Transportation Network Equilibrium --
The Formalism for Networks Today from the Internet to Electric Power Supply Chains and Financial Networks:
What the World Should Learn from Regional Scientists

Anna Nagurney
John F. Smith Memorial Professor
Department of Finance and Operations Management
University of Massachusetts
Amherst, MA 01003

November 20, 2008
Outline of Presentation

• Background
• The Transportation Network Equilibrium Problem and Methodological Tools
• The Braess Paradox
• Transportation and Complex Networks
• A New Network Performance/Efficiency Measure with Applications to Network Systems
• What About Dynamic Networks?
• Evolutionary Variational Inequalities, the Internet, an the Time-Dependent (Demand-Varying) Braess Paradox
• Extension of the Efficiency Measure to Dynamic Networks
• An Empirical Case Study to Real-World Supply Chains
• Mergers and Acquisitions, Supply Chains, and a Paradox
• Humanitarian Logistics
Interdisciplinary Impact of Regional Science

**Economics**
- Interregional Trade
- Industrial Organization
- Location Theory
- Spatial Econometrics
- Input/Output Analysis

**Engineering - OR/MS**
- Transportation
- Telecommunications
- Energy
- Supply Chains

**Sociology**
- Social Networks
- Organizational Theory

**Computer Science**
- Routing Algorithms
- Price of Anarchy

**Physics**
- Complex Systems and Networks
We Are in a New Era of Decision-Making:

- **complex interactions** among decision-makers in organizations;
- alternative and at times **conflicting criteria** used in decision-making;
- **constraints on resources**: natural, human, financial, time, etc.;
- **global reach** of many decisions in which **spatial issues are critical**;
- **high impact** of many decisions, and
- the **importance of dynamics** and realizing a fast and sound response to evolving events.
Network problems are their own class of problems and they come in various forms and formulations, i.e., as optimization (linear or nonlinear) problems or as equilibrium problems and even dynamic network problems.

Complex network problems, with a focus on transportation, will be the focus of this talk.
Transportation, Communication, and Energy Networks

Bus Network

Rail Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks

British Electricity Grid
## Components of Common Physical Networks

<table>
<thead>
<tr>
<th>Network System</th>
<th>Nodes</th>
<th>Links</th>
<th>Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transportation</td>
<td>Intersections, Homes, Workplaces, Airports, Railyards</td>
<td>Roads, Airline Routes, Railroad Track</td>
<td>Automobiles, Trains, and Planes,</td>
</tr>
<tr>
<td>Manufacturing and logistics</td>
<td>Workstations, Distribution Points</td>
<td>Processing, Shipment</td>
<td>Components, Finished Goods</td>
</tr>
<tr>
<td>Communication</td>
<td>Computers, Satellites, Telephone Exchanges</td>
<td>Fiber Optic Cables, Radio Links</td>
<td>Voice, Data, Video</td>
</tr>
<tr>
<td>Energy</td>
<td>Pumping Stations, Plants</td>
<td>Pipelines, Transmission Lines</td>
<td>Water, Gas, Oil, Electricity</td>
</tr>
</tbody>
</table>
US Railroad Freight Flows

Railroad Freight Density (million gross tons):
- Blue: Under 10 mgt
- Green: 10 to 20 mgt
- Yellow: 20 to 40 mgt
- Orange: 40 to 60 mgt
- Red: 60 to 100 mgt
- Red: Over 100 mgt

Natural Gas Pipeline Network in the US
World Oil Trading Network
The study of the efficient operation on transportation networks dates to *ancient Rome* with a classical example being the publicly provided Roman road network and the *time of day chariot policy*, whereby chariots were banned from the ancient city of Rome at particular times of day.
Characteristics of Networks Today

• *large-scale nature* and complexity of network topology;

• *congestion*;

• the *interactions among networks* themselves such as in transportation versus telecommunications;

• *policies* surrounding networks today may have a *major impact* not only economically but also environmentally, socially, politically, and security-wise;
• alternative behaviors of the users of the network

– system-optimized versus

– user-optimized (network equilibrium),

which may lead to

paradoxical phenomena.
There are *two fundamental principles of travel behavior* (Wardrop (1952)):

- User-optimization (or network equilibrium)
- System-optimization

(Beckmann, McGuire, and Winsten (1956), Dafermos and Sparrow (1969)).

These concepts correspond to decentralized versus centralized decision-making and are extremely relevant in today's networked economies and societies.
In a *user-optimized (network equilibrium) problem*, each user of a network system seeks to determine his/her cost-minimizing route of travel between an origin/destination pair, until an equilibrium is reached, in which no user can decrease his/her cost of travel by unilateral action.

In a *system-optimized network problem*, users are allocated among the routes so as to minimize the total cost in the system.

Both classes of problems, under certain imposed assumptions, possess convex optimization formulations.
Capturing Link Congestion

For a typical user link travel time function, where the free flow travel time refers to the travel time to traverse a link when there is zero flow on the link (or zero vehicles).

Link travel time [minutes]

Free flow travel time

capacity

Link flow [vehicles/hour]
BPR Link Cost Function

A common link performance function is the Bureau of Public Roads (BPR) cost function developed in 1964. This equation is given by

\[ c_a = c_a^0 \left[ 1 + \alpha \left( \frac{f_a}{t'_a} \right)^\beta \right], \]

where, \( c_a \) and \( f_a \) are the travel time and link flow, respectively, on link \( a \), \( c_a^0 \) is the free-flow travel time, and \( t'_a \) is the “practical capacity” of link \( a \). The quantities \( \alpha \) and \( \beta \) are model parameters, for which the values \( \alpha = 0.15 \) minutes and \( \beta = 4 \) are typical values. For example, these values imply that the practical capacity of a link is the flow at which the travel time is 15% greater than the free-flow travel time.
The Transportation Network Equilibrium (TNE) Problem and Methodological Tools
Transportation applications have motivated the development of methodological tools in different disciplines, many of which have been motivated and derived from the book, *Studies in the Economics of Transportation*, Beckmann, McGuire, and Winsten (1956); see Boyce, Mahmassani, and Nagurney, *Papers in Regional Science* 84 (2005), 85-103.
On Saturday, November 22, 2003 at the 50th North American Meeting of the Regional Science Association International, a Special Panel was held to recognize the impacts and significance of *Studies in the Economics of Transportation*.

Chair and Discussant:
Suzanne Evans, Birbeck College, London

Panelists:
David E. Boyce, University of Illinois at Chicago (Emeritus) (Northwestern University (2008))
Anna Nagurney, University of Massachusetts at Amherst
Hani Mahmassani, University of Maryland (Northwestern University (2008))
The Transportation Social - Knowledge Network

On the Beach in Mallacoota, Australia

Professors Beckmann and Dafermos at Anna Nagurney’s Post-Ph.D. Defense Party in Barus Holley

Professor Beckmann with Professor Michael Florian of Montreal

Professors Beckmann and McGuire

INFORMS Honoring the 50th Anniversary of the Publication of Studies in the Economics of Transportation
Dafermos (1980) showed that the transportation network equilibrium (also referred to as user-optimization) conditions as formulated by Smith (1979) could be formulated as a finite-dimensional variational inequality.


Daniele, Maugeri, and Oettli (1998, 1999) introduced evolutionary variational inequalities for time-dependent (dynamic) traffic network equilibrium problems.


Bar-Gera and Boyce (2003) and Boyce and Bar-Gera (2003) developed and applied origin-based algorithms for large-scale transportation networks.
Consider a general network \( G = [N, L] \), where \( N \) denotes the set of nodes, and \( L \) the set of directed links. Let \( a \) denote a link of the network connecting a pair of nodes, and let \( p \) denote a path consisting of a sequence of links connecting an O/D pair. \( P_w \) denotes the set of paths, assumed to be acyclic, connecting the O/D pair of nodes \( w \) and \( P \) the set of all paths.

Let \( x_p \) represent the flow on path \( p \) and \( f_a \) the flow on link \( a \). The following conservation of flow equation must hold:

\[
 f_a = \sum_{p \in P} x_p \delta_{ap},
\]

where \( \delta_{ap} = 1 \), if link \( a \) is contained in path \( p \), and 0, otherwise. This expression states that the load on a link \( a \) is equal to the sum of all the path flows on paths \( p \) that contain (traverse) link \( a \).
Moreover, if we let $d_w$ denote the demand associated with O/D pair $w$, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair $w$ must be equal to the given demand $d_w$.

Let $c_a$ denote the user cost associated with traversing link $a$, which is assumed to be continuous, and $C_p$ the user cost associated with traversing the path $p$. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising the path.
Transportation Network Equilibrium Conditions

The network equilibrium conditions are then given by:
For each path $p \in P_w$ and every O/D pair $w$:

$$
C_p \left\{ \begin{array}{ll}
\lambda_w, & \text{if } x_p^* > 0 \\
\geq \lambda_w, & \text{if } x_p^* = 0
\end{array} \right.
$$

where $\lambda_w$ is an indicator, whose value is not known a priori. These equilibrium conditions state that the user costs on all used paths connecting a given O/D pair will be minimal and equalized.
As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that \[ \frac{\partial c_b}{\partial c_a} = \frac{\partial c_a}{\partial c_b} \] for all links \( a, b \) in the network then the solution to the above network equilibrium problem can be reformulated as the solution to an associated optimization problem. For example, if we have that \( c_a = c_a(f_a), \forall a \in L, \) then the solution can be obtained by solving:

\[
\text{Minimize} \quad \sum_{a \in L} \int_{0}^{f_a} c_a(y) dy
\]

subject to:

\[
d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,
\]

\[
f_a = \sum_{p \in P} x_p, \quad \forall a \in L,
\]

\[
x_p \geq 0, \quad \forall p \in P.
\]
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: \( p_1=(a,c) \) and \( p_2=(b,d) \).

For a travel demand of 6, the equilibrium path flows are \( x_{p_1}^* = x_{p_2}^* = 3 \) and

The equilibrium path travel cost is

\( C_{p_1} = C_{p_2} = 83. \)

\[
\begin{align*}
c_a(f_a) &= 10 f_a \\
c_b(f_b) &= f_b + 50 \\
c_c(f_c) &= f_c + 50 \\
c_d(f_d) &= 10 f_d
\end{align*}
\]
Adding a new link creates a new path \( p_3=(a,e,d) \).

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path \( p_3 \), \( C_{p_3} = 70 \).

The new equilibrium flow pattern network is

\[
\begin{align*}
x_{p_1}^* &= x_{p_2}^* = x_{p_3}^* = 2.
\end{align*}
\]

The equilibrium path travel costs:

\[
C_{p_1} = C_{p_2} = C_{p_3} = 92.
\]
The 1968 Braess article has been translated from German to English and appears as:

The System-Optimization (S-O) Problem

The above discussion focused on the user-optimized (U-O) problem. We now turn to the system-optimized (S-O) problem in which a central controller, say, seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in L} \hat{c}_a(f_a)$$

where it is assumed that the total cost function on a link $a$ is defined as:

$$\hat{c}_a(f_a) \equiv c_a(f_a) \times f_a,$$

subject to the conservation of flow constraints, and the nonnegativity assumption on the path flows. Here separable link costs have been assumed, for simplicity, and other total cost expressions may be used, as mandated by the particular application.
The S-O Optimality Conditions

Under the assumption of strictly increasing user link cost functions, the optimality conditions are: For each path $p \in P_w$, and every O/D pair $w$:

$$\hat{C}_p' \begin{cases} = \mu_w, & \text{if } x_p > 0 \\ \geq \mu_w, & \text{if } x_p = 0, \end{cases}$$

where $\hat{C}_p'$ denotes the marginal total cost on path $p$, given by:

$$\hat{C}_p' = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}.$$ 

The above conditions correspond to Wardrop’s second principle of travel behavior.
What is the S-O solution for the two Braess networks (before and after the addition of a new link e)?

Before the addition of the link e, we may write:

\[ \hat{c}_a' = 20f_a, \quad \hat{c}_b' = 2f_b + 50, \]
\[ \hat{c}_c' = 2f_c + 50, \quad \hat{c}_d' = 20f_d. \]

It is easy to see that, in this case, the S-O solution is identical to the U-O solution with \( x_{p_1} = x_{p_2} = 3 \) and \( \hat{C}_{p_1}' = \hat{C}_{p_2}' = 116 \). Furthermore, after the addition of link e, we have that \( \hat{c}_e' = 2f_e + 10 \). The new path \( p_3 \) is not used in the S-O solution, since with zero flow on path \( p_3 \), we have that \( \hat{C}_{p_3}' = 170 \) and \( \hat{C}_{p_1}' = \hat{C}_{p_2}' \) remains at 116.
If the symmetry assumption does not hold for the user link costs functions, then the transportation network equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a variational inequality problem!
VI Formulation of TNE
Dafermos (1980), Smith (1979)

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequality problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset \mathbb{R}^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in $\mathbb{R}^n$ and $K$ is closed and convex.
A Geometric Interpretation of a Variational Inequality and a Projected Dynamical System

Dupuis and Nagurney (1993)
Nagurney and Zhang (1996)
The variational inequality problem, contains, as special cases, such classical problems as:

- systems of equations
- optimization problems
- complementarity problems

and is also closely related to fixed point problems.

Hence, it is a unifying mathematical formulation for a variety of mathematical programming problems.
Transportation and Complex Network Systems
The TNE Paradigm is the Unifying Paradigm for Complex Network Problems:

• Transportation Networks
• Supply Chain Networks
• Electric Power Supply Chains
• Financial Networks
• The Internet
The Equivalence of Supply Chains and Transportation Networks

Supply Chain - Transportation Supernetwork Representation

--- Two-way information exchanges between specific decision-makers

The fifth chapter of Beckmann, McGuire, and Winsten’s book, *Studies in the Economics of Transportation* (1956) describes some *unsolved problems* including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

Specifically, they asked whether electric power generation and distribution networks can be reformulated as transportation network equilibrium problems.
Electric Power Supply Chains
The Electric Power Supply Chain Network

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks

Electric Power Supply Chain Network with Fuel Suppliers

In 1952, Copeland wondered whether money flows like water or electricity.
The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation

We have shown that *money* as well as *electricity* flow like *transportation* and have answered questions posed fifty years ago by Copeland and by Beckmann, McGuire, and Winsten!
We are now using the connections between TNE and electric power supply chains for energy studies:

Empirical Case Study

- New England electric power market and fuel markets
- 82 generators who own and operate 573 power plants
- 5 types of fuels: natural gas, residual fuel oil, distillate fuel oil, jet fuel, and coal
- Hourly demand/price data of July 2006 (24 × 31 = 744 scenarios)
- 6 blocks (L1 = 94 hours, and Lw = 130 hours; w = 2, ..., 6)
1. Maine
2. New Hampshire
3. Vermont
4. Connecticut (excluding Southwestern Connecticut)
5. Southwestern Connecticut (excluding the Norwalk-Stamford area)
6. Norwalk-Stamford area
7. Rhode Island
8. Southeastern Massachusetts
9. Western and Central Massachusetts
10. Boston/Northeastern Massachusetts
The New England Electric Power Supply Chain Network with Fuel Suppliers
Predicted Prices vs. Actual Prices ($/Mwh)
Recent disasters have demonstrated the importance and the vulnerability of network systems.

Examples:

- 9/11 Terrorist Attacks, September 11, 2001;
- The biggest blackout in North America, August 14, 2003;
- Two significant power outages in September 2003 -- one in the UK and the other in Italy and Switzerland;
- Hurricane Katrina, August 23, 2005;
- The Minneapolis I-35 Bridge Collapse, August 1, 2007;
Disasters in Transportation Networks

www.salem-news.com

www.boston.com
Communication Network Disasters

www.tx.mb21.co.uk

www.w5jgv.com

www.wirelessestimator.com
Electric Power Network Disasters

www.cellar.org

media.collegepublisher.com

www.crh.noaa.gov
Recent Literature on Network Vulnerability

- Holme, Kim, Yoon and Han (2002)
- Taylor and D’este (2004)
- Chassin and Posse (2005)
- Barrat, Barthélemy and Vespignani (2005)
- Sheffi (2005)
- Dall’Asta, Barrat, Barthélemy and Vespignani (2006)
- Jenelius, Petersen and Mattson (2006)
- Taylor and D’Este (2007)
Our Research on Network Efficiency, Vulnerability, and Robustness


Robustness of Transportation Networks Subject to Degradable Links, Nagurney and Qiang, *Europhysics Letters* 80, December (2007).

A New Performance/Efficiency Measure with Applications to Complex Networks which Exploits the TNE Paradigm
The network performance/efficiency measure $\mathcal{E}(G,d)$, for a given network topology $G$ and fixed demand vector $d$, is defined as

$$
\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_w},
$$

where $n_w$ is the number of O/D pairs in the network and $\lambda_w$ is the equilibrium disutility for O/D pair $w$.

**Importance of a Network Component**

**Definition: Importance of a Network Component**

The importance, $I(g)$, of a network component $g \in G$ is measured by the relative network efficiency drop after $g$ is removed from the network:

$$I(g) = \frac{\Delta \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, d) - \mathcal{E}(G - g, d)}{\mathcal{E}(G, d)},$$

where $G - g$ is the resulting network after component $g$ is removed.
The Latora and Marchiori (L-M) Network Efficiency Measure

Definition: The L-M Measure

The network performance/efficiency measure, $E(G)$ for a given network topology, $G$, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where $n$ is the number of nodes in the network and $d_{ij}$ is the shortest path length between node $i$ and node $j$. 
The L-M Measure vs. the N-Q Measure

Theorem:

If positive demands exist for all pairs of nodes in the network, \( G \), and each of demands is equal to 1, and if \( d_{ij} \) is set equal to \( \lambda_w \), where \( w=(i,j) \), for all \( w \in W \), then the N-Q and L-M network efficiency measures are one and the same.
The Approach to Study the Importance of Network Components

The elimination of a link is treated in the N-Q network efficiency measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks.

The measure generalizes the Latora and Marchiori network measure for complex networks.
Example 1

Assume a network with two O/D pairs: $w_1=(1,2)$ and $w_2=(1,3)$ with demands: $d_{w_1}=100$ and $d_{w_2}=20$.

The paths are:
for $w_1$, $p_1=a$; for $w_2$, $p_2=b$.

The equilibrium path flows are:
$\mathbf{x}_{p_1}^* = 100$, $\mathbf{x}_{p_2}^* = 20$.

The equilibrium path travel costs are:
$C_{p_1} = C_{p_2} = 20$.
# Importance and Ranking of Links and Nodes

<table>
<thead>
<tr>
<th>Link</th>
<th>Importance Value from Our Measure</th>
<th>Importance Ranking from Our Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.8333</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>0.1667</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Node</th>
<th>Importance Value from Our Measure</th>
<th>Importance Ranking from Our Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.8333</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0.1667</td>
<td>3</td>
</tr>
</tbody>
</table>
Example 2

The network is given by:

\[
\begin{align*}
  w_1 &= (1, 20) \\
  w_2 &= (1, 19) \\
  d_{w_1} &= 100 \\
  d_{w_2} &= 100
\end{align*}
\]

## Example 2: Link Cost Functions

<table>
<thead>
<tr>
<th>Link $a$</th>
<th>Link Cost Function $c_a(f_a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00005$f_1^4 + 5f_1 + 500$</td>
</tr>
<tr>
<td>2</td>
<td>0.00003$f_2^4 + 4f_2 + 200$</td>
</tr>
<tr>
<td>3</td>
<td>0.00005$f_3^4 + 3f_3 + 350$</td>
</tr>
<tr>
<td>4</td>
<td>0.00003$f_4^4 + 6f_4 + 400$</td>
</tr>
<tr>
<td>5</td>
<td>0.00006$f_5^4 + 6f_5 + 600$</td>
</tr>
<tr>
<td>6</td>
<td>7$f_6 + 500$</td>
</tr>
<tr>
<td>7</td>
<td>0.00008$f_7^4 + 8f_7 + 400$</td>
</tr>
<tr>
<td>8</td>
<td>0.00004$f_8^4 + 5f_8 + 650$</td>
</tr>
<tr>
<td>9</td>
<td>0.00001$f_9^4 + 6f_9 + 700$</td>
</tr>
<tr>
<td>10</td>
<td>4$f_{10} + 800$</td>
</tr>
<tr>
<td>11</td>
<td>0.00007$f_{11}^4 + 7f_{11} + 650$</td>
</tr>
<tr>
<td>12</td>
<td>8$f_{12} + 700$</td>
</tr>
<tr>
<td>13</td>
<td>0.00001$f_{13}^4 + 7f_{13} + 600$</td>
</tr>
<tr>
<td>14</td>
<td>8$f_{14} + 500$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Link $a$</th>
<th>Link Cost Function $c_a(f_a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.00003$f_{15}^4 + 9f_{15} + 200$</td>
</tr>
<tr>
<td>16</td>
<td>$8f_{16} + 300$</td>
</tr>
<tr>
<td>17</td>
<td>0.00003$f_{17}^4 + 7f_{17} + 450$</td>
</tr>
<tr>
<td>18</td>
<td>$5f_{18} + 300$</td>
</tr>
<tr>
<td>19</td>
<td>$8f_{19} + 600$</td>
</tr>
<tr>
<td>20</td>
<td>0.00003$f_{20}^4 + 6f_{20} + 300$</td>
</tr>
<tr>
<td>21</td>
<td>0.00004$f_{21}^4 + 4f_{21} + 400$</td>
</tr>
<tr>
<td>22</td>
<td>0.00002$f_{22}^4 + 6f_{22} + 500$</td>
</tr>
<tr>
<td>23</td>
<td>0.00003$f_{23}^4 + 9f_{23} + 350$</td>
</tr>
<tr>
<td>24</td>
<td>0.00002$f_{24}^4 + 8f_{24} + 400$</td>
</tr>
<tr>
<td>25</td>
<td>0.00003$f_{25}^4 + 9f_{25} + 450$</td>
</tr>
<tr>
<td>26</td>
<td>0.00006$f_{26}^4 + 7f_{26} + 300$</td>
</tr>
<tr>
<td>27</td>
<td>0.00003$f_{27}^4 + 8f_{27} + 500$</td>
</tr>
<tr>
<td>28</td>
<td>0.00003$f_{28}^4 + 7f_{28} + 650$</td>
</tr>
</tbody>
</table>
### Example 2: Importance and Ranking of Links

<table>
<thead>
<tr>
<th>Link $a$</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9086</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.8984</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.8791</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>0.8672</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>0.8430</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>0.8226</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>0.7750</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>0.5483</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>0.0362</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>0.6641</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>12</td>
<td>0.0006</td>
<td>20</td>
</tr>
<tr>
<td>13</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>14</td>
<td>0.0000</td>
<td>22</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Link $a$</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>16</td>
<td>0.0001</td>
<td>21</td>
</tr>
<tr>
<td>17</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>18</td>
<td>0.0175</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>0.0362</td>
<td>17</td>
</tr>
<tr>
<td>20</td>
<td>0.6641</td>
<td>14</td>
</tr>
<tr>
<td>21</td>
<td>0.7537</td>
<td>13</td>
</tr>
<tr>
<td>22</td>
<td>0.8333</td>
<td>10</td>
</tr>
<tr>
<td>23</td>
<td>0.8598</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>0.8939</td>
<td>5</td>
</tr>
<tr>
<td>25</td>
<td>0.4162</td>
<td>16</td>
</tr>
<tr>
<td>26</td>
<td>0.9203</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>0.9213</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>0.0155</td>
<td>19</td>
</tr>
</tbody>
</table>
Example 2: Link Importance Rankings
The network data are from LeBlanc, Morlok, and Pierskalla (1975).

The network has 528 O/D pairs, 24 nodes, and 76 links.

The user link cost functions are of Bureau of Public Roads (BPR) form.
Example 3 - Sioux Falls Network
Link Importance Rankings
Example 4: An Electric Power Supply Chain Network

Nagurney and Liu (2006) and Nagurney, Liu, Cojocaru, and Daniele (2007) have shown that an electric power supply chain network can be transformed into an equivalent transportation network problem.
Supernetwork Transformation

Figure 3: Electric Power Supply Chain Network and the Corresponding Supernetwork

Five Demand Ranges

- Demand Range I: \( d_w \in [0, 1] \)
- Demand Range II: \( d_w \in (1, \frac{4}{3}] \)
- Demand Range III: \( d_w \in (\frac{4}{3}, \frac{7}{3}] \)
- Demand Range IV: \( d_w \in (\frac{7}{3}, \frac{11}{3}] \)
- Demand Range V: \( d_w \in (\frac{11}{3}, \infty) \)
Importance Ranking of Links in the Electric Power Supply Chain Network
Importance Ranking of Nodes in the Electric Power Supply Chain Network

- Power Generator 1
- Power Supplier 1
- Power Supplier 2
- Power Supplier 3
- Demand Market 1
The Advantages of the N-Q Network Efficiency Measure

- The measure captures demands, flows, costs, and behavior of users, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
- It can be applied to assess the efficiency/performance of a wide range of network systems.
- It is applicable also to elastic demand networks (Qiang and Nagurney, *Optimization Letters* (2008)).
- It has been extended to dynamic networks (Nagurney and Qiang, *Netnomics*, in press).
What About Dynamic Networks?
We are using evolutionary variational inequalities to model dynamic networks with:

- **dynamic (time-dependent)** supplies and demands
- **dynamic (time-dependent)** capacities
- **structural changes** in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).
2005-2006 Radcliffe Institute for Advanced Study Fellowship Year at Harvard Collaboration with Professor David Parkes of Harvard University and Professor Patrizia Daniele of the University of Catania
A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks. (page 10 of Roughgarden’s (2005) book, Selfish Routing and the Price of Anarchy).

A Dynamic Model of the Internet

We now define the feasible set \( \mathcal{K} \). We consider the Hilbert space \( \mathcal{L} = L^2([0, T], R^{Kn_p}) \) (where \([0, T]\) denotes the time interval under consideration) given by

\[
\mathcal{K} = \left\{ x \in L^2([0, T], R^{Kn_p}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T]; \sum_{p \in P_w} x^k_p(t) = d^k_w(t), \forall w, \forall k \text{ a.e. in } [0, T] \right\}.
\]

We assume that the capacities \( \mu^k_r(t) \), for all \( r \) and \( k \), are in \( \mathcal{L} \), and that the demands, \( d^k_w \geq 0 \), for all \( w \) and \( k \), are also in \( \mathcal{L} \). Further, we assume that

\[
0 \leq d(t) \leq \Phi \mu(t), \text{ a.e. on } [0, T],
\]

where \( \Phi \) is the \( Kn_W \times Kn_P \)-dimensional O/D pair-route incidence matrix, with element \((kw, kr)\) equal to 1 if route \( r \) is contained in \( P_w \), and 0, otherwise. The feasible set \( \mathcal{K} \) is nonempty. It is easily seen that \( \mathcal{K} \) is also convex, closed, and bounded.

The dual space of \( \mathcal{L} \) will be denoted by \( \mathcal{L}^* \). On \( \mathcal{L} \times \mathcal{L}^* \) we define the canonical bilinear form by

\[
\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.
\]
Furthermore, the cost mapping $C : \mathcal{K} \to \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

The conditions below are a generalization of the Wardrop’s (1952) first principle of traffic behavior.

**Definition: Dynamic Multiclass Network Equilibrium**

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop’s first principle) if, for every O/D pair $w \in W$, every route $r \in P_w$, every class $k; k = 1, \ldots, K$, and a.e. on $[0,T]$:

$$C^k_r(x^*(t)) - \lambda^k_w(t) \begin{cases} 
\leq 0, & \text{if } x^k_r(t) = \mu^k_r(t), \\
= 0, & \text{if } 0 < x^k_r(t) < \mu^k_r(t), \\
\geq 0, & \text{if } x^k_r(t) = 0. 
\end{cases}$$
The standard form of the EVI that we work with is:

determine $x^* \in \mathcal{K}$ such that $\langle \langle F(x^*), x - x^* \rangle \rangle \geq 0$, $\forall x \in \mathcal{K}$.

Theorem (Nagurney, Parkes, Daniele (2007))

$x^* \in \mathcal{K}$ is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$
The Time-Dependent (Demand-Varying) Braess Paradox and Evolutionary Variational Inequalities
Recall the Braess Network where we add the link e.
The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)
In Demand Regime I, only the new path is used.
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off!
In Demand Regime III, only the original paths are used.

Network 1 is the Original Braess Network - Network 2 has the added link.
The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!
Extension of the Network Efficiency Measure to Dynamic Networks

An Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities with Applications to the Internet and Vulnerability Analysis, Nagurney and Qiang, Netnomics, in press.
The network efficiency for the network $G$ with time-varying demand $d$ for $t \in [0, T]$, denoted by $E(G, d, T)$, is defined as follows:

$$E(G, d, T) = \frac{\int_{0}^{T} \left[ \sum_{w} d_{w}(t) \lambda_{w}(t) \right] / n_{W} \, dt}{T}.$$ 

The above measure is the average network performance over time of the dynamic network.
Network Efficiency Measure for Dynamic Networks - Discrete Time

Let $d^1_w, d^2_w, \ldots, d^H_w$ denote demands for O/D pair $w$ in $H$ discrete time intervals, given, respectively, by:

$[t_0, t_1], (t_1, t_2], \ldots, (t_{H-1}, t_H]$, where $t_H \equiv T$. We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair $w$ at the $H$ different time intervals by: $\lambda^1_w, \lambda^2_w, \ldots, \lambda^H_w$. The demand vector $d$, in this special discrete case, is a vector in $\mathbb{R}^{n_W \times H}$. The dynamic network efficiency measure in this case is as follows:

**Dynamic Network Efficiency: Discrete Time Version**

*The network efficiency for the network $(G, d)$ over $H$ discrete time intervals:

$[t_0, t_1], (t_1, t_2], \ldots, (t_{H-1}, t_H]$, where $t_H \equiv T$, and with the respective constant demands:

$d^1_w, d^2_w, \ldots, d^H_w$ for all $w \in W$ is defined as follows:

$$
\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d^i_w}{\lambda^i_w})(t_i - t_{i-1})/n_w]}{t_H}.
$$

*
Importance of a Network Component

The importance of a network component \( g \) of network \( G \) with demand \( d \) over time horizon \( T \) is defined as follows:

\[
I(g, d, T) = \frac{\mathcal{E}(G, d, T) - \mathcal{E}(G - g, d, T)}{\mathcal{E}(G, d, T)}
\]

where \( \mathcal{E}(G-g,d,T) \) is the dynamic network efficiency after component \( g \) is removed.
**Importance of Nodes and Links in the Dynamic Braess Network Using the N-Q Measure when T=10**

<table>
<thead>
<tr>
<th>Link</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$c$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$d$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$e$</td>
<td>-0.1341</td>
<td>3</td>
</tr>
</tbody>
</table>

Link $e$ is never used after $t = 8.89$ and in the range $t \in [2.58, 8.89]$, it increases the cost, so the fact that link $e$ has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!

<table>
<thead>
<tr>
<th>Node</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.2604</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0.2604</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1.0000</td>
<td>1</td>
</tr>
</tbody>
</table>
The other side of the network vulnerability is that of possible synergy.

Let us consider now Mergers & Acquisitions in a network/supply chain formalism.
According to Kusstatscher and Cooper (2005) there were five major waves of Merger & Acquisition (M & A) activity:

The First Wave: 1898-1902: an increase in horizontal mergers that resulted in many US industrial groups;

The Second Wave: 1926-1939: mainly public utilities;

The Third Wave: 1969-1973: *diversification* was the driving force;

The Fourth Wave: 1983-1986: the goal was efficiency;

The Fifth Wave: 1997 until the early years of the 21st century: *globalization* was the motto.

In 1998, M&As reached $2.1 trillion worldwide; in 1999, the activity exceeded $3.3 trillion, and in 2000, almost $3.5 was reached.
A survey of 600 executives involved in their companies’ mergers and acquisitions (M&A) conducted by Accenture and the Economist Unit (see Byrne (2007)) found that less than half (45%) achieved expected cost-saving synergies.

Langabeer and Seifert (2003) determined a direct correlation between how effectively supply chains of merged firms are integrated and how successful the merger is. They concluded, based on the empirical findings of Langabeer (2003), who analyzed hundreds of mergers over the preceding decade, that

**Improving Supply Chain Integration between Merging Companies is the Key to Improving the Likelihood of Post-Merger Success!**
Recently, we introduced a system-optimization perspective for supply chains in which firms are engaged in multiple activities of production, storage, and distribution to the demand markets and proposed a cost synergy measure associated with evaluating proposed mergers:


In that paper, the merger of two firms was modeled and the demands for the product at the markets, which were distinct for each firm prior to the merger, were assumed to be fixed.
Figure: Case 0: Firms A and B Prior to Horizontal Merger (Nagurney (2008a))
Figure: Case 1: Firms A and B Merge (Nagurney (2008a))
Figure: Case 2: Firms A and B Merge (Nagurney (2008a))
Figure: Case 3: Firms A and B Merge (Nagurney (2008a))
The measure that we utilized in Nagurney (2008a) to capture the gains, if any, associated with a horizontal merger Case $i; i = 1, 2, 3$ is as follows:

$$S^i = \left[ \frac{TC^0 - TC^i}{TC^0} \right] \times 100\%,$$

where $TC^i$ is the total cost associated with the value of the objective function $\sum_{a \in L^i} \hat{c}_a(f_a)$ for $i = 0, 1, 2, 3$ evaluated at the optimal solution for Case $i$. Note that $S^i; i = 1, 2, 3$ may also be interpreted as synergy.
The Supply Chain Network Oligopoly Model (Nagurney (2008b))

Figure: Supply Chain Network Structure of the Oligopoly
It is interesting to relate this supply chain network oligopoly model to the spatial oligopoly model proposed by Dafermos and Nagurney (1987), which is done in the following corollary.

**Corollary 1: Relationship to the Spatial Oligopoly Model**

Assume that there are $I$ firms in the supply chain network oligopoly model and that each firm has a single manufacturing plant and a single distribution center. Assume also that the distribution costs from each manufacturing plant to the distribution center and the storage costs are all equal to zero. Then the resulting model is isomorphic to the spatial oligopoly model of Dafermos and Nagurney (1987) whose underlying network structure is given in Figure 6.

Figure: Network Structure of the Spatial Oligopoly
The relationship between the supply chain network oligopoly model to the classical Cournot (1838) oligopoly model is now given (see also Gabay and Moulin (1982) and Nagurney (1993)).

**Corollary 2: Relationship to Classical Oligopoly Model**

Assume that there is a single manufacturing plant associated with each firm in the above model, and a single distribution center. Assume also that there is a single demand market. Assume also that the manufacturing cost of each manufacturing firm depends only upon its own output. Then, if the storage and distribution cost functions are all identically equal to zero the above model collapses to the classical oligopoly model in quantity variables. Furthermore, if \( I = 2 \), one then obtains the classical duopoly model.
Figure: Network Structure of the Classical Oligopoly
Mergers Through Coalition Formation

Figure: Mergers of the First $n_1'$ Firms and the Next $n_2'$ Firms
This framework can also be applied to teeming of humanitarian organizations in the case of humanitarian logistics operations.

_Humanitarian Logistics: Networks for Africa_

Rockefeller Foundation Bellagio Center Conference, Bellagio, Lake Como, Italy

May 5-9, 2008

Conference Organizer: Anna Nagurney, John F. Smith Memorial Professor
University of Massachusetts at Amherst

http://hlogistics.som.umass.edu
The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an Interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.
Thank you!

For more information, see http://supernet.som.umass.edu

Thanks also to the National Science Foundation, the AT&T Foundation, the John F. Smith Memorial Fund, the Rockefeller Foundation, and the Radcliffe Institute for Advanced Study for funding support.