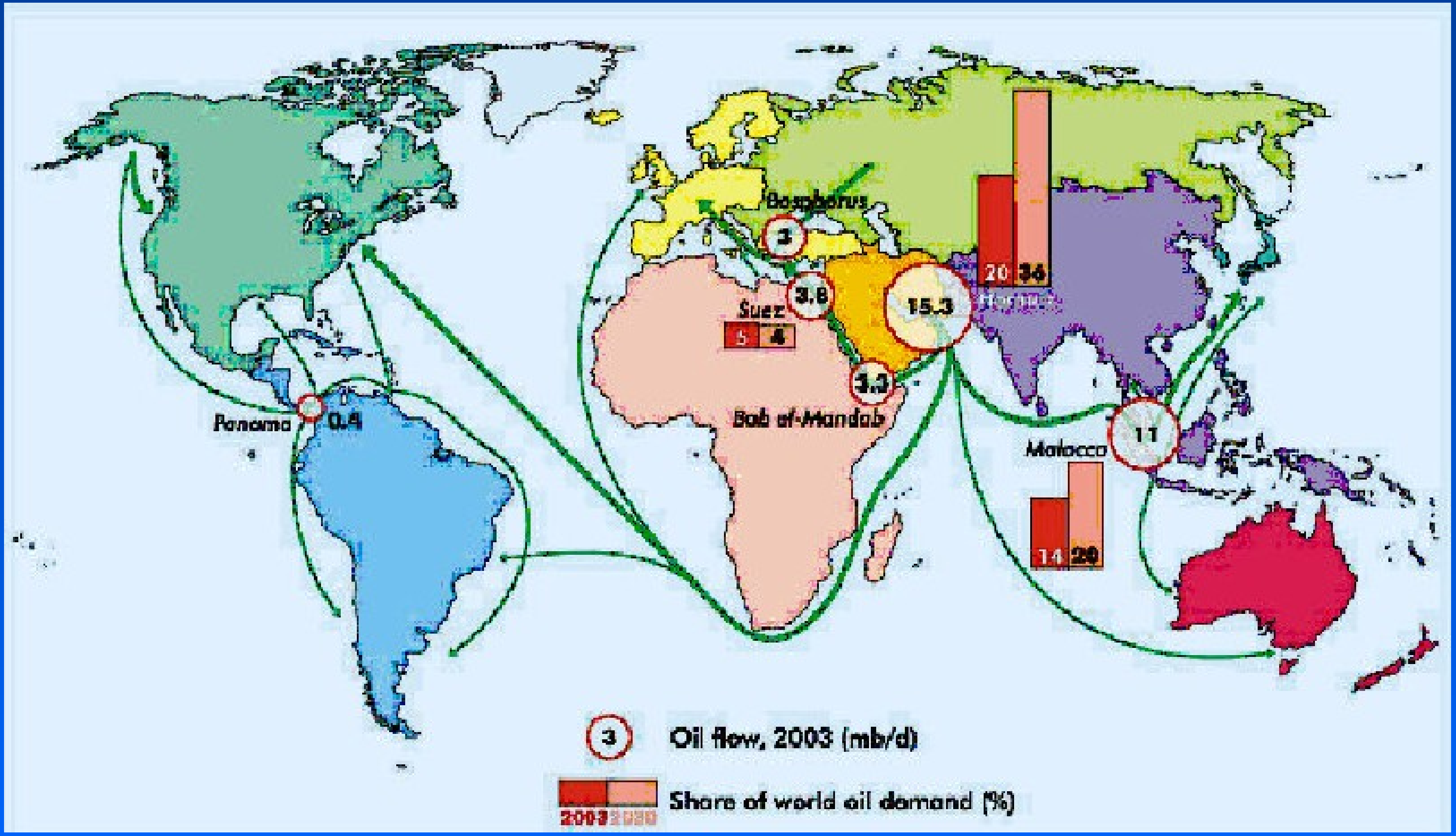
US Railroad Freight Flows



Natural Gas Pipeline Network in the US



World Oil Trading Network



The study of the efficient operation on transportation networks dates to *ancient Rome* with a classical example being the publicly provided Roman road network and the *time of day chariot policy*, whereby chariots were banned from the ancient city of Rome at particular times of day.

Characteristics of Networks Today

- *large-scale nature* and complexity of network topology;
- *congestion*;
- the *interactions among networks* themselves such as in transportation versus telecommunications;
- *policies* surrounding networks today may have a *major impact* not only economically but also environmentally, *socially, politically, and security-wise*.

- *alternative behaviors of the users of the network*

  – system-optimized versus

  – user-optimized (network equilibrium),

  which may lead to

  paradoxical phenomena.

There are *two fundamental principles of travel behavior*, due to Wardrop (1952), which we refer to as user-optimization (or network equilibrium) or system-optimization. These terms were coined by Dafermos and Sparrow (1969); see also Beckmann, McGuire, and Winsten (1956).

In a *user-optimized (network equilibrium) problem*, each user of a network system seeks to determine his/her cost-minimizing route of travel between an origin/destination pair, until an equilibrium is reached, in which no user can decrease his/her cost of travel by unilateral action.

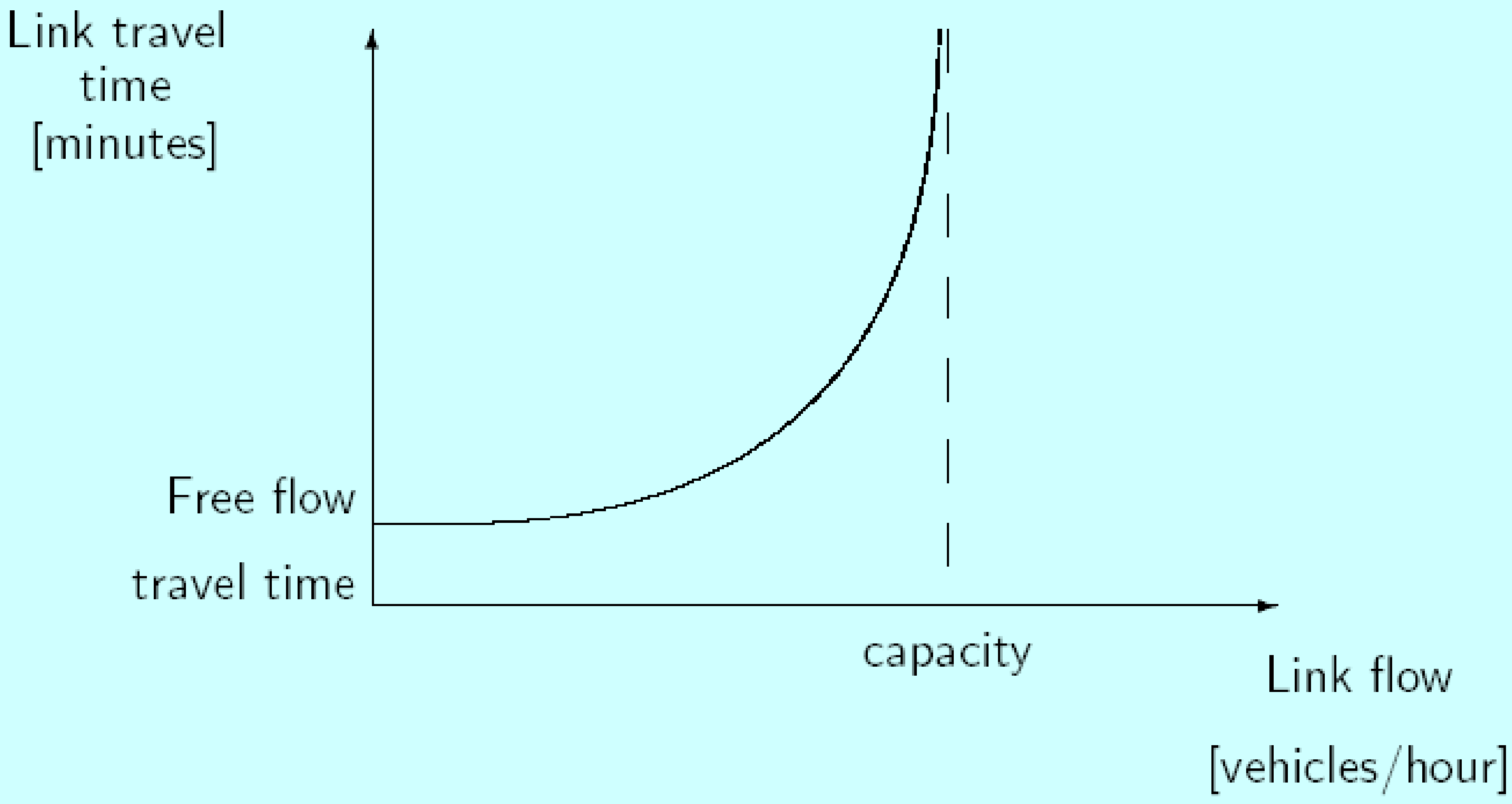
In a *system-optimized network problem*, users are allocated among the routes so as to minimize the total cost in the system. Both classes of problems, under certain imposed assumptions, possess optimization formulations.



Traffic Congestion



Capturing Link Congestion



For a typical user link travel time function, where the free flow travel time refers to the travel time to traverse a link when there is zero flow on the link (or zero vehicles).

BPR Link Cost Function

A common link performance function is the Bureau of Public Roads (BPR) cost function developed in 1964.
This equation is given by

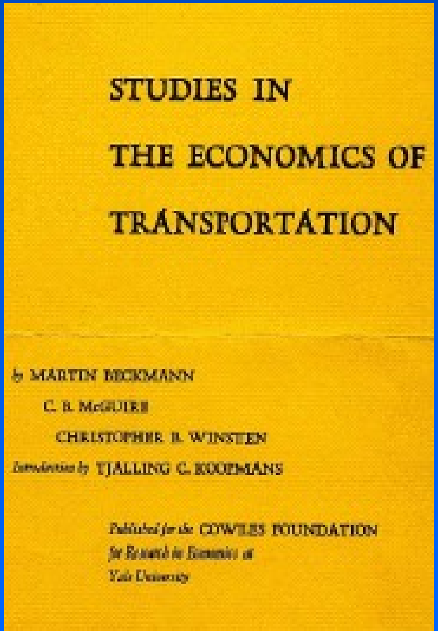
$$c_a = c_a^0 \left[1 + \alpha (\frac{f_a}{t'_a})^{\beta} \right] ,$$

where, c_a and f_a are the travel time and link flow, respectively, on link a, c_a^0 is the free-flow travel time, and t'_a is the "practical capacity" of link a. The quantities α and β are model parameters, for which the values $\alpha = 0.15$ minutes and $\beta = 4$ are typical values. For example, these values imply that the practical capacity of a link is the flow at which the travel time is 15% greater than the free-flow travel time.

The Transportation Network Equilibrium (TNE) Problem and Methodological Tools

Transportation applications have motivated the development of methodological tools in Operations Research (and in Economics).

An example is the book, *Studies in the Economics of Transportation*, by Beckmann, McGuire, and Winsten (1956).



Dafermos (1980) showed that the transportation network equilibrium (also referred to as user-optimization) conditions as formulated by **Smith (1979)** were a finite-dimensional variational inequality. In 1981, **Dafermos** proposed a multicriteria transportation network equilibrium model in which the costs were flow-dependent.

In 1993, **Dupuis and Nagurney** proved that the set of solutions to a variational inequality problem coincided with the set of solutions to a projected dynamical system (PDS) in R^n.

In 1996, **Nagurney and Zhang** published ***Projected Dynamical Systems and Variational Inequalities***.

Daniele, Maugeri, and Oettli (1998, 1999) introduced evolutionary variational inequalities for time-dependent (dynamic) traffic network equilibrium problems.

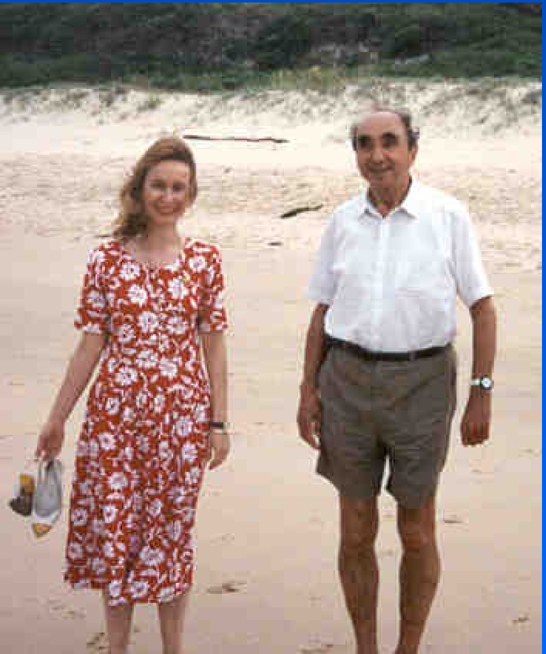
In 2002, **Nagurney and Dong** published ***Supernetworks: Decision-Making for the Information Age***.

The Transportation Social - Knowledge Network

Professors Beckmann and Dafermos at Anna Nagurney's Post-Ph.D. Defense Party in Barus Holley

Professor Beckmann with Professor Michael Florian of Montreal

On the Beach in Mallacoota, Austrailia



INFORMS Honoring the 50th Anniversary of the Publication of ***Studies in the Economics of Transportation***

Professors Beckman and McGuire

Transportation Network Equilibrium User-Optimization (U-O) Problem

Consider a general network $G = [N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. P_w denotes the set of paths, assumed to be acyclic, connecting the O/D pair of nodes w and P the set of all paths.

Let x_p represent the flow on path p and f_a the flow on link a. The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. This expression states that the load on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if we let d_w denote the demand associated with O/D pair w, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair w must be equal to the given demand d_w.

Let c_a denote the user cost associated with traversing link a, which is assumed to be continuous, and C_p the user cost associated with traversing the path p. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising the path.

Transportation Network Equilibrium Conditions

The network equilibrium conditions are then given by: For each path $p \in P_w$ and every O/D pair w:

$$C_p \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0 \end{cases}$$

where λ_w is an indicator, whose value is not known a priori. These equilibrium conditions state that the user costs on all used paths connecting a given O/D pair will be minimal and equalized.

As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that $[\frac{\partial c_b}{\partial c_a} = \frac{\partial c_a}{\partial c_b}]$ for all links a, b in the network then the solution to the above network equilibrium problem can be reformulated as the solution to an associated optimization problem. For example, if we have that $c_a = c_a(f_a)$, $\forall a \in L$, then the solution can be obtained by solving:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y) dy$$

subject to:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in L,$$

$$x_p \geq 0, \quad \forall p \in P.$$

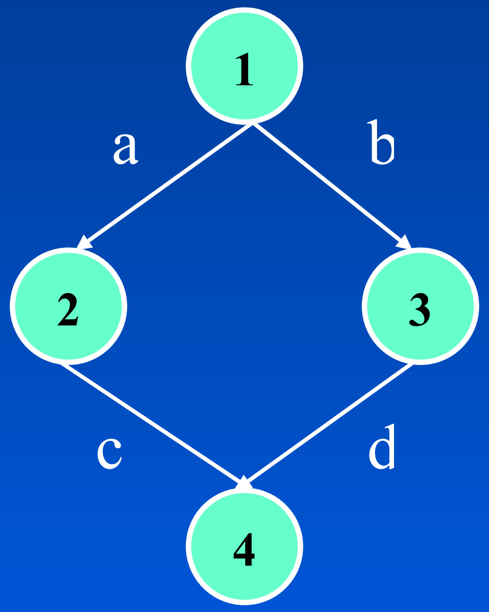
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of **6,** the equilibrium path flows are $\mathbf{x_{p_1}^* = x_{p_2}^* = 3}$ and

The equilibrium path travel cost is

$\mathbf{C_{p_1} = C_{p_2} = 83}$.



$c_a(f_a)=10\ f_a$ $\quad c_b(f_b) = f_b+50$

$c_c(f_c) = f_c+50$ $\quad c_d(f_d) = 10\ f_d$

Adding a Link Increases Travel Cost for All!

Adding a new link creates a new path **p_3=(a,e,d).**

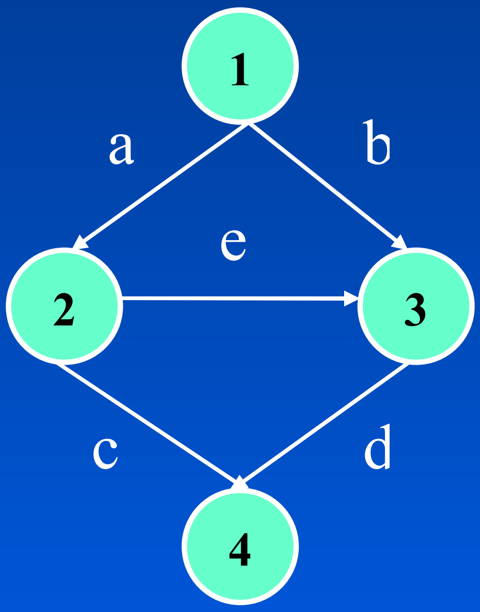
The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path **p_3, C_{p_3}=70.**

The new equilibrium flow pattern network is

$x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2$.

The equilibrium path travel costs:

$C_{p_1} = C_{p_2} = C_{p_3} = 92$.



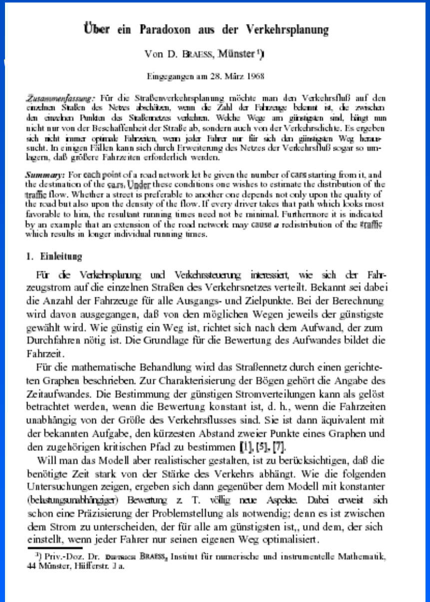
$c_e(f_e) = f_e + 10$

The 1968 Braess article has been translated from German to English and appears as

On a Paradox of Traffic Planning

by Braess, Nagurney, Wakolbinger

in the November 2005 issue of *Transportation*



Über ein Paradoxon aus der Verkehrsplanung

Von D. Braess, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

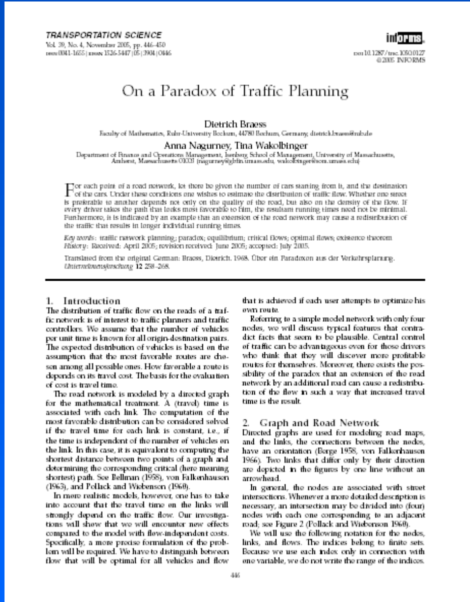
1. Einleitung

Für die Verkehrsplanung und Verkehrssteuerung interessiert, wie sich der Fahrzeugstrom auf die einzelnen Straßen des Verkehrsnetzes verteilt. Bekannt sei dabei die Anzahl der Fahrzeuge für alle Ausgangs- und Zielpunkte. Bei der Berechnung wird davon ausgegangen, daß von den möglichen Wegen jeweils der günstigste gewählt wird. Wie günstig ein Weg ist, richtet sich nach dem Aufwand, der zum Durchfahren nötig ist. Die Grundlage für die Bewertung des Aufwandes bildet die Fahrzeit.

Für die mathematische Behandlung wird das Straßennetz durch einen gerichteten Graphen beschrieben. Zur Charakterisierung der Bögen gehört die Angabe des Zeitaufwandes. Die Bestimmung der günstigen Stromverteilungen kann als gelöst betrachtet werden, wenn die Bewertung konstant ist, d. h., wenn die Fahrzeiten unabhängig von der Größe des Verkehrsflusses sind. Sie ist dann äquivalent mit der bekannten Aufgabe, den kürzesten Abstand zweier Punkte eines Graphen und den zugehörigen kritischen Pfad zu bestimmen [1], [5], [7].

Will man das Modell aber realistischer gestalten, ist zu berücksichtigen, daß die benötigte Zeit stark von der Stärke des Verkehrs abhängt. Wie die folgenden Untersuchungen zeigen, ergeben sich dann gegenüber dem Modell mit konstanter (belastungsunabhängiger) Bewertung z. T. völlig neue Aspekte. Dabei erweist sich schon eine Präzisierung der Problemstellung als notwendig; denn es ist zwischen dem Strom zu unterscheiden, der für alle am günstigsten ist,, und dem, der sich einstellt, wenn jeder Fahrer nur seinen eigenen Weg optimalisiert.

[1]) Priv.-Doz. Dr. Dietrich Braess, Institut für numerische und instrumentelle Mathematik, 44 Münster, Hüfferstr. 3 a.



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On a Paradox of Traffic Planning

Dietrich Braess
Faculty of Mathematics, Ruhr-University Bochum, 44780 Bochum, Germany

Anna Nagurney, Tina Wakolbinger
Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts 01003

For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Key words: traffic network planning; paradox; equilibrium; critical flows; optimal flows; existence theorem
History: Received: April 2005; revision received: June 2005; accepted: July 2005.

Translated from the original German: Braess, Dietrich. 1968. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* 12 258–268.

1. Introduction

The System-Optimization (S-O) Problem

The above discussion focused on the user-optimized (U-O) problem. We now turn to the system-optimized (S-O) problem in which a central controller, say, seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in L} \hat{c}_a(f_a)$$

where it is assumed that the total cost function on a link a is defined as:

$$\hat{c}_a(f_a) \equiv c_a(f_a) \times f_a,$$

subject to the conservation of flow constraints, and the nonnegativity assumption on the path flows. Here separable link costs have been assumed, for simplicity, and other total cost expressions may be used, as mandated by the particular application.

The S-O Optimality Conditions

Under the assumption of strictly increasing user link cost functions, the optimality conditions are: For each path $p \in P_w$, and every O/D pair w:

$$\hat{C}'_p \begin{cases} = \mu_w, & \text{if} \quad x_p > 0 \\ \geq \mu_w, & \text{if} \quad x_p = 0, \end{cases}$$

where $\hat{C}'_p$ denotes the marginal total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}.$$

The above conditions correspond to Wardrop's second principle of travel behavior.

What is the S-O solution for the two Braess networks (before and after the addition of a new link e)?

Before the addition of the link e, we may write:

$$\hat{c}'_a = 20f_a, \quad \hat{c}'_b = 2f_b + 50,$$

$$\hat{c}'_c = 2f_c + 50, \quad \hat{c}'_d = 20f_d.$$

It is easy to see that, in this case, the S-O solution is identical to the U-O solution with $x_{p_1} = x_{p_2} = 3$ and $\hat{C}'_{p_1} = \hat{C}'_{p_2} = 116$. Furthermore, after the addition of link e, we have that $\hat{c}'_e = 2f_e + 10$. The new path p_3 is not used in the S-O solution, since with zero flow on path p_3, we have that $\hat{C}'_{p_3} = 170$ and $\hat{C}'_{p_1} = \hat{C}'_{p_2}$ remains at 116.

If the symmetry assumption does not hold for the user link costs functions, then the transportation network equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a *variational inequality problem!*

VI Formulation of TNE
Dafermos (1980), Smith (1979)

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

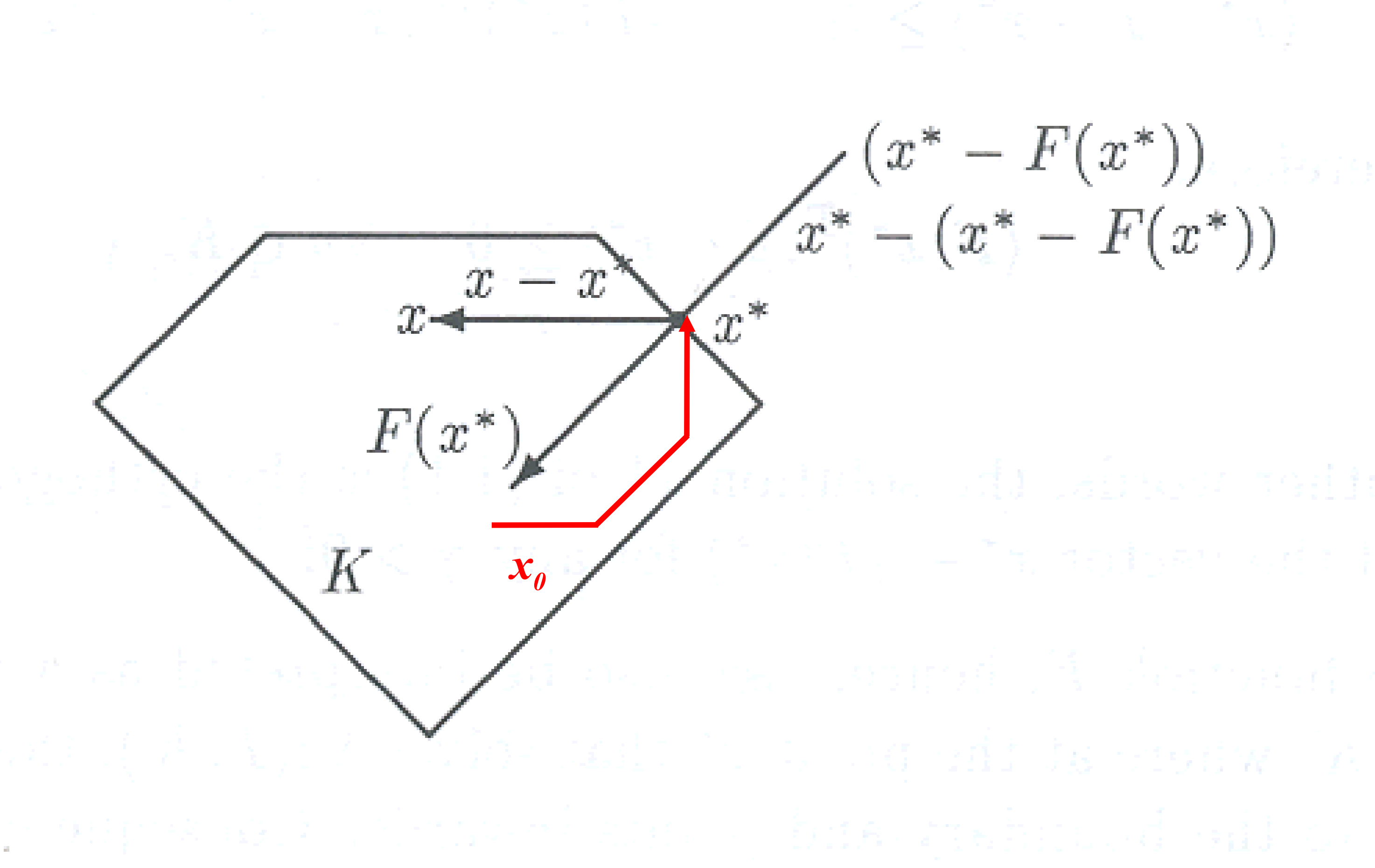
$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

A Geometric Interpretation of a Variational Inequality and a Projected Dynamical System

Dupuis and Nagurney (1993)

Nagurney and Zhang (1996)



The variational inequality problem, contains, as special cases, such classical problems as:

- systems of equations
- optimization problems
- complementarity problems

and is also closely related to fixed point problems.

Hence, it is a unifying mathematical formulation for a variety of mathematical programming problems.

Transportation and Complex Network Systems

Extensions of the fixed demand models in which the cost on a link depends on the flow on a link have been made to capture multiple modes of transportation as well as elastic demands.

- For example, one may have that the cost on a link as experienced by a mode of transportation (or a class of user) depends, in general, on the flow of all the modes (or classes) on all the links on the network.

- To handle elastic demand associated with travel between origin/destination pairs, we introduce a travel disutility associated with traveling between each O/D pair which can be a function of the travel demands associated with all the O/D pairs (and all modes in a multimodal case).

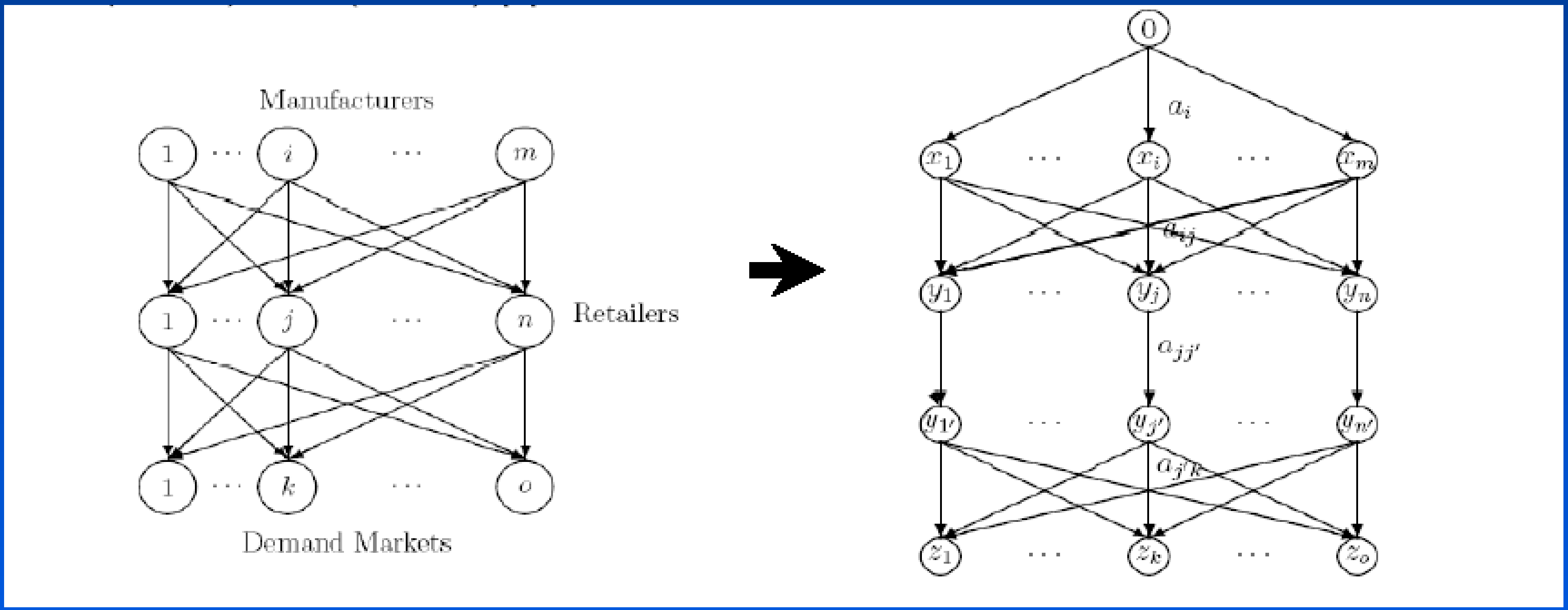
- The U-O and the S-O conditions are then generalized to include the multiple modes/classes of transportation as well as the travel disutilities, which are now functions and are associated with the different modes/classes.

- For a variety of such models, along with references see the books by Nagurney (1999, 2000).

The TNE Paradigm is the Unifying Paradigm for Complex Network Problems:

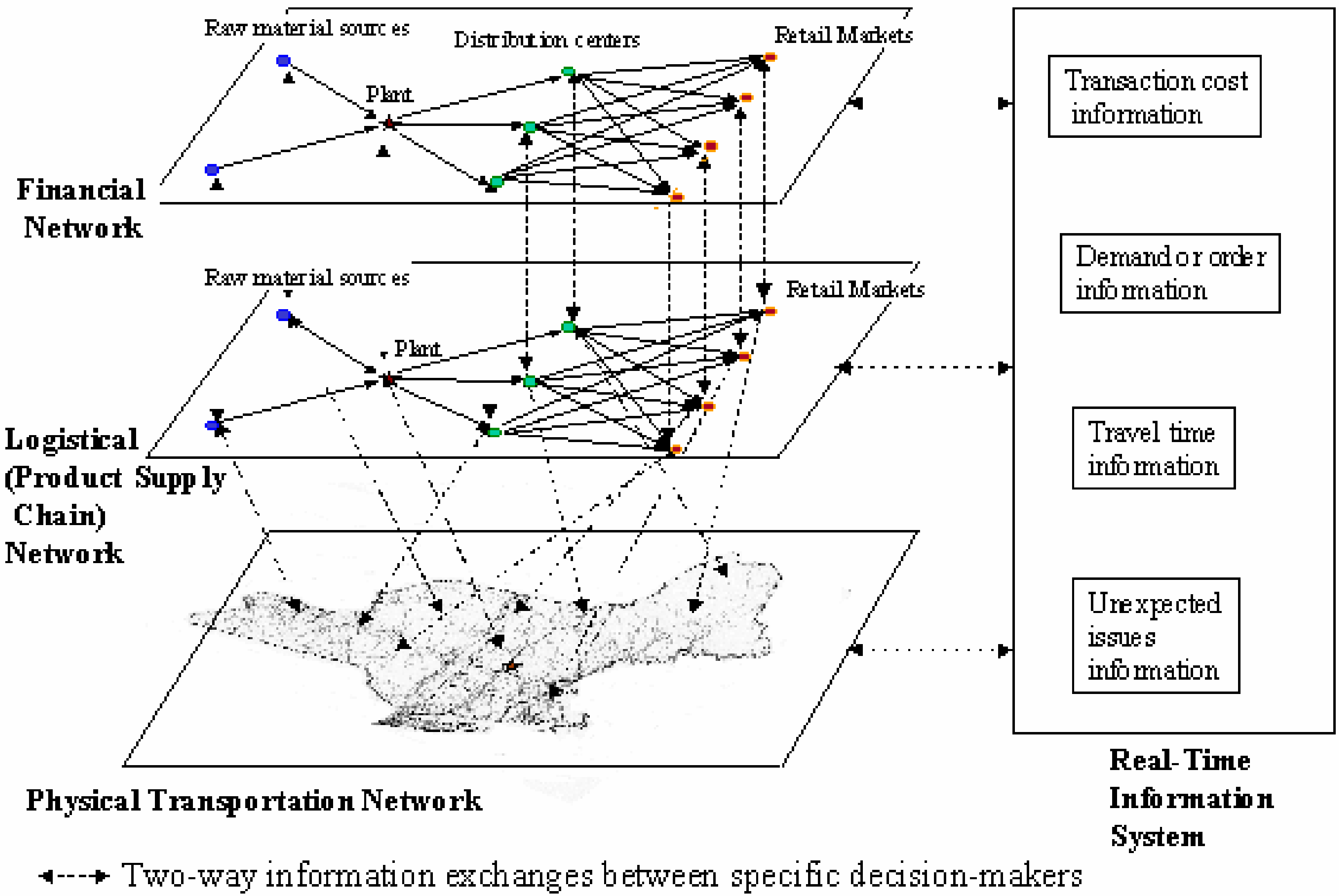
- Transportation Networks
- The Internet
- Financial Networks
- Electric Power Supply Chains.

The Equivalence of Supply Chains and Transportation Networks



Nagurney, *Transportation Research E* (2006).

Supply Chain -Transportation Supernetwork Representation



Nagurney, Ke, Cruz, Hancock, Southworth, *Environment and Planning B* (2002).

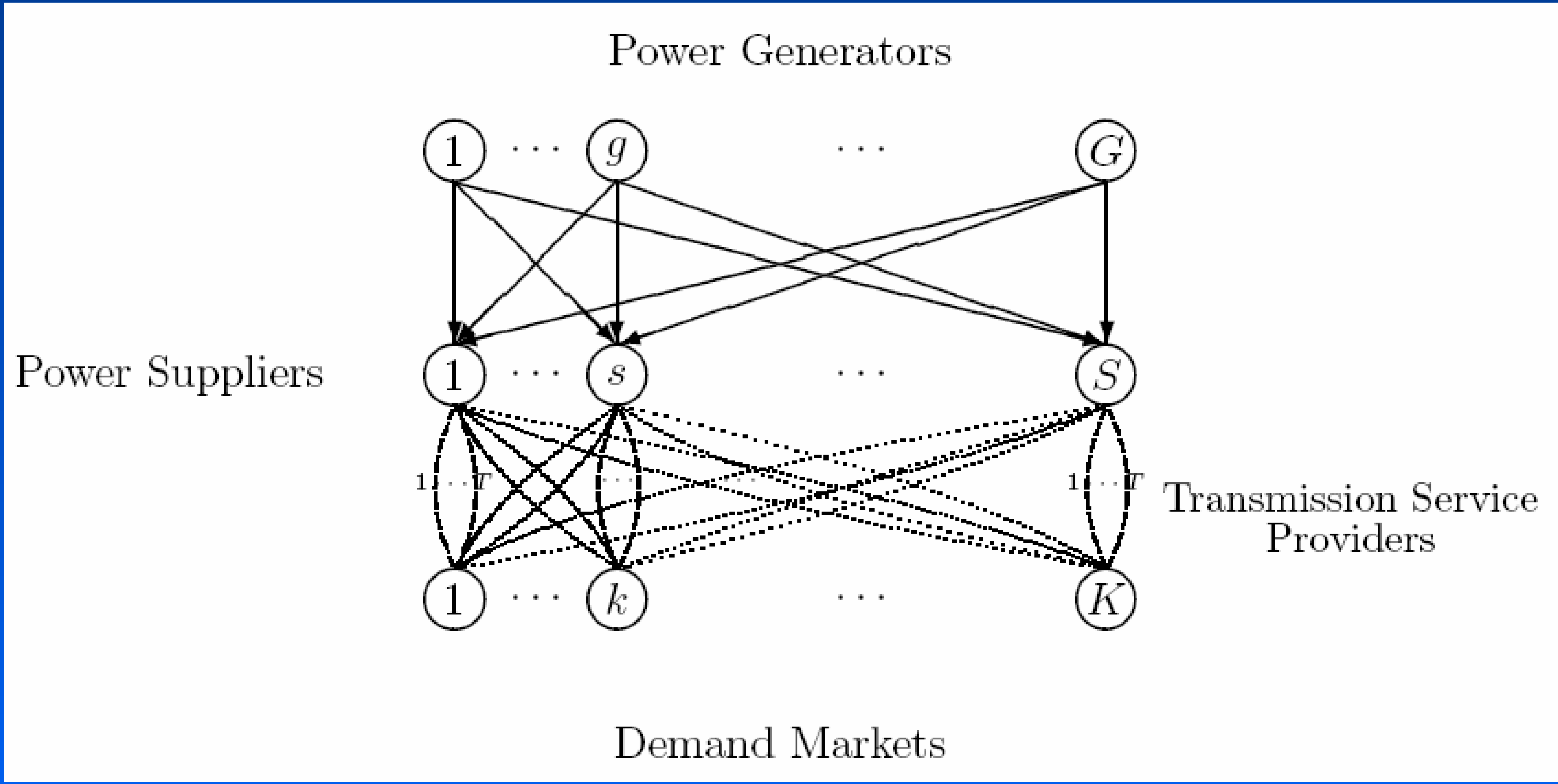
The fifth chapter of Beckmann, McGuire, and Winsten's book, ***Studies in the Economics of Transportation*** (1956) describes some *unsolved problems* including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

Specifically, they asked whether electric power generation and distribution networks can be reformulated as transportation network equilibrium problems.

From: http://www.nasa.gov

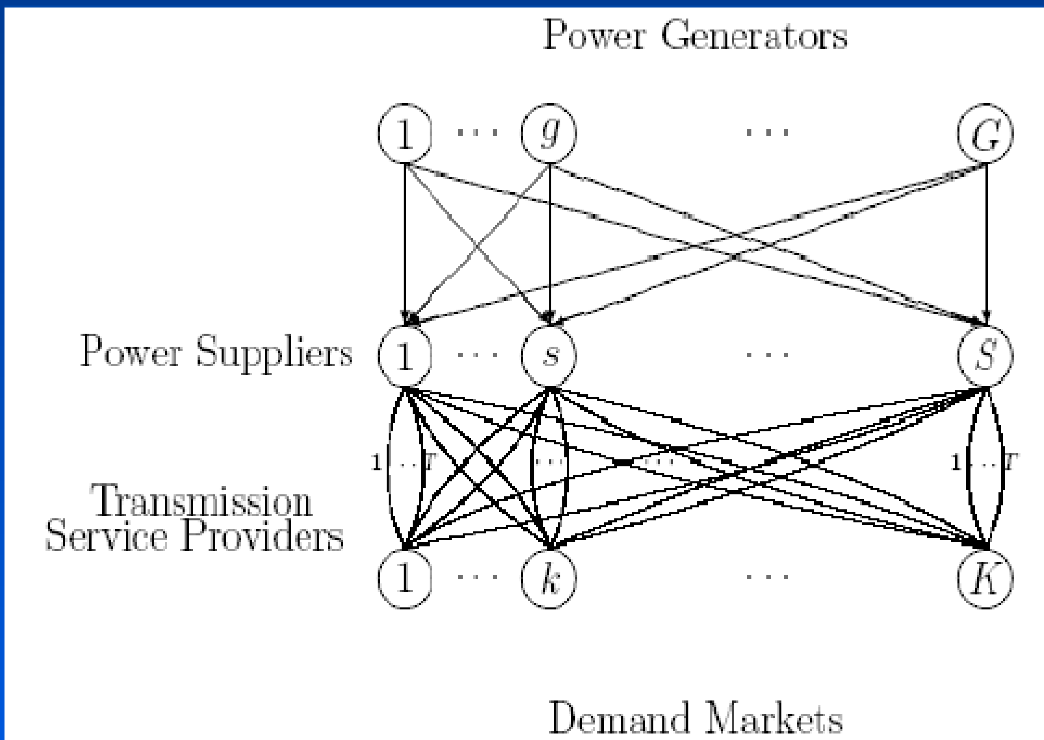
Electric Power Supply Chains

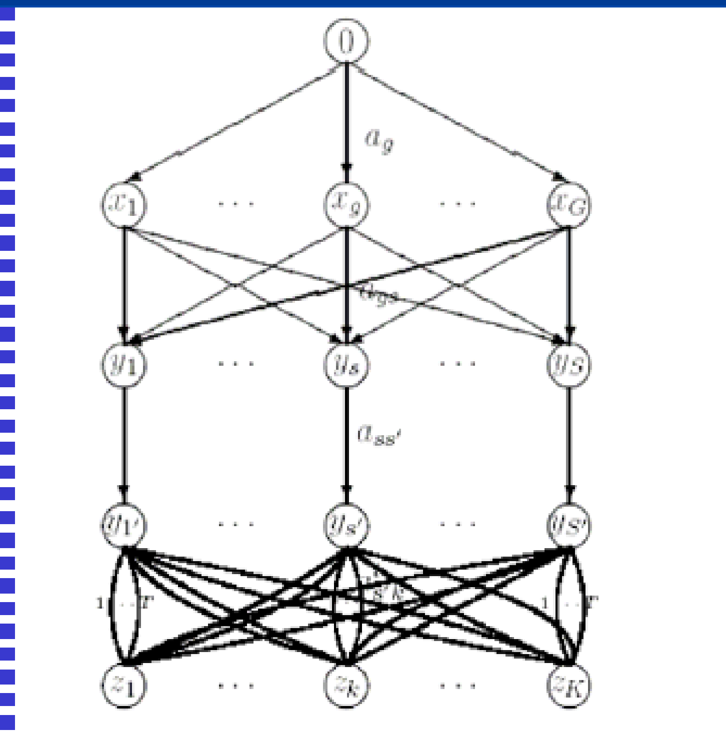
The Electric Power Supply Chain Network



Nagurney and Matsypura, *Proceedings of the CCCT* (2004).

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks



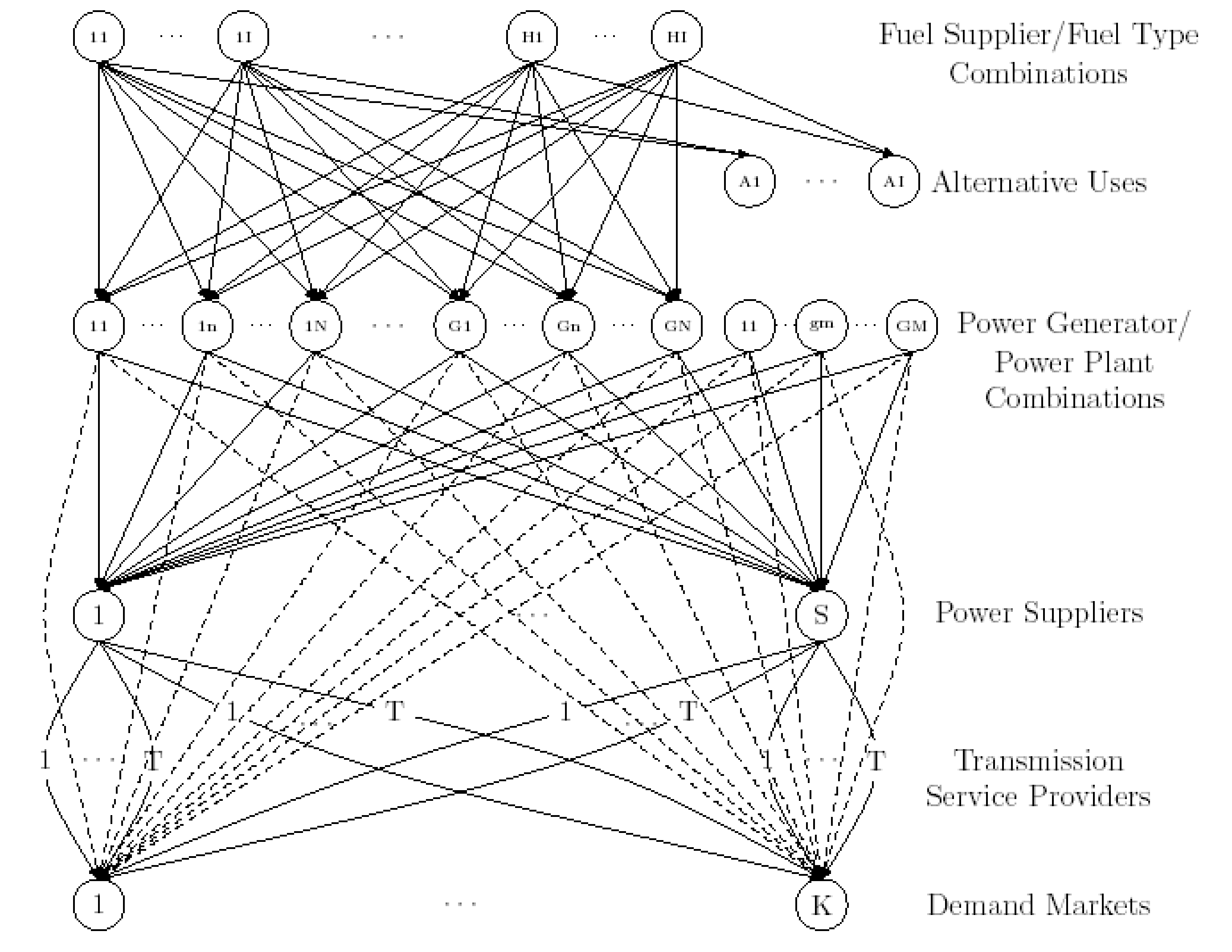


Electric Power Supply Network

Transportation Chain Network

Nagurney, Liu, Cojocaru, and Daniele, *Transportation Research E* (2007).

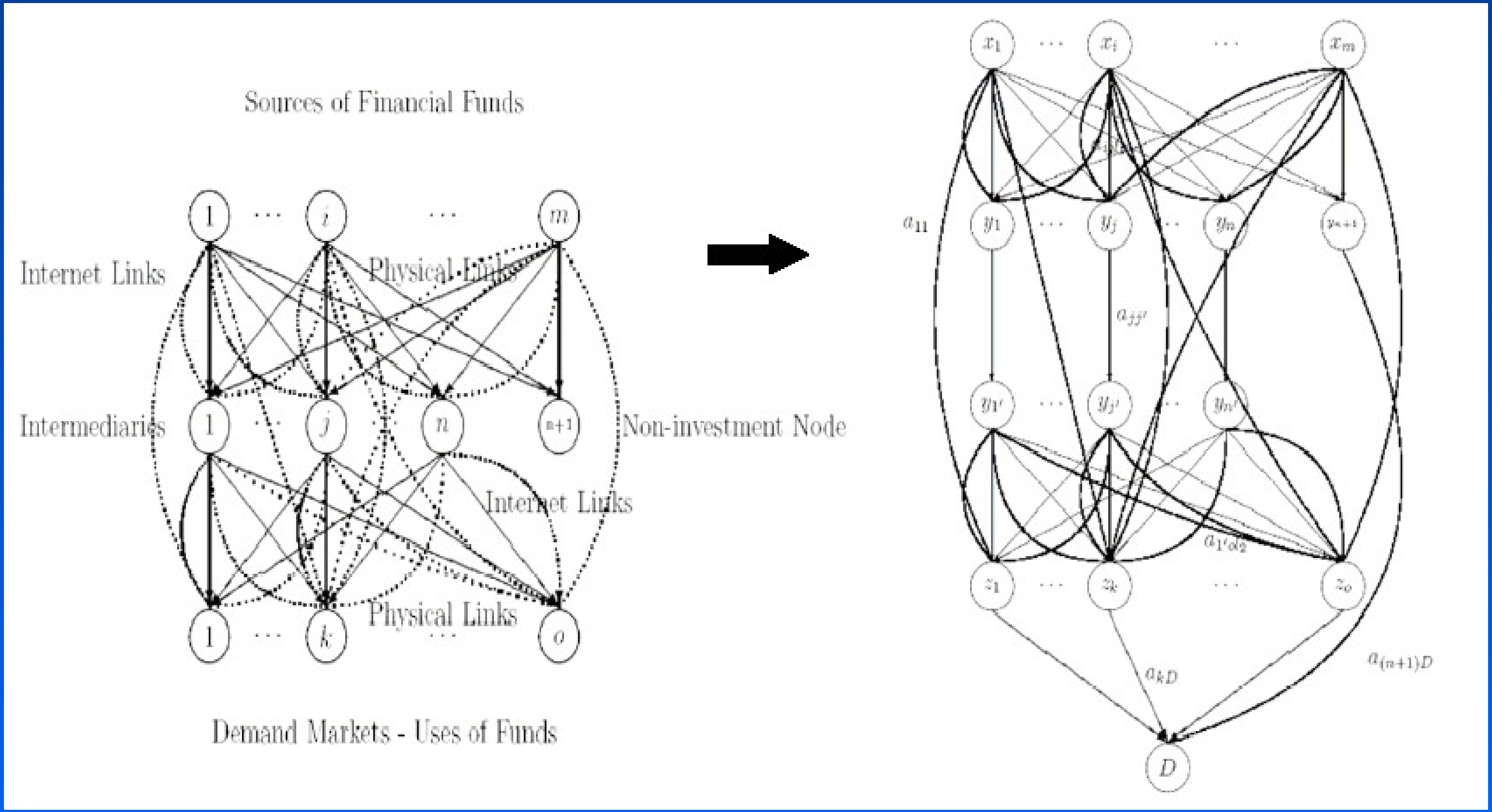
Electric Power Supply Chain Network with Fuel Suppliers



Matsypura, Nagurney, and Liu, *International Journal of Emerging Power Systems* (2007).

In 1952, Copeland wondered whether money flows like water or electricity.

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation



Liu and Nagurney, *Computational Management Science* (2007).

We have shown that *money* as well as *electricity* flow like *transportation* and have answered questions posed fifty years ago by Copeland and by Beckmann, McGuire, and Winsten!

Recent disasters have demonstrated the importance and the vulnerability of network systems.

Examples:

- 9/11 Terrorist Attacks, September 11, 2001;
- The biggest blackout in North America, August 14, 2003;
- Two significant power outages in September 2003 -- one in the UK and the other in Italy and Switzerland;
- Hurricane Katrina, August 23, 2005;
- The Minneapolis I-35 Bridge Collapse, August 1, 2007;
- Mediterranean Sea telecommunications cable destruction – January 30, 2008.

Disasters in Transportation Networks

www.salem-news.com

www.boston.com

Communication Network Disasters

www.tx.mb21.co.uk

www.w5jgv.com

www.wirelessestimator.com

Electric Power Network Disasters

media.collegepublisher.com

www.cellar.org

www.crh.noaa.gov

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Holme, Kim, Yoon and Han (2002)
- Taylor and D'este (2004)
- Murray-Tuite and Mahmassani (2004)
- Chassin and Posse (2005)
- Barrat, Barthélemy and Vespignani (2005)
- Sheffi (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Jenelius, Petersen and Mattson (2006)
- Taylor and D'Este (2007)

Our Research on Network Efficiency, Vulnerability, and Robustness

A Network Efficiency Measure for Congested Networks, Nagurney and Qiang, *Europhysics Letters,* **79**, August (2007).

A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability, Nagurney and Qiang, *Proceedings of the POMS 18th Annual Conference,* Dallas, Texas (2007).

A Network Efficiency Measure with Application to Critical Infrastructure Networks, Nagurney and Qiang, *Journal of Global Optimization* (2008).

Robustness of Transportation Networks Subject to Degradable Links, Nagurney and Qiang, *Europhysics Letters,* **80**, December (2007).

A Unified Network Performance Measure with Importance Identification and the Ranking of Network Components, Qiang and Nagurney, *Optimization Letters,* **2** (2008).

A New Performance/Efficiency Measure with Applications to Complex Networks

The Nagurney and Qiang (N-Q) Network Efficiency Measure

The network performance/efficiency measure $\mathcal{E}(G,d)$, for a given network topology G and fixed demand vector d, is defined as

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where n_w is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

Nagurney and Qiang, *Europhysics Letters,* **79** (2007).

Importance of a Network Component

Definition: Importance of a Network Component

The importance, *I(g),* of a network component *gεG* is measured by the relative network efficiency drop after *g* is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where *G-g* is the resulting network after component g is removed.

The Latora and Marchiori (L-M) Network Efficiency Measure

Definition: The L-M Measure

The network performance/efficiency measure, *E(G)* for a given network topology, *G*, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

The L-M Measure vs. the N-Q Measure

Theorem:

If positive demands exist for all pairs of nodes in the network, *G*, and each of demands is equal to 1, and if d_{ij} is set equal to λ_w, where $w=(i,j)$, for all $w \varepsilon W$, then the N-Q and L-M network efficiency measures are one and the same.

The Approach to Study the Importance of Network Components

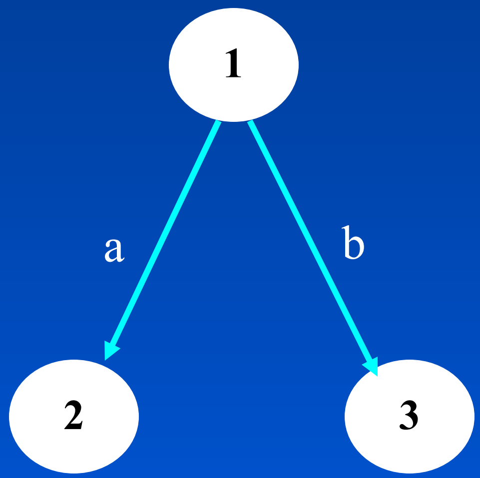
The elimination of a link is treated in the N-Q network efficiency measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks.

The measure generalizes the Latora and Marchiori network measure for complex networks.

Example 1

Assume a network with two O/D pairs:
$\boldsymbol{w_1}$=(1,2) and $\boldsymbol{w_2}$=(1,3) with demands:
$\boldsymbol{d_{w_1}}$=100 and $\boldsymbol{d_{w_2}}$=20.



$c_a(f_a)=0.01f_a+19$
$c_b(f_b)=0.05f_b+19$

The paths are:

for $\boldsymbol{w_1}$, $\boldsymbol{p_1=a}$; for $\boldsymbol{w_2}$, $\boldsymbol{p_2=b}$.

The equilibrium path flows are:
$\boldsymbol{x_{p_1}^*}$= 100, $\boldsymbol{x_{p_2}^*}$=20.

The equilibrium path travel costs are:
$\boldsymbol{C_{p_1}=C_{p_2}}$=20.

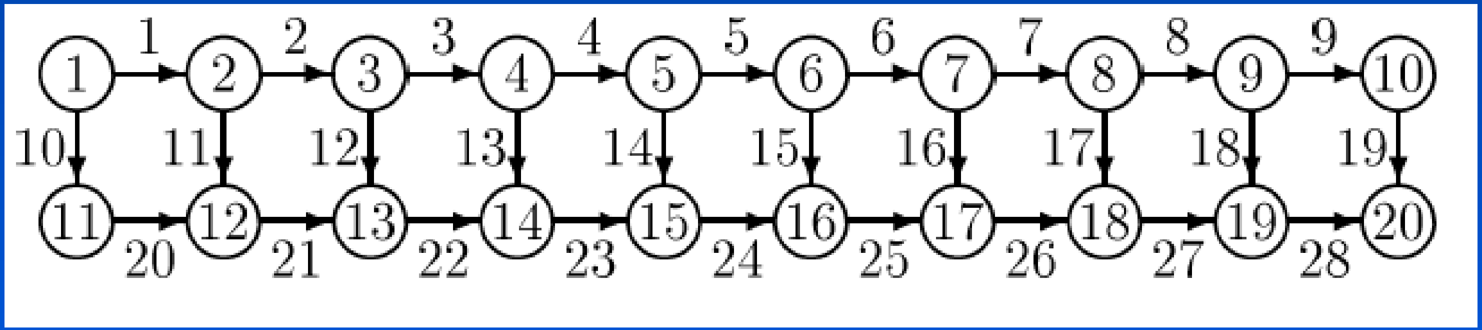
Importance and Ranking of Links and Nodes

Link	Importance Value from Our Measure	Importance Ranking from Our Measure
a	0.8333	1
b	0.1667	2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure
1	1	1
2	0.8333	2
3	0.1667	3

Example 2

The network is given by:



From: Nagurney,

Transportation Research B (1984)

w_1=(1,20) w_2=(1,19)

d_{w_1} = 100 d_{w_2} = 100

Example 2: Link Cost Functions

Link a	Link Cost Function $c_a(f_a)$
1	$.00005f_1^4 + 5f_1 + 500$
2	$.00003f_2^4 + 4f_2 + 200$
3	$.00005f_3^4 + 3f_3 + 350$
4	$.00003f_4^4 + 6f_4 + 400$
5	$.00006f_5^4 + 6f_5 + 600$
6	$7f_6 + 500$
7	$.00008f_7^4 + 8f_7 + 400$
8	$.00004f_8^4 + 5f_8 + 650$
9	$.00001f_9^4 + 6f_9 + 700$
10	$4f_{10} + 800$
11	$.00007f_{11}^4 + 7f_{11} + 650$
12	$8f_{12} + 700$
13	$.00001f_{13}^4 + 7f_{13} + 600$
14	$8f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003f_{15}^4 + 9f_{15} + 200$
16	$8f_{16} + 300$
17	$.00003f_{17}^4 + 7f_{17} + 450$
18	$5f_{18} + 300$
19	$8f_{19} + 600$
20	$.00003f_{20}^4 + 6f_{20} + 300$
21	$.00004f_{21}^4 + 4f_{21} + 400$
22	$.00002f_{22}^4 + 6f_{22} + 500$
23	$.00003f_{23}^4 + 9f_{23} + 350$
24	$.00002f_{24}^4 + 8f_{24} + 400$
25	$.00003f_{25}^4 + 9f_{25} + 450$
26	$.00006f_{26}^4 + 7f_{26} + 300$
27	$.00003f_{27}^4 + 8f_{27} + 500$
28	$.00003f_{28}^4 + 7f_{28} + 650$

Algorithms for Solution

The projection method (cf. Dafermos (1980) and Nagurney (1999)) embedded with the equilibration algorithm of Dafermos and Sparrow (1969) was used for the computations.

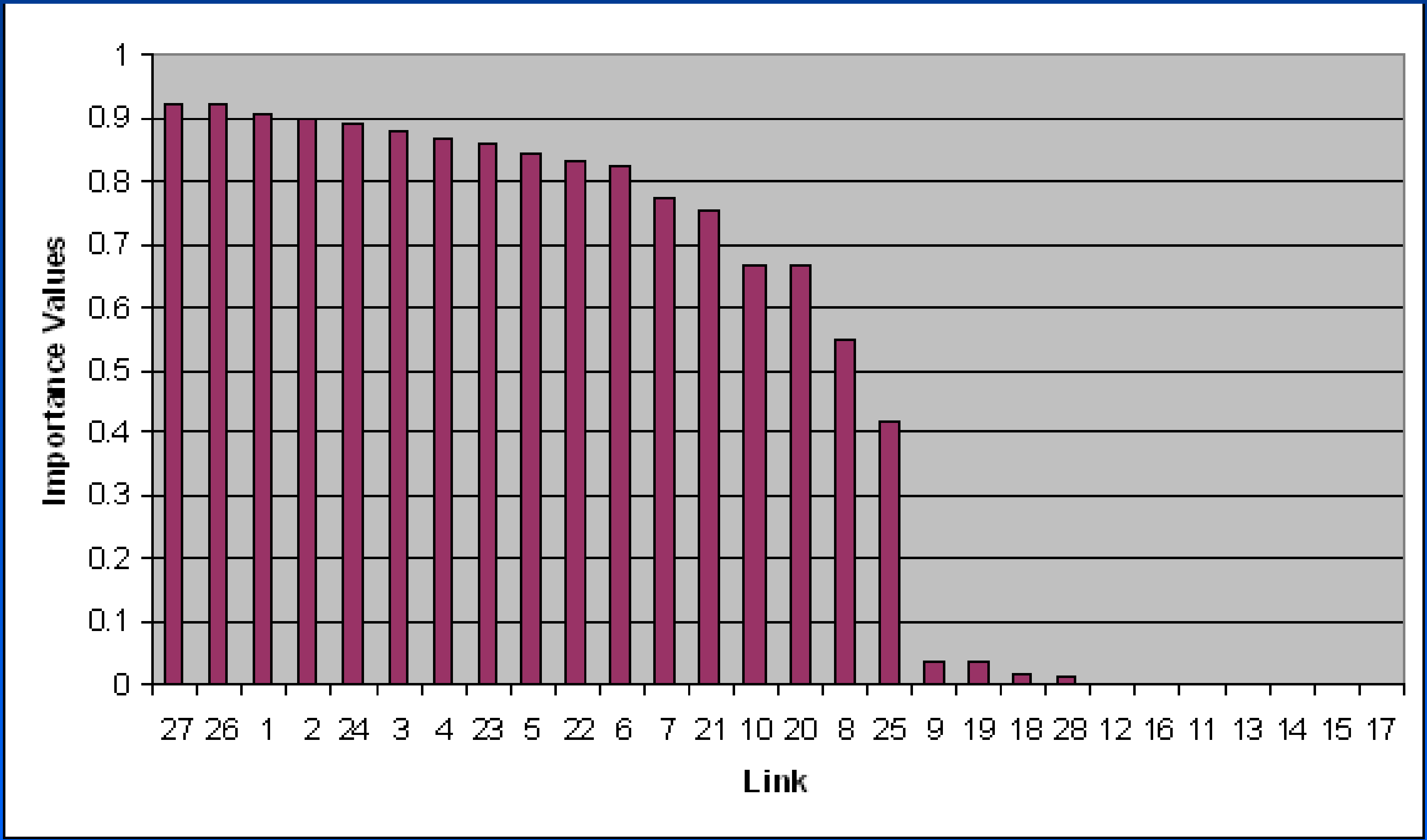
In addition, the column generation method of Leventhal, Nemhauser, and Trotter (1973) was implemented to generate paths, as needed, in the case of the large-scale Sioux Falls network example.

Example 2: Importance and Ranking of Links

Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19

Example 2: Link Importance Rankings

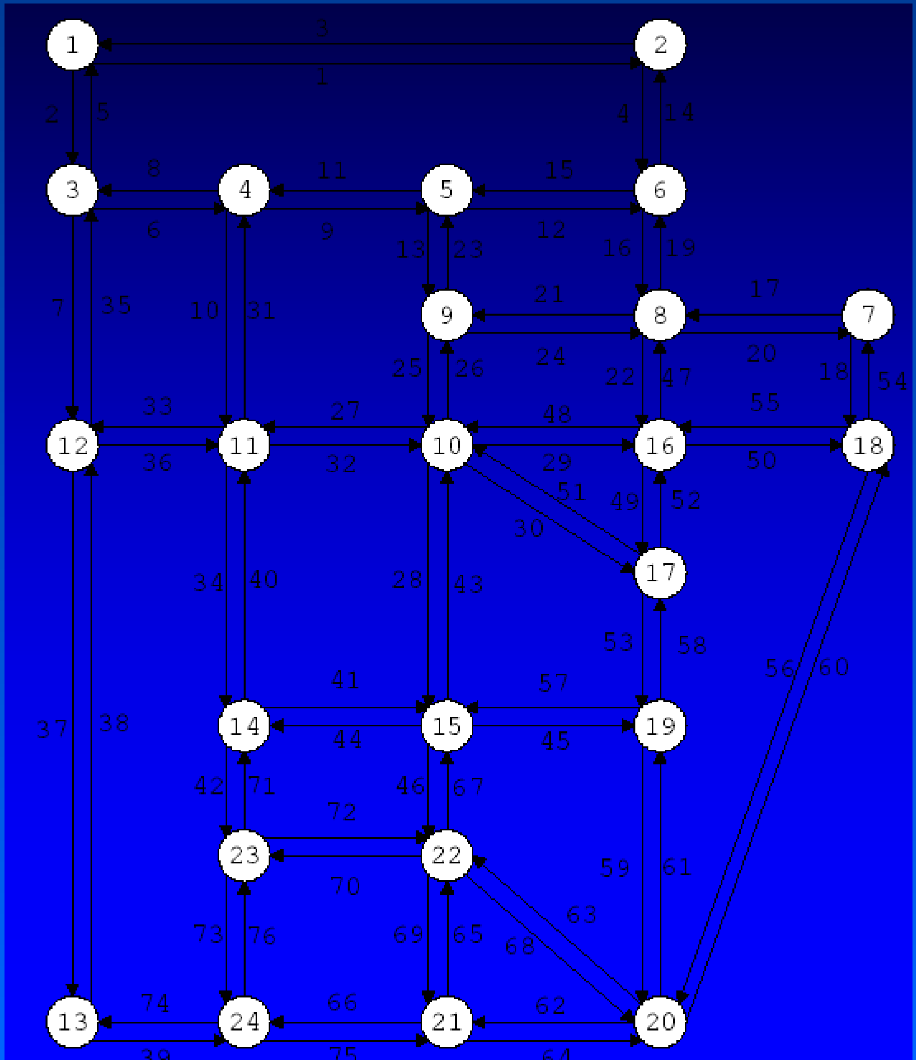


Example 3 - Sioux Falls Network

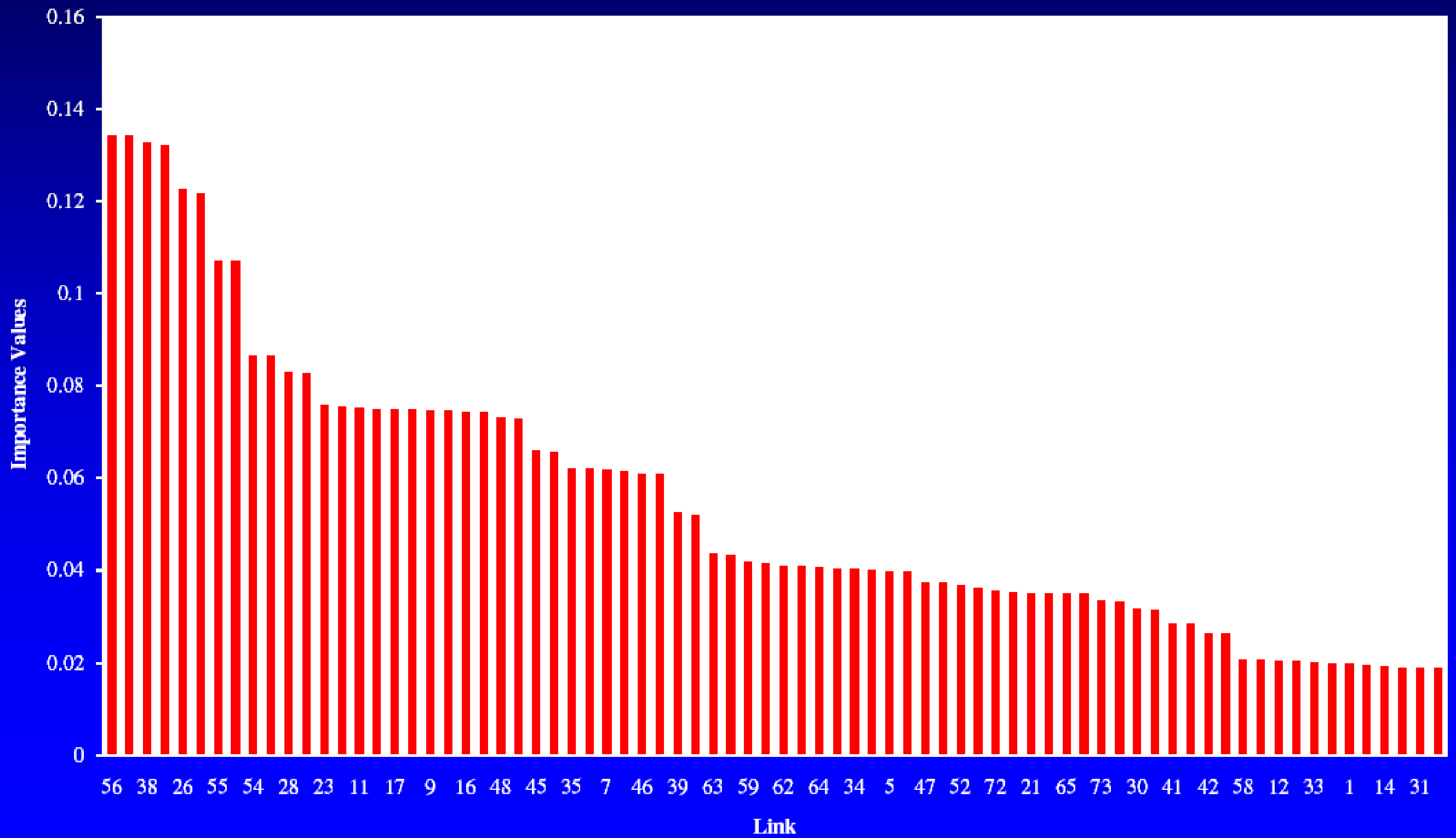
The network data are from LeBlanc, Morlok, and Pierskalla (1975).

The network has 528 O/D pairs, 24 nodes, and 76 links.

The user link cost functions are of Bureau of Public Roads (BPR) form.



Example 3 - Sioux Falls Network Link Importance Rankings



Example 4: An Electric Power Supply Chain Network

Nagurney and Liu (2006) and Nagurney, Liu, Cojocaru and Daniele (2007) have shown that an electric power supply chain network can be transformed into an equivalent transportation network problem.

Supernetwork Transformation

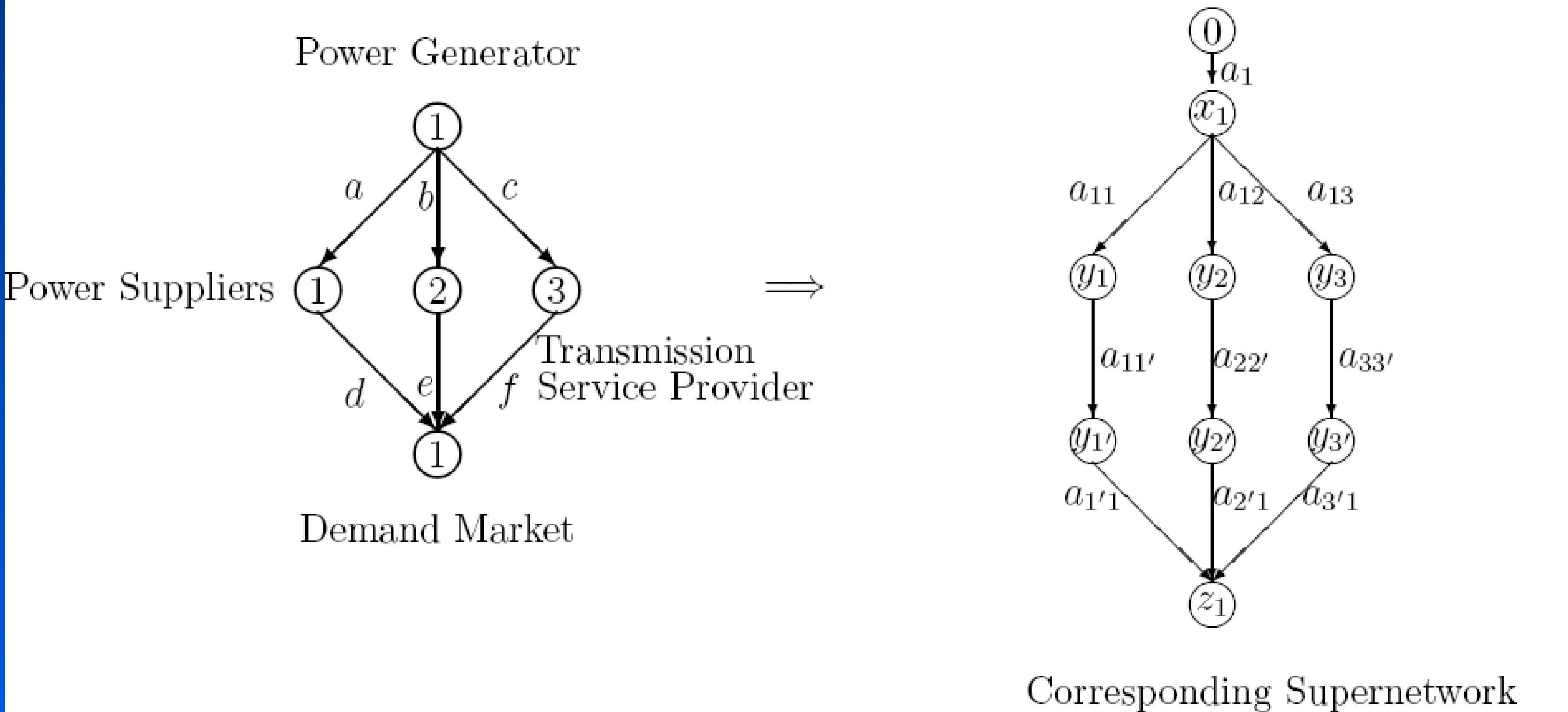


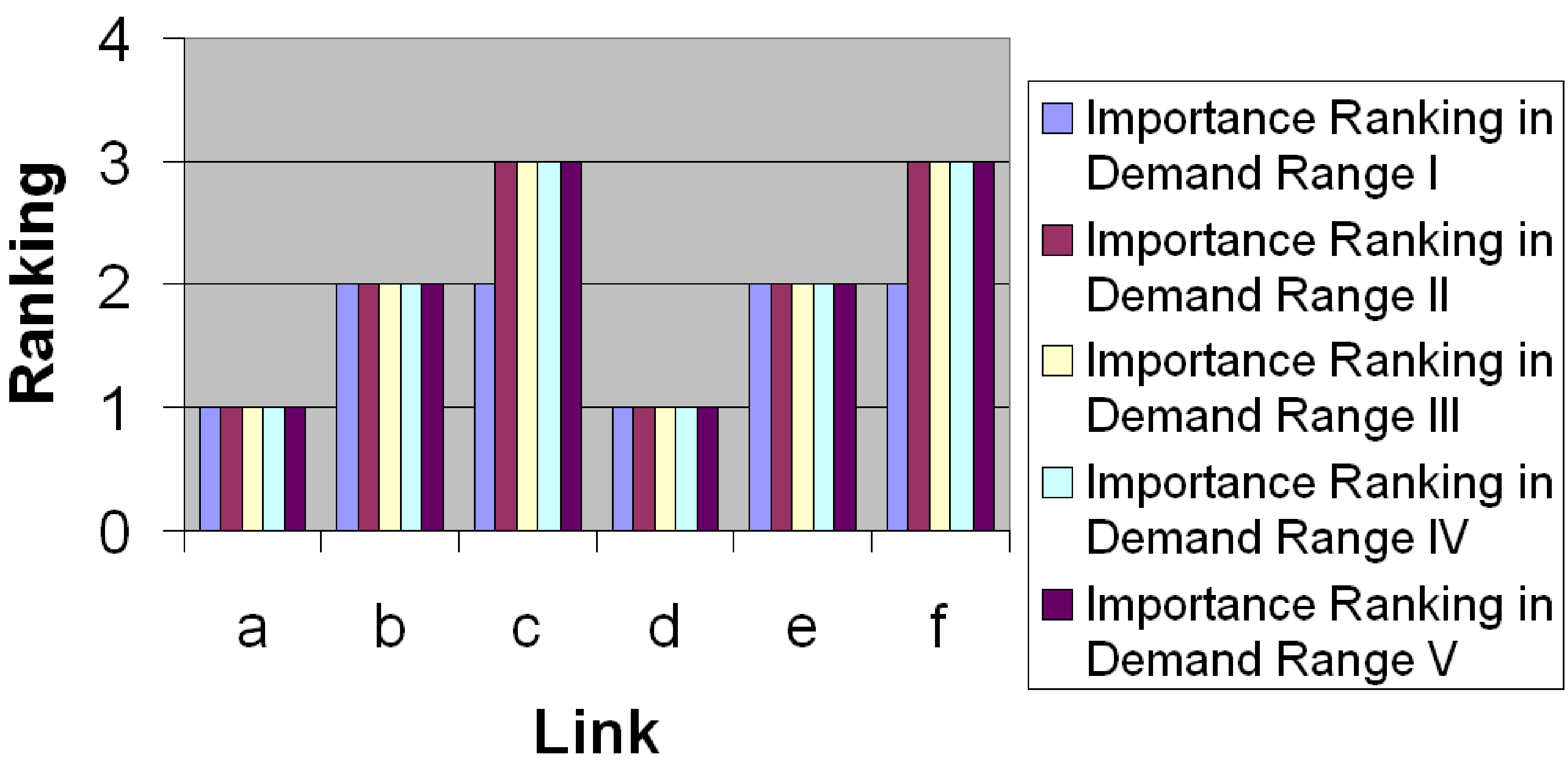
Figure 3: Electric Power Supply Chain Network and the Corresponding Supernetwork

Nagurney, Liu, Cojocaru and Daniele, *Transportation Research E* (2007). Example taken from Nagurney and Qiang, *JOGO*, in press.

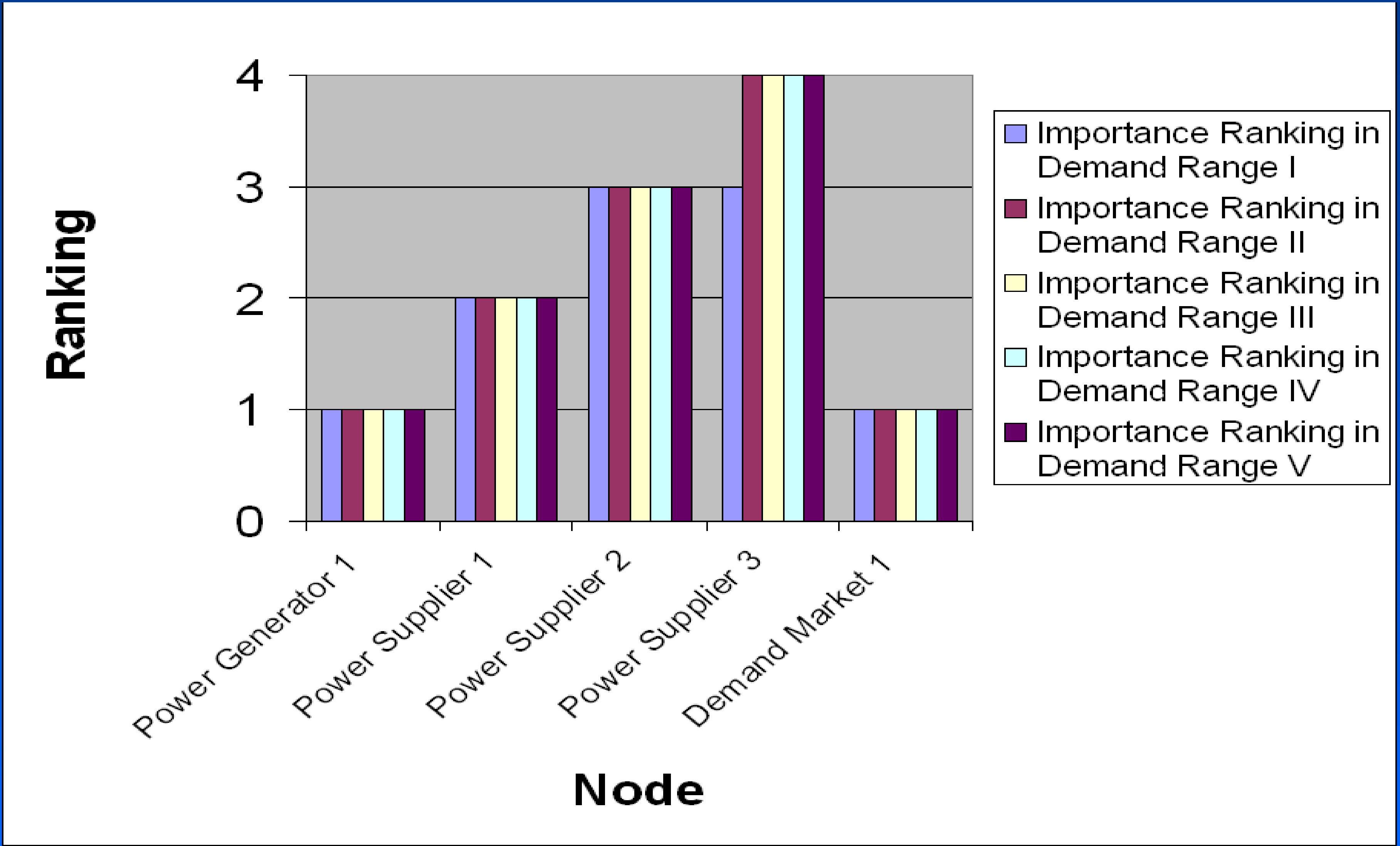
Five Demand Ranges

- Demand Range I: d_w Є [0, 1]
- Demand Range II: d_w Є (1,4/3]
- Demand Range III: d_w Є (4/3,7/3]
- Demand Range IV: d_w Є (7/3, 11/3]
- Demand Range V: d_w Є (11/3, \infty)

Importance Ranking of Links in the Electric Power Supply Chain Network



Importance Ranking of Nodes in the Electric Power Supply Chain Network



The Advantages of the N-Q Network Efficiency Measure

- The measure captures demands, flows, costs, and behavior of users, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
- It can be applied to assess the efficiency/performance of a wide range of network systems.
- It is applicable also to elastic demand networks (Qiang and Nagurney, *Optimization Letters* (2008)).
- It has been extended to dynamic networks (Nagurney and Qiang, *Netnomics*, in press).

What About Dynamic Networks?

We are using evolutionary variational inequalities to model dynamic networks with:

- *dynamic (time-dependent)* supplies and demands
- *dynamic (time-dependent)* capacities
- *structural changes* in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).

Evolutionary Variational Inequalities

Evolutionary variational inequalities, which are infinite dimensional, were originally introduced by Lions and Stampacchia (1967) and by Brezis (1967) in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems.

Steinbach (1998) studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity.

Daniele, Maugeri, and Oettli (1998, 1999), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others. See also Ran and Boyce(1996).

2005-2006 Radcliffe Institute for Advanced Study Fellowship Year at Harvard Collaboration with Professor David Parkes of Harvard University and Professor Patrizia Daniele of the University of Catania

A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks. (page 10 of Roughgarden's (2005) book, *Selfish Routing and the Price of Anarchy*).

A Dynamic Model of the Internet

The Internet, Evolutionary Variational Inequalities, and the Time-Dependent Braess Paradox, Nagurney, Parkes, and Daniele, *Computational Management Science* **4** (2007), 355-375.

We now define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0,T], R^{Kn_P})$ (where $[0,T]$ denotes the time interval under consideration) given by

$$\mathcal{K} = \Big\{ x \in L^2([0,T], R^{Kn_P}) : 0 \le x(t) \le \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k \text{ a.e. in } [0,T] \Big\}.$$

We assume that the capacities $\mu_r^k(t)$, for all r and k, are in $\mathcal{L}$, and that the demands, $d_w^k \ge 0$, for all w and k, are also in $\mathcal{L}$. Further, we assume that

$$0 \le d(t) \le \Phi\mu(t), \text{a.e. on } [0,T],$$

where Φ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element (kw, kr) equal to 1 if route r is contained in P_w, and 0, otherwise. The feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \to \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

The conditions below are a generalization of the Wardrop's (1952) first principle of traffic behavior.

Definition: Dynamic Multiclass Network Equilibrium

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle) if, for every O/D pair $w \in W$, every route $r \in P_w$, every class k; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if } \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if } \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if } \quad x_r^{k*}(t) = 0. \end{cases}$$

The standard form of the EVI that we work with is:

determine $x^* \in \mathcal{K}$ such that $\langle\langle F(x^*), x - x^* \rangle\rangle \geq 0$, $\forall x \in \mathcal{K}$.

Theorem (Nagurney, Parkes, Daniele (2007))

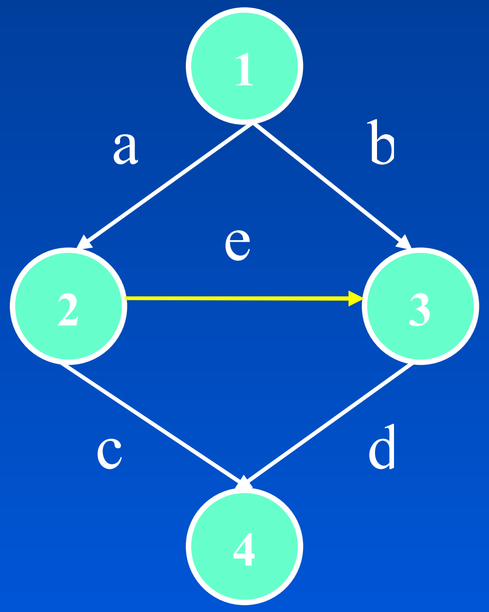
$x^* \in \mathcal{K}$ is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

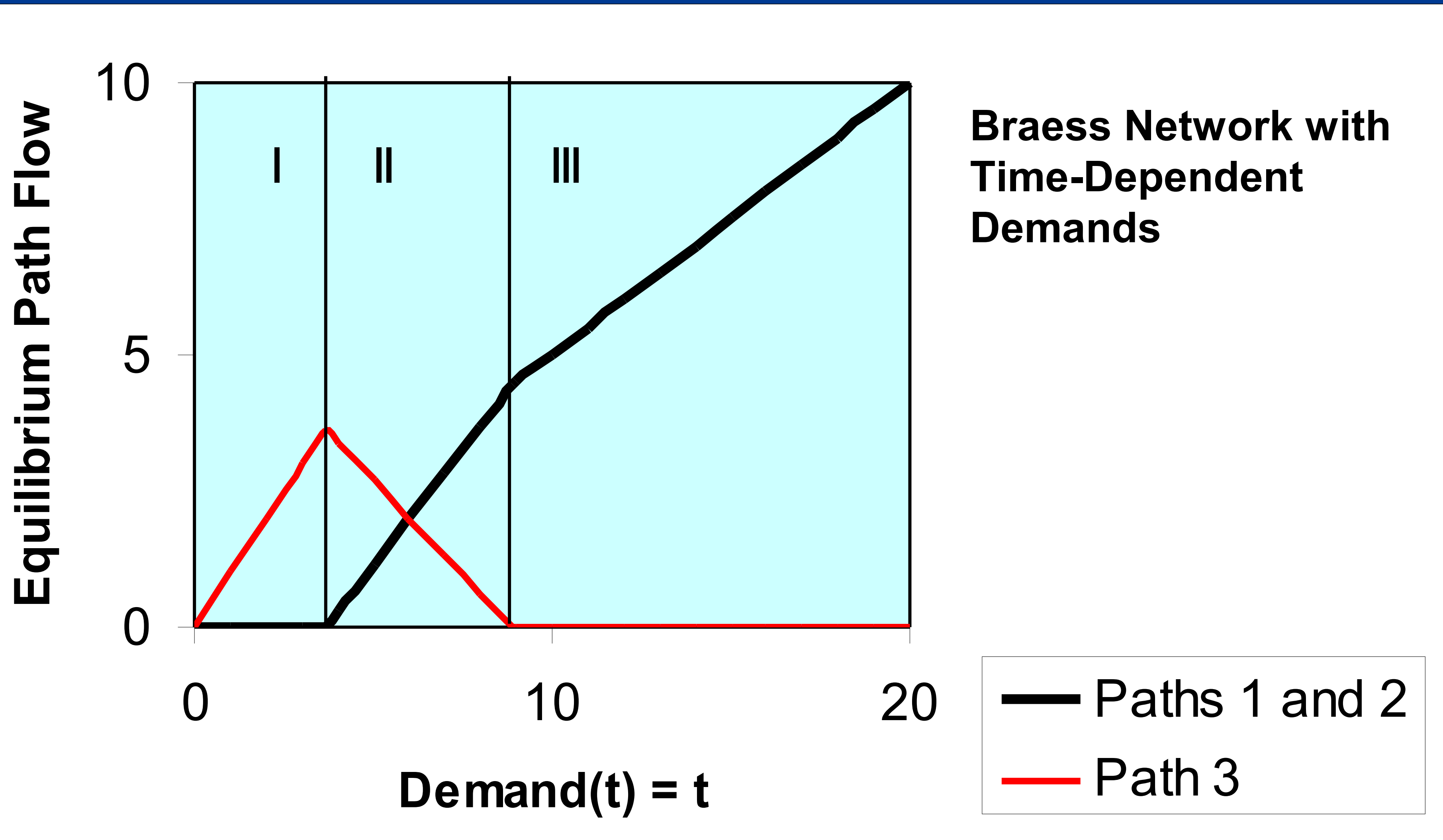
Nagurney, Parkes, and Daniele, *Computational Management Science* (2007).

The Time-Dependent (Demand-Varying) Braess Paradox and Evolutionary Variational Inequalities

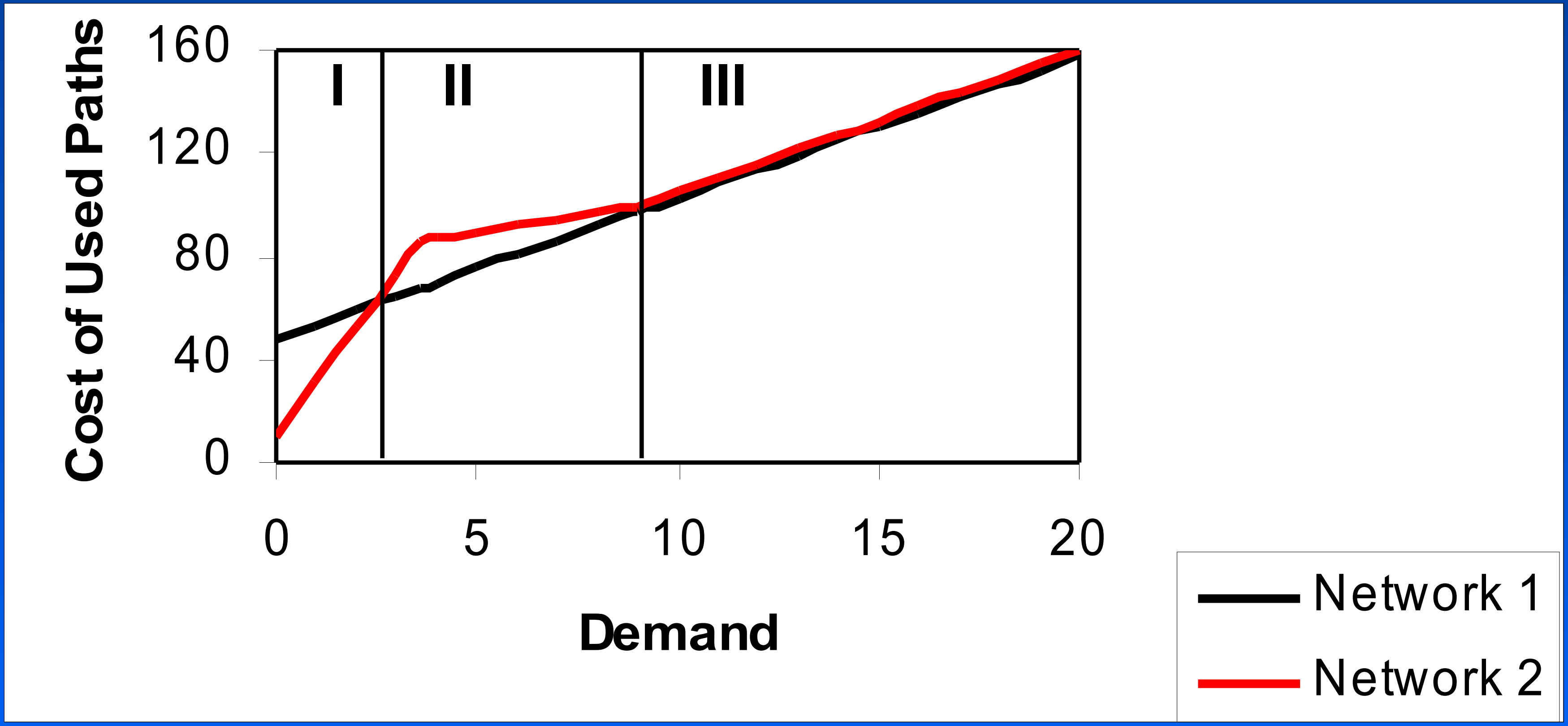
Recall the Braess Network
where we add the link e.



The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)



In Demand Regime I, only the new path is used.
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off!
In Demand Regime III, only the original paths are used.



Network 1 is the Original Braess Network - Network 2 has the added link.

The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!

Extension of the Network Efficiency Measure to Dynamic Networks

An Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities with Applications to the Internet and Vulnerability Analysis, Nagurney and Qiang, *Netnomics*, in press.

Network Efficiency Measure for Dynamic Networks - Continuous Time

The network efficiency for the network G with time-varying demand d for $t \in [0, T]$, denoted by $\mathcal{E}(G, d, T)$, is defined as follows:

$$\mathcal{E}(G, d, T) = \frac{\int_0^T [\sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)}]/n_W \, dt}{T}.$$

The above measure is the average network performance over time of the dynamic network.

Network Efficiency Measure for Dynamic Networks - Discrete Time

Let $d_w^1, d_w^2, ..., d_w^H$ denote demands for O/D pair w in H discrete time intervals, given, respectively, by:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$. We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair w at the H different time intervals by: $\lambda_w^1, \lambda_w^2, ..., \lambda_w^H$. The demand vector d, in this special discrete case, is a vector in $R^{n_W \times H}$. The dynamic network efficiency measure in this case is as follows:

Dynamic Network Efficiency: Discrete Time Version

The network efficiency for the network (G, d) over H discrete time intervals:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, *where* $t_H \equiv T$, *and with the respective constant demands:*
$d_w^1, d_w^2, ..., d_w^H$ *for all* $w \in W$ *is defined as follows:*

$$\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.$$

Importance of a Network Component

The importance of a network component *g* of network *G* with demand *d* over time horizon *T* is defined as follows:

$$I(g,d,T) = \frac{\mathcal{E}(G,d,T) - \mathcal{E}(G-g,d,T)}{\mathcal{E}(G,d,T)}$$

where $\mathcal{E}(G-g,d,T)$ is the dynamic network efficiency after component *g* is removed.

Importance of Nodes and Links in the Dynamic Braess Network Using the N-Q Measure when T=10

Link	Importance Value	Importance Ranking
a	0.2604	1
b	0.1784	2
c	0.1784	2
d	0.2604	1
e	-0.1341	3

Node	Importance Value	Importance Ranking
1	1.0000	1
2	0.2604	2
3	0.2604	2
4	1.0000	1

Link e is never used after $t = 8.89$ and in the range $t \in [2.58, 8.89]$, it increases the cost, so the fact that link e has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!

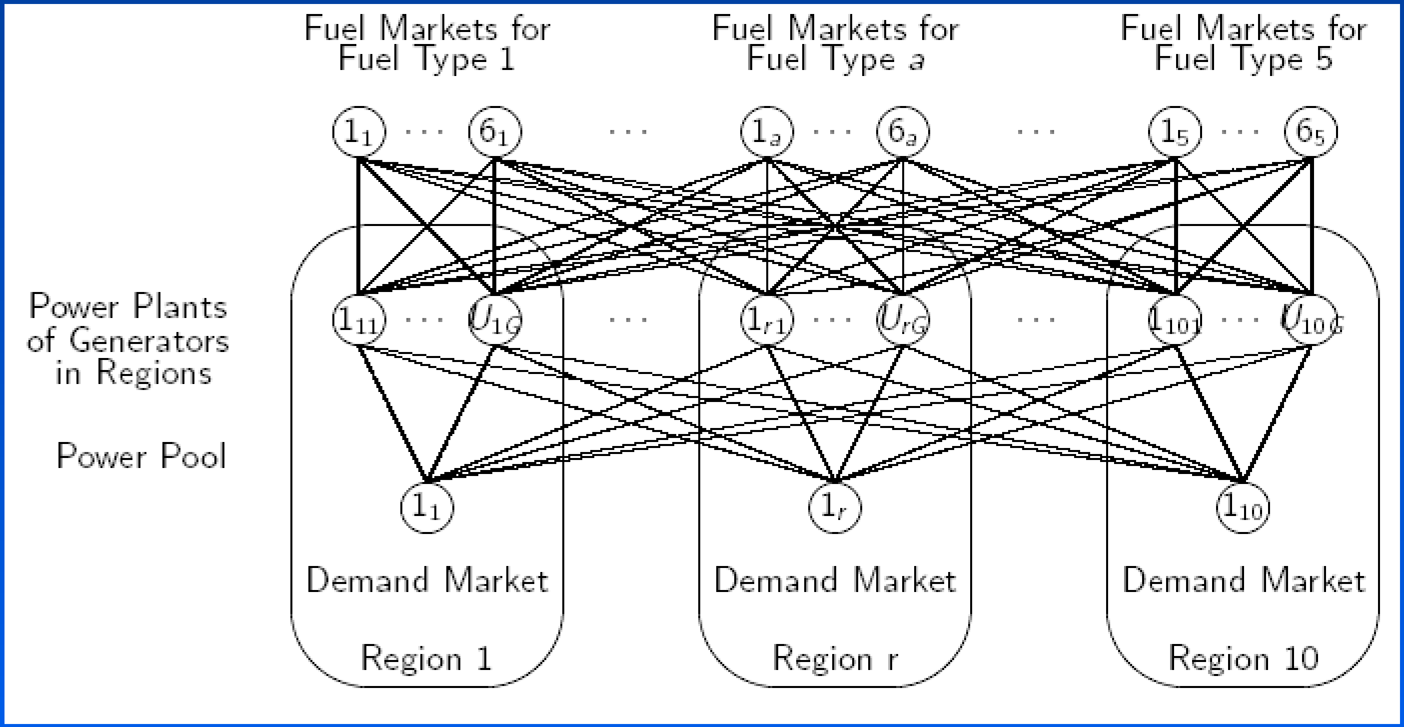
Where Are We Now?

An Integrated Electric Power Supply Chain and Fuel Market Network Framework: Theoretical Modeling with Empirical Analysis for New England, Liu and Nagurney (2007).

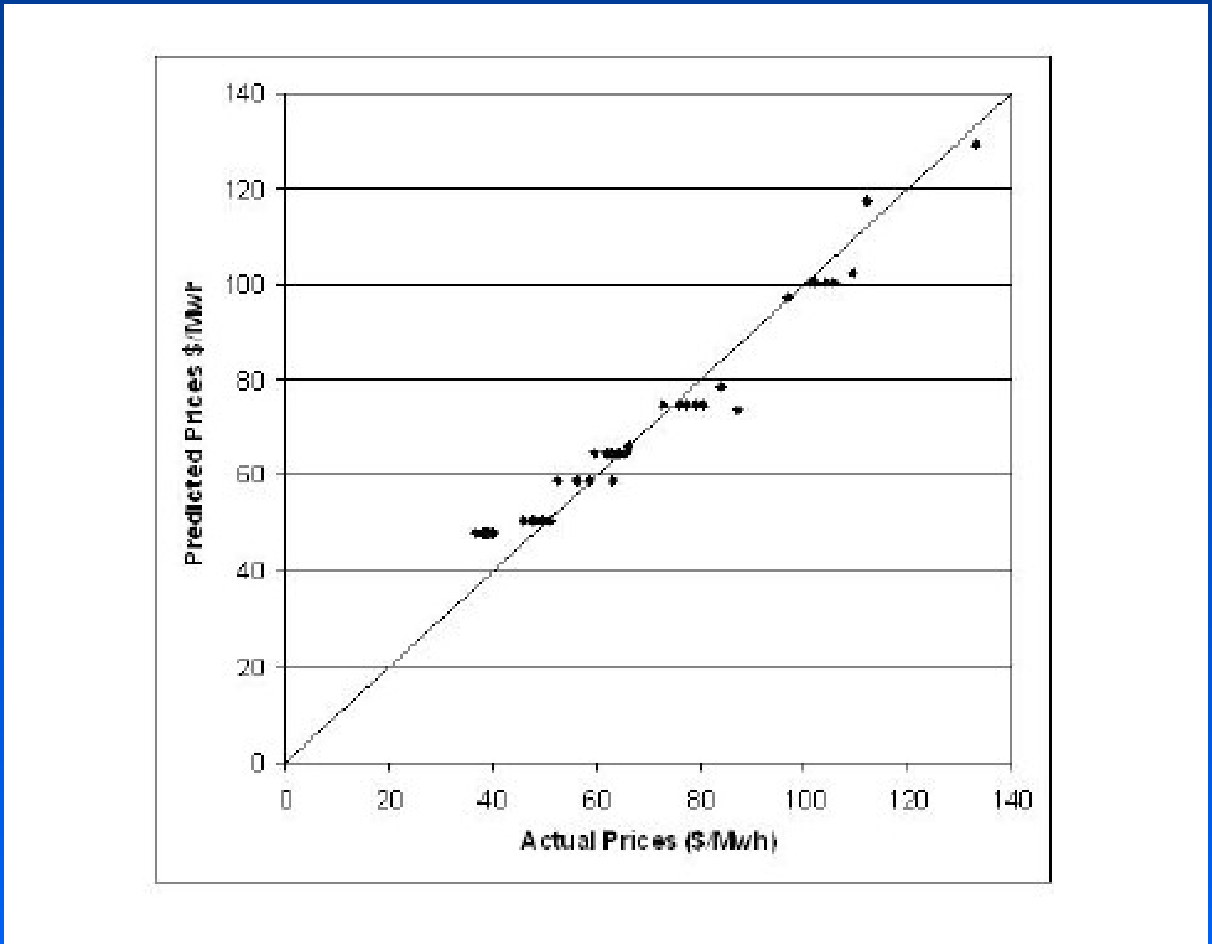
Empirical Case Study

- New England electric power market and fuel markets
- 82 generators who own and operate 573 power plants
- 5 types of fuels: natural gas, residual fuel oil, distillate fuel oil, jet fuel, and coal
- Ten regions (R=10): 1. Maine, 2. New Hampshire, 3. Vermont, 4. Connecticut(excluding Southwest Connecticut), 5. Southwest Connecticut(excluding Norwalk-Stamford area), 6. Norwalk-Stamford area, 7. Rhode Island, 8. Southeast Massachusetts, 9. West and Central Massachusetts, 10. Boston/Northeast Massachusetts
- Hourly demand/price data of July 2006 ($24 \times 31 = 744$ scenarios)
- 6 blocks ($L1 = 94$ hours, and $Lw = 130$ hours; $w = 2, ..., 6$)

The New England Electric Power Supply Chain Network with Fuel Suppliers



Predicted Prices vs. Actual Prices ($/Mwh)



Summary and Conclusions

We have described a new network efficiency/performance measure that can be applied to fixed demand, elastic demand as well as dynamic network problems to identify the importance and rankings of network components.

We also demonstrated through a variety of complex network applications the suitability of the measure to investigate vulnerability as well as robustness of complex networks with a focus on transportation and related applications, including the Internet and electric power supply chains.

An analogue of the measure has been developed and applied to financial networks with intermediation and electronic commerce by Nagurney and Qiang -- in *Computational Methods in Financial Engineering* (2008), Kontoghiorghes, Rustem, and Winker, editors, Springer,

Ongoing Research and Questions

- How can time delays be incorporated into the measure?
- How do we capture multiclass user behavior; equivalently, behavior in multimodal networks?
- Can the framework be generalized to capture multicriteria decision-making?
- What happens if either system-optimizing (S-O) or user-optimizing (U-O) behavior needs to be assessed from a network system performance angle? We have some results in this dimension in terms of vulnerability and robustness analysis as well as from an environmental (emissions generated) perspective.
- Can we identify the most important nodes and links in large-scale electric power supply chains as in our empirical case study?

http://supernet.som.umass.edu

Thank you!

For more information, see

http://supernet.som.umass.edu

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks