Transportation Science and the Dynamics of Critical Infrastructure Networks with Applications to Performance Measurement and Vulnerability Analysis

Anna Nagurney

John F. Smith Memorial Professor

Isenberg School of Management

University of Massachusetts - Amherst

The Warren Lecture

Department of Civil Engineering - University of Minnesota

October 5, 2007

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks

Funding for this research has been provided by:

National Science Foundation

AT&T Foundation

John F. Smith Memorial Fund - University of Massachusetts at Amherst

THE ROCKEFELLER FOUNDATION

RADCLIFFE INSTITUTE FOR ADVANCED STUDY

HARVARD UNIVERSITY

Outline of Presentation:

- Background
- The Transportation Network Equilibrium Problem and Methodological Tools
- The Braess Paradox
- Some Interesting Applications of Variational Inequalities
- The Time-Dependent (Demand-Varying) Braess Paradox and Evolutionary Variational Inequalities
- A New Network Performance/Efficiency Measure with Applications to Critical Infrastructure Networks
- Double-Layered Dynamics
- A New Approach to Transportation Network Robustness

We are in a New Era of Decision-Making Characterized by:

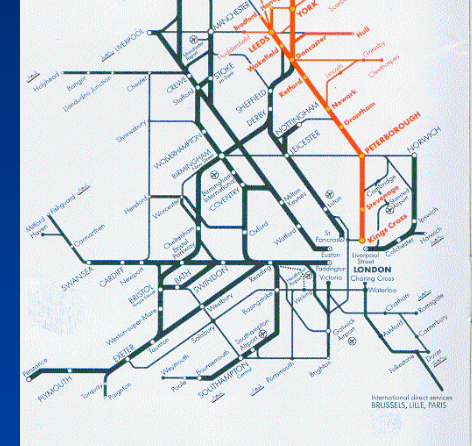
- complex interactions among decision-makers in organizations;
- alternative and at times conflicting criteria used in decision-making;
- constraints on resources: natural, human, financial, time, etc.;
- global reach of many decisions;
- high impact of many decisions;
- increasing risk and uncertainty, and
- the *importance of dynamics* and realizing a fast and sound response to evolving events.

Network problems are their own class of problems and they come in various forms and formulations, i.e., as optimization (linear or nonlinear) problems or as equilibrium problems and even dynamic network problems.

Critical infrastructure network problems, with an emphasis on Transportation, will be the focus of this talk.

Transportation, Communication, and Energy Networks

Bus Network

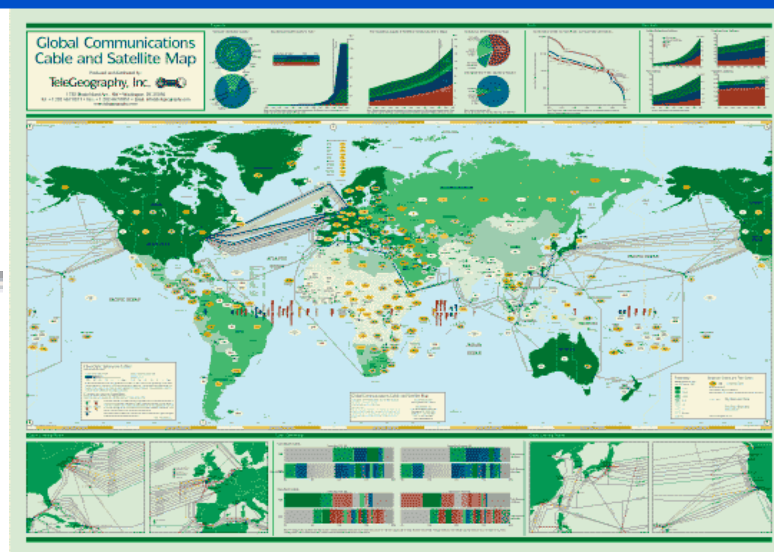


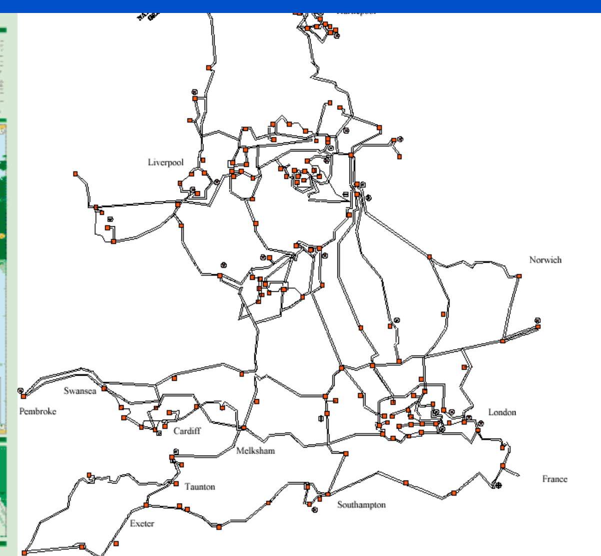
Rail Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks

British Electricity Grid

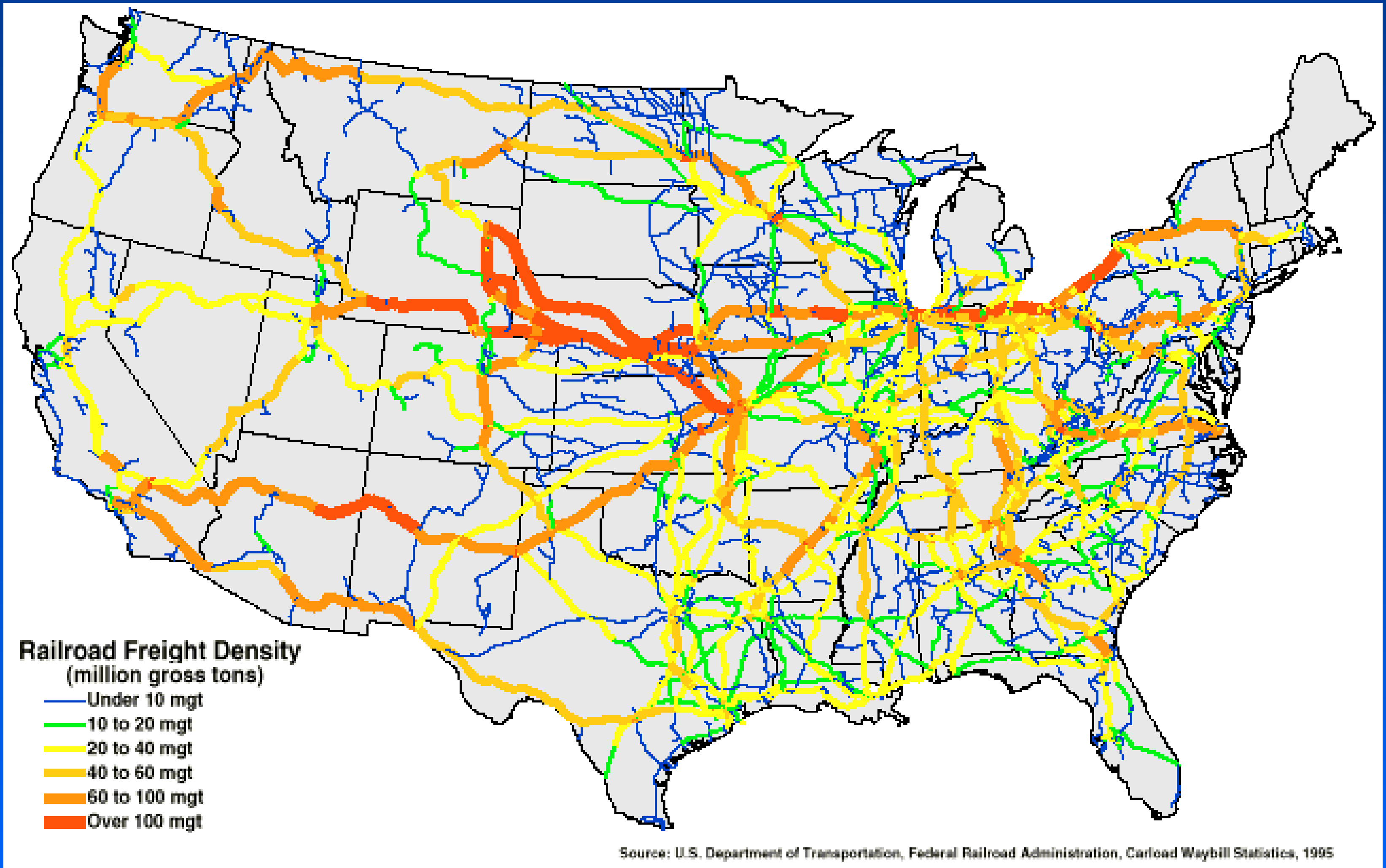




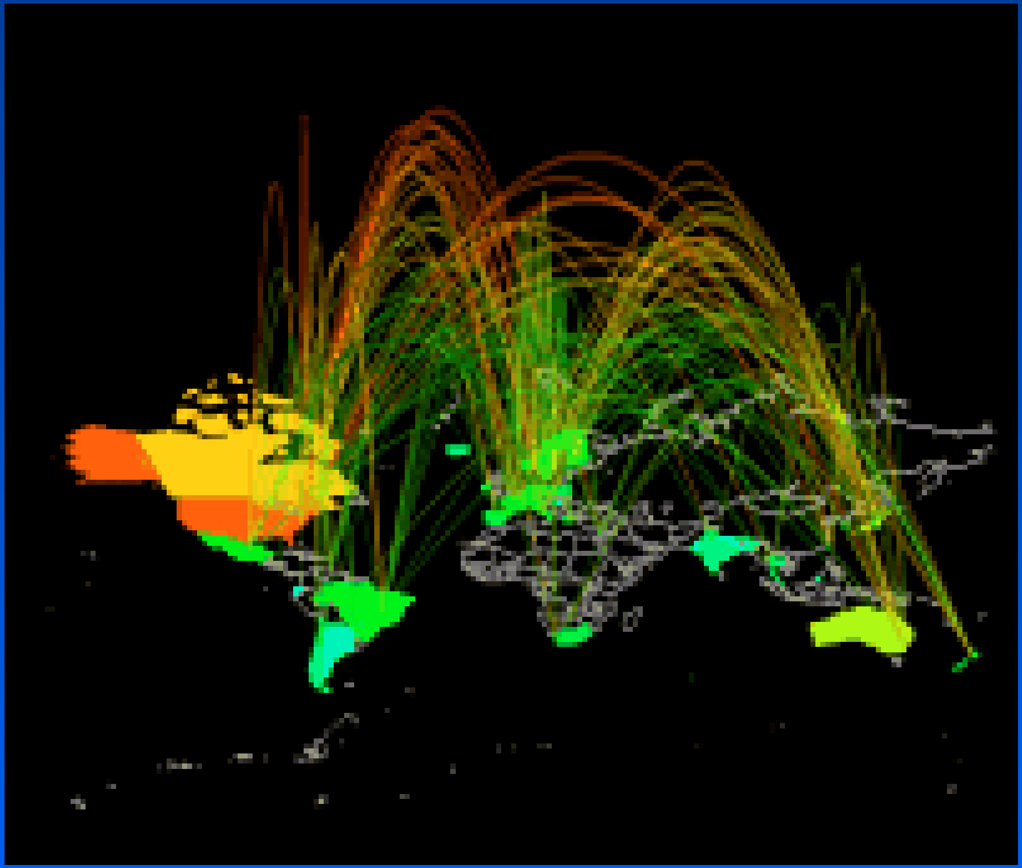
Components of Common Physical Networks

Network System	Nodes	Links	Flows
Transportation	Intersections, Homes, Workplaces, Airports, Railyards	Roads, Airline Routes, Railroad Track	Automobiles, Trains, and Planes,
Manufacturing and logistics	Workstations, Distribution Points	Processing, Shipment	Components, Finished Goods
Communication	Computers, Satellites, Telephone Exchanges	Fiber Optic Cables Radio Links	Voice, Data, Video
Energy	Pumping Stations, Plants	Pipelines, Transmission Lines	Water, Gas, Oil, Electricity

US Railroad Freight Flows

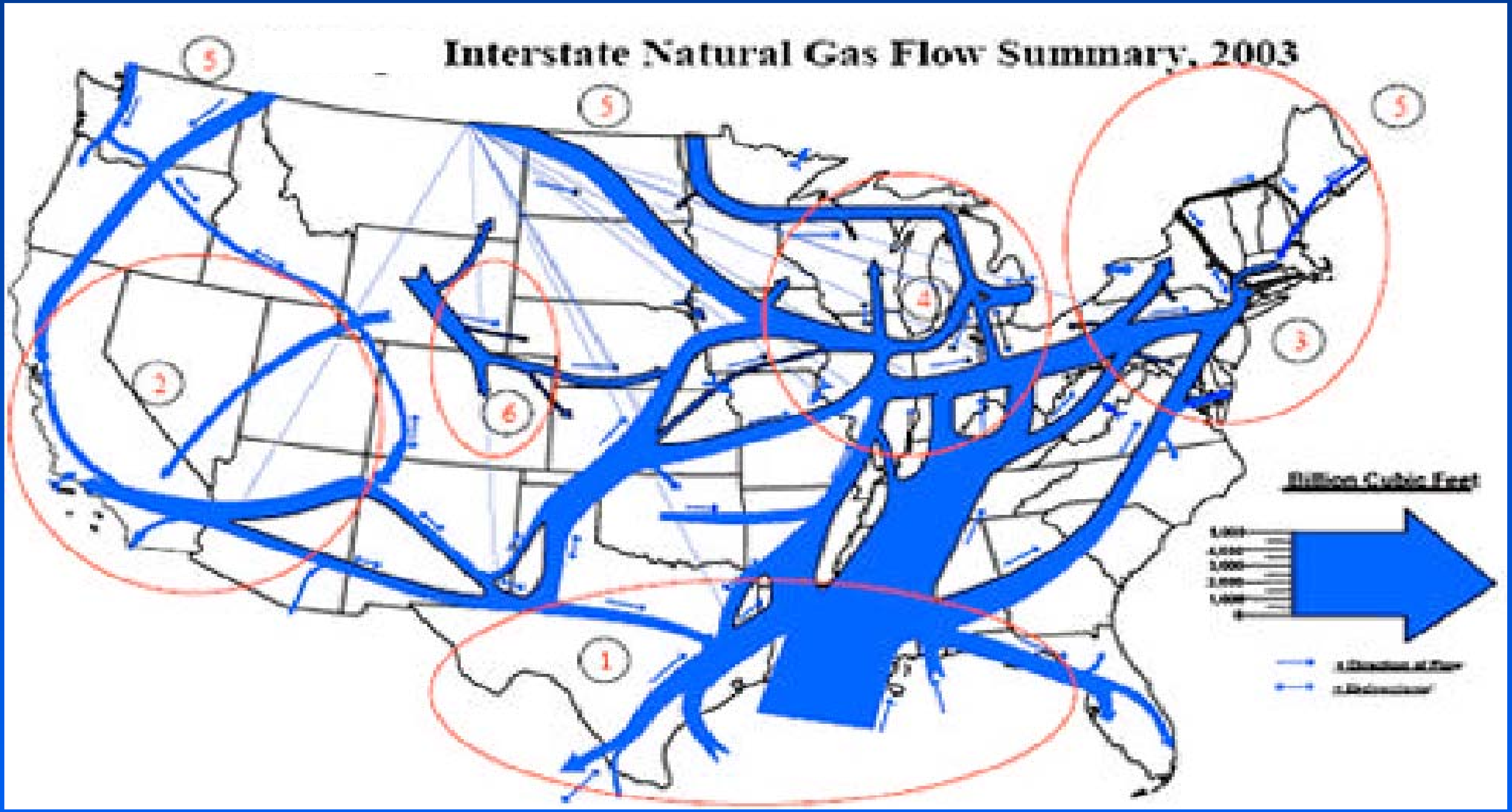


Internet Traffic Flows Over One 2 Hour Period

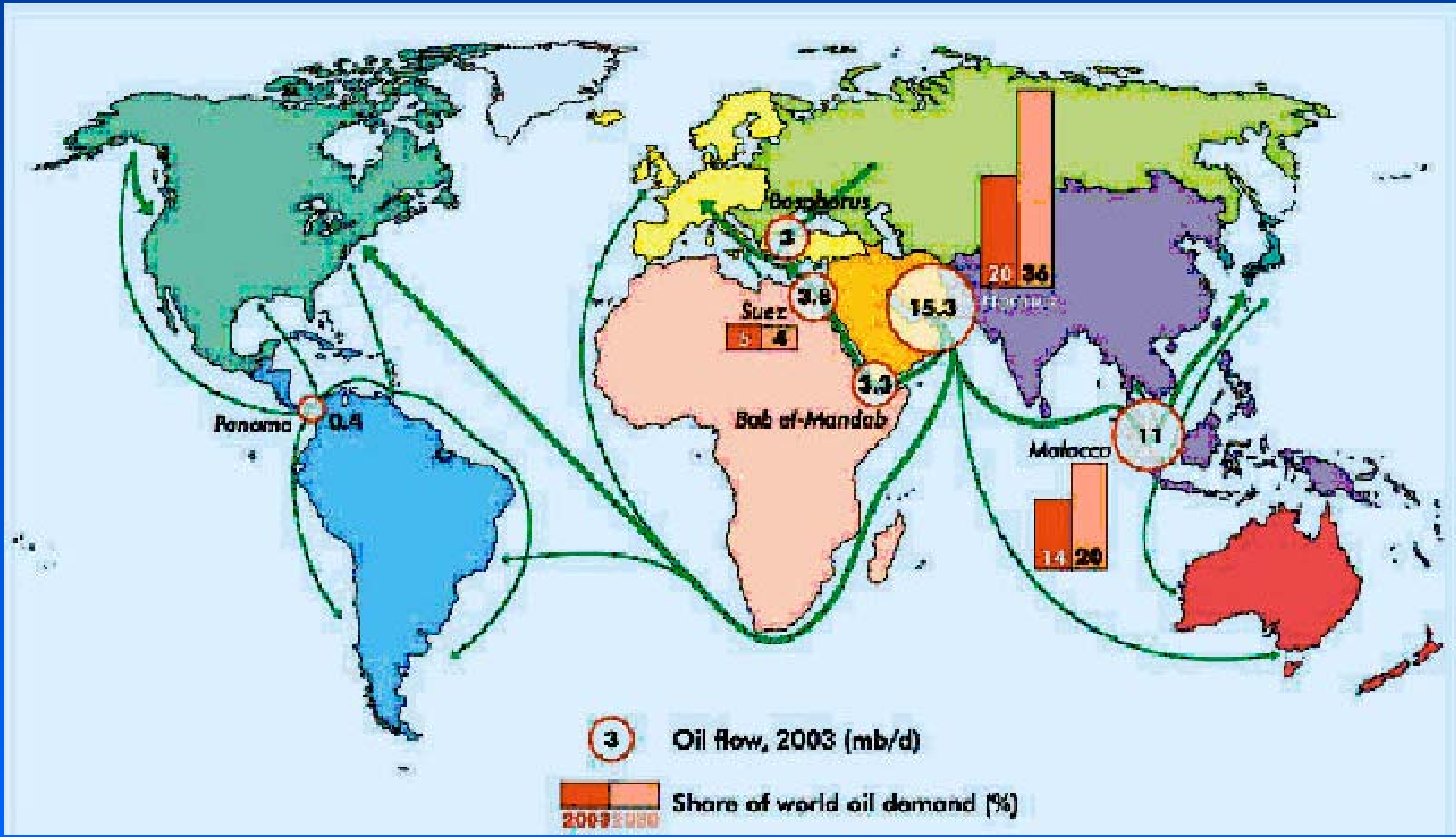


from Stephen Eick, Visual Insights

Natural Gas Pipeline Network in the US



World Oil Trading Network



The study of the efficient operation on transportation networks dates to *ancient Rome* with a classical example being the publicly provided Roman road network and the *time of day chariot policy*, whereby chariots were banned from the ancient city of Rome at particular times of day.

Characteristics of Networks Today

- *large-scale nature* and complexity of network topology;
- *congestion*;
- the *interactions among networks* themselves such as in transportation versus telecommunications;
- *policies* surrounding networks today may have a *major impact* not only economically but also *socially, politically, and security-wise*.

- *alternative behaviors of the users of the network*

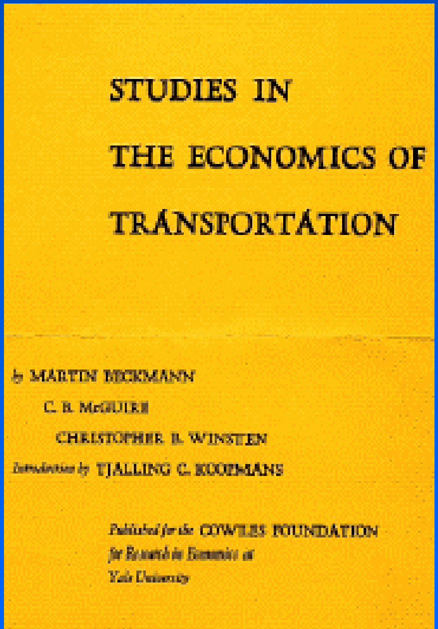
  – system-optimized versus

  – user-optimized (network equilibrium),

  which may lead to

  paradoxical phenomena.

Transportation science has historically been the discipline that has pushed the frontiers in terms of methodological developments for such problems (which are often large-scale) beginning with the book, *Studies in the Economics of Transportation*, by Beckmann, McGuire, and Winsten (1956).



Dafermos (1980) showed that the transportation network equilibrium (also referred to as user-optimization) conditions as formulated by Smith (1979) were a finite-dimensional variational inequality.

In 1993, Dupuis and Nagurney proved that the set of solutions to a variational inequality problem coincided with the set of solutions to a projected dynamical system (PDS) in R^n.

In 1996, Nagurney and Zhang published ***Projected Dynamical Systems and Variational Inequalities.***

Transportation Network Equilibrium Problem

Consider a general network $G = [N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. P_w denotes the set of paths, assumed to be acyclic, connecting the O/D pair of nodes w and P the set of all paths.

Let x_p represent the flow on path p and f_a the flow on link a. The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. This expression states that the load on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if we let d_w denote the demand associated with O/D pair w, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair w must be equal to the given demand d_w.

Let c_a denote the user cost associated with traversing link a, which is assumed to be continuous, and C_p the user cost associated with traversing the path p. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising the path.

Transportation Network Equilibrium

The network equilibrium conditions are then given by: For each path $p \in P_w$ and every O/D pair w:

$$C_p \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0 \end{cases}$$

where λ_w is an indicator, whose value is not known a priori. These equilibrium conditions state that the user costs on all used paths connecting a given O/D pair will be minimal and equalized.

As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that $[\frac{\partial c_b}{\partial c_a} = \frac{\partial c_a}{\partial c_b}]$ for all links a, b in the network then the solution to the above network equilibrium problem can be reformulated as the solution to an associated optimization problem. For example, if we have that $c_a = c_a(f_a)$, $\forall a \in L$, then the solution can be obtained by solving:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y)dy$$

subject to:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in L,$$

$$x_p \geq 0, \quad \forall p \in P.$$

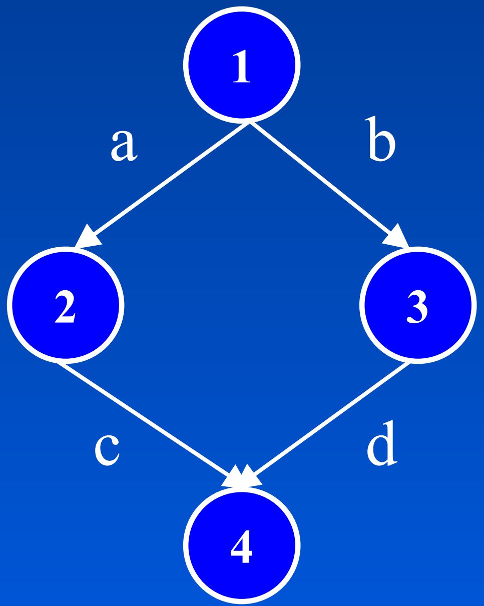
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of **6,** the equilibrium path flows are $\mathbf{x_{p_1}^* = x_{p_2}^* = 3}$ and

The equilibrium path travel cost is

$C_{p_1} = C_{p_2} = 83.$



$$c_a(f_a)=10\ f_a \quad c_b(f_b) = f_b+50$$

$$c_c(f_c) = f_c+50 \quad c_d(f_d) = 10\ f_d$$

Adding a Link Increases Travel Cost for All!

Adding a new link creates a new path **p_3=(a,e,d).**

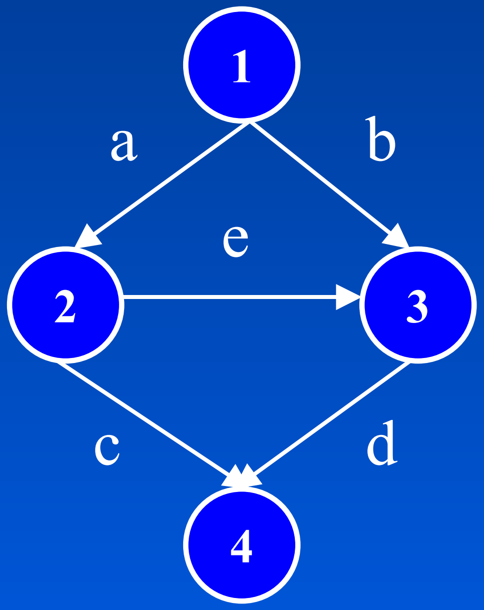
The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path **p_3, C_{p_3}=70.**

The new equilibrium flow pattern network is

$x_{p_1}{}^* = x_{p_2}{}^* = x_{p_3}{}^* = 2.$

The equilibrium path travel costs:

$C_{p_1} = C_{p_2} = C_{p_3} = 92.$



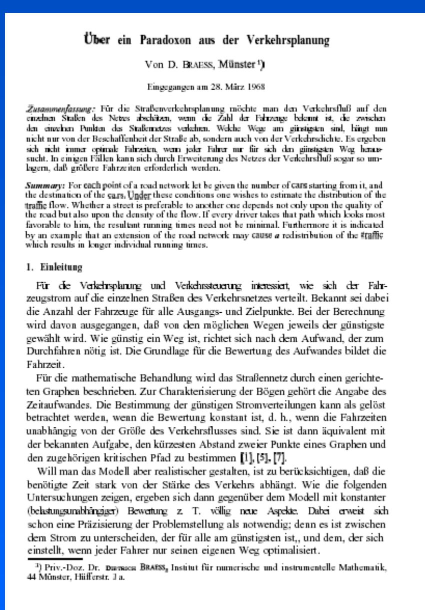
$c_e(f_e) = f_e + 10$

The 1968 Braess article has been translated from German to English and appears as

On a Paradox of Traffic Planning

by Braess, Nagurney, Wakolbinger

in the November 2005 issue of *Transportation Science.*



Über ein Paradoxon aus der Verkehrsplanung

Von D. Braess, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

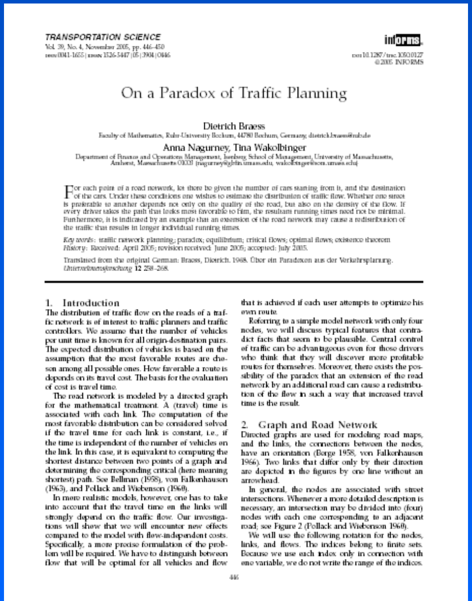
1. Einleitung

Für die Verkehrsplanung und Verkehrssteuerung interessiert, wie sich der Fahrzeugstrom auf die einzelnen Straßen des Verkehrsnetzes verteilt. Bekannt sei dabei die Anzahl der Fahrzeuge für alle Ausgangs- und Zielpunkte. Bei der Berechnung wird davon ausgegangen, daß von den möglichen Wegen jeweils der günstigste gewählt wird. Wie günstig ein Weg ist, richtet sich nach dem Aufwand, der zum Durchfahren nötig ist. Die Grundlage für die Bewertung des Aufwandes bildet die Fahrzeit.

Für die mathematische Behandlung wird das Straßennetz durch einen gerichteten Graphen beschrieben. Zur Charakterisierung der Bögen gehört die Angabe des Zeitaufwandes. Die Bestimmung der günstigen Stromverteilungen kann als gelöst betrachtet werden, wenn die Bewertung konstant ist, d. h., wenn die Fahrzeiten unabhängig von der Größe des Verkehrsflusses sind. Sie ist dann äquivalent mit der bekannten Aufgabe, den kürzesten Abstand zweier Punkte eines Graphen und den zugehörigen kritischen Pfad zu bestimmen [1], [5], [7].

Will man das Modell aber realistischer gestalten, ist zu berücksichtigen, daß die benötigte Zeit stark von der Stärke des Verkehrs abhängt. Wie die folgenden Untersuchungen zeigen, ergeben sich dann gegenüber dem Modell mit konstanter (belastungsunabhängiger) Bewertung z. T. völlig neue Aspekte. Dabei erweist sich schon eine Präzisierung der Problemstellung als notwendig; denn es ist zwischen dem Strom zu unterscheiden, der für alle am günstigsten ist,, und dem, der sich einstellt, wenn jeder Fahrer nur seinen eigenen Weg optimalisiert.

[1]) Priv.-Doz. Dr. Dietrich Braess, Institut für numerische und instrumentelle Mathematik, 44 Münster, Hüfferstr. 3 a.



TRANSPORTATION SCIENCE
Vol. 39, No. 4, November 2005, pp. 446–450
ISSN 0041-1655 | EISSN 1526-5447 | 05 | 3904 | 0446
DOI 10.1287/trsc.1050.0127

On a Paradox of Traffic Planning

Dietrich Braess
Faculty of Mathematics, Ruhr-University Bochum, 44780 Bochum, Germany, dietrich.braess@rub.de
Anna Nagurney, Tina Wakolbinger
Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts 01003 (nagurney@gbfin.umass.edu, wakolbinger@som.umass.edu)

For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Key words: traffic network planning; paradox; equilibrium; critical flows; optimal flows; existence theorem
History: Received: April 2005; revision received: June 2005; accepted: July 2005.

Translated from the original German: Braess, Dietrich. 1968. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* **12** 258–268.

1. Introduction

The distribution of traffic flow on the roads of a traffic network is of interest to traffic planners and traffic controllers. We assume that the number of vehicles per unit time is known for all origin-destination pairs. The expected distribution of vehicles is based on the assumption that the most favorable routes are chosen among all possible ones. How favorable a route is depends on its travel cost. The basis for the evaluation of cost is travel time.

The road network is modeled by a directed graph for the mathematical treatment. A (travel) time is associated with each link. The computation of the most favorable distribution can be considered solved if the travel time for each link is constant, i.e., if the time is independent of the number of vehicles on the link. In this case, it is equivalent to computing the shortest distance between two points of a graph and determining the corresponding critical (here meaning shortest) path. See Bellman (1958), von Falkenhausen (1963), and Pollack and Wiebenson (1960).

In more realistic models, however, one has to take into account that the travel time on the links will strongly depend on the traffic flow. Our investigations will show that we will encounter new effects compared to the model with flow-independent costs. Specifically, a more precise formulation of the problem will be required. We have to distinguish between flow that will be optimal for all vehicles and flow that is achieved if each user attempts to optimize his own route.

Referring to a simple model network with only four nodes, we will discuss typical features that contradict facts that seem to be plausible. Central control of traffic can be advantageous even for those drivers who think that they will discover more profitable routes for themselves. Moreover, there exists the possibility of the paradox that an extension of the road network by an additional road can cause a redistribution of the flow in such a way that increased travel time is the result.

2. Graph and Road Network

Directed graphs are used for modeling road maps, and the links, the connections between the nodes, have an orientation (Berge 1958, von Falkenhausen 1966). Two links that differ only by their direction are depicted in the figures by one line without an arrowhead.

In general, the nodes are associated with street intersections. Whenever a more detailed description is necessary, an intersection may be divided into (four) nodes with each one corresponding to an adjacent road; see Figure 2 (Pollack and Wiebenson 1960).

We will use the following notation for the nodes, links, and flows. The indices belong to finite sets. Because we use each index only in connection with one variable, we do not write the range of the indices.

If no such symmetry assumption holds for the user link costs functions, then the equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a *variational inequality problem!*

Smith (1979), Dafermos (1980)

VI Formulation of Transportation Network Equilibrium
(Dafermos (1980), Smith (1979))

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

The variational inequality problem, contains, as special cases, such classical problems as:

- systems of equations
- optimization problems
- complementarity problems

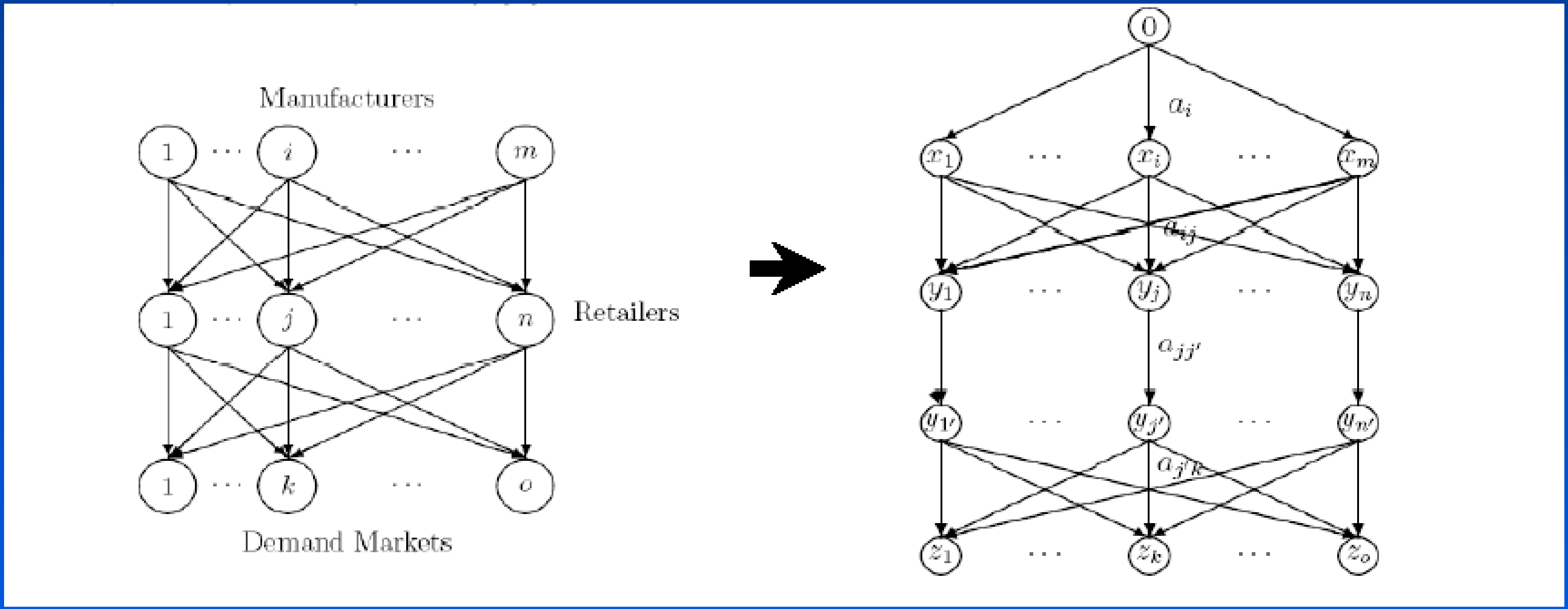
and is also closely related to fixed point problems.

Hence, it is a unifying mathematical formulation for a variety of mathematical programming problems.

The Transportation Network Equilibrium Paradigm is the unifying paradigm for Critical Infrastructure Problems:

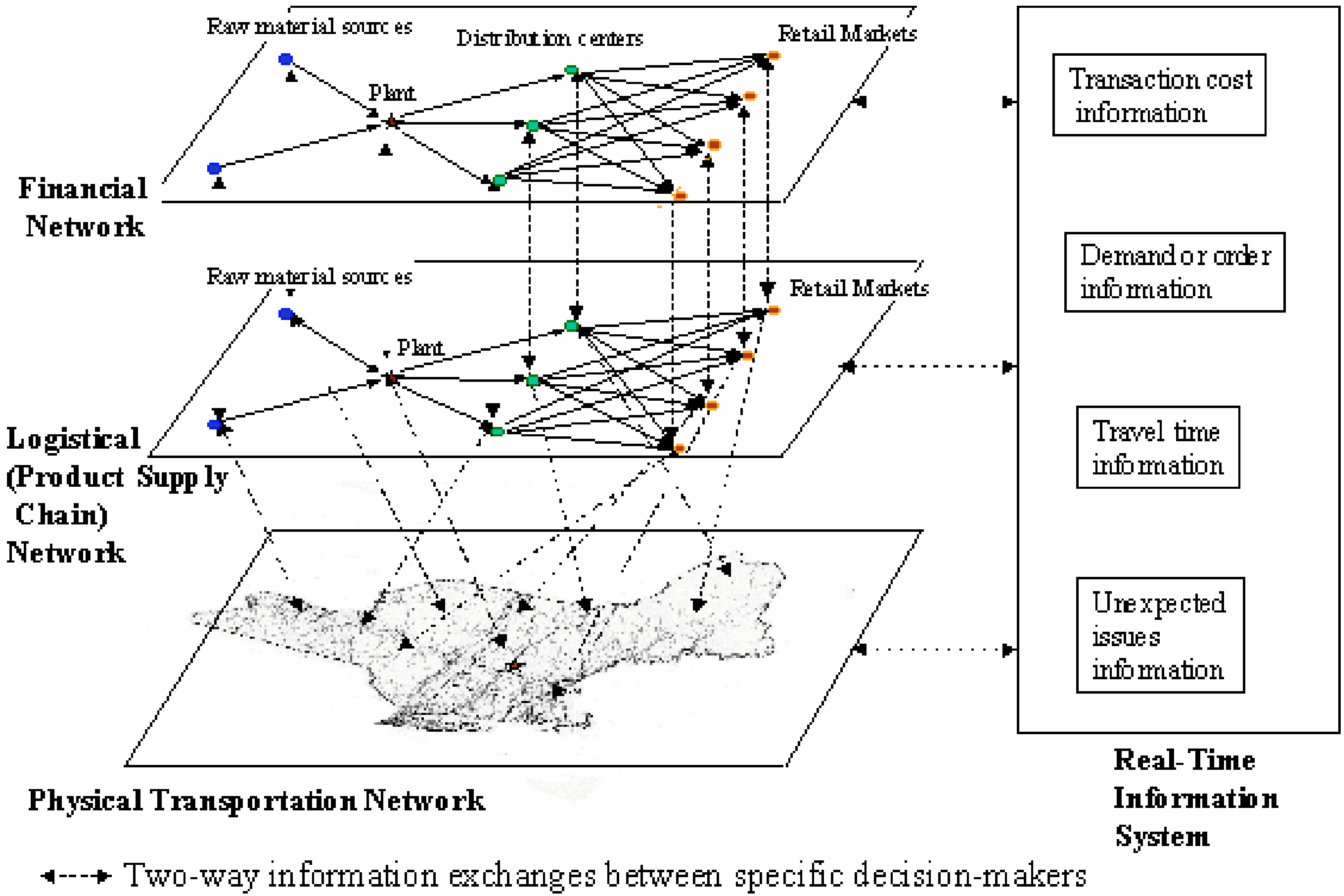
- Transportation Networks
- Internet
- Financial Networks
- Electric Power Supply Chains.

The Equivalence of Supply Chain Networks and Transportation Networks



Nagurney, *Transportation Research E* (2006).

Supply Chain -Transportation Supernetwork Representation

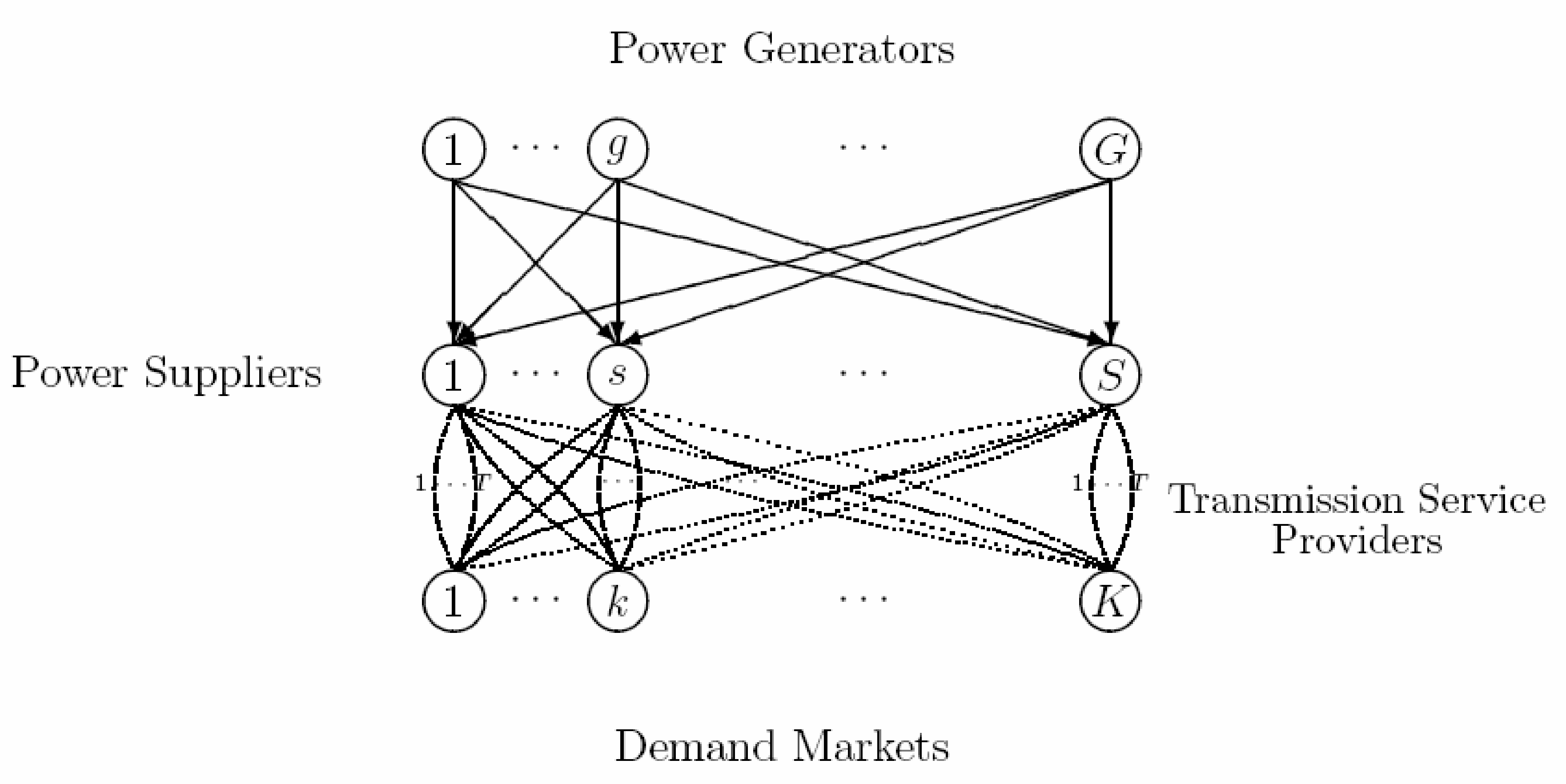


Nagurney, Ke, Cruz, Hancock, Southworth, *Environment and Planning B* (2002)

The fifth chapter of Beckmann, McGuire, and Winsten’s book, ***Studies in the Economics of Transportation*** (1956) describes some *unsolved problems* including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

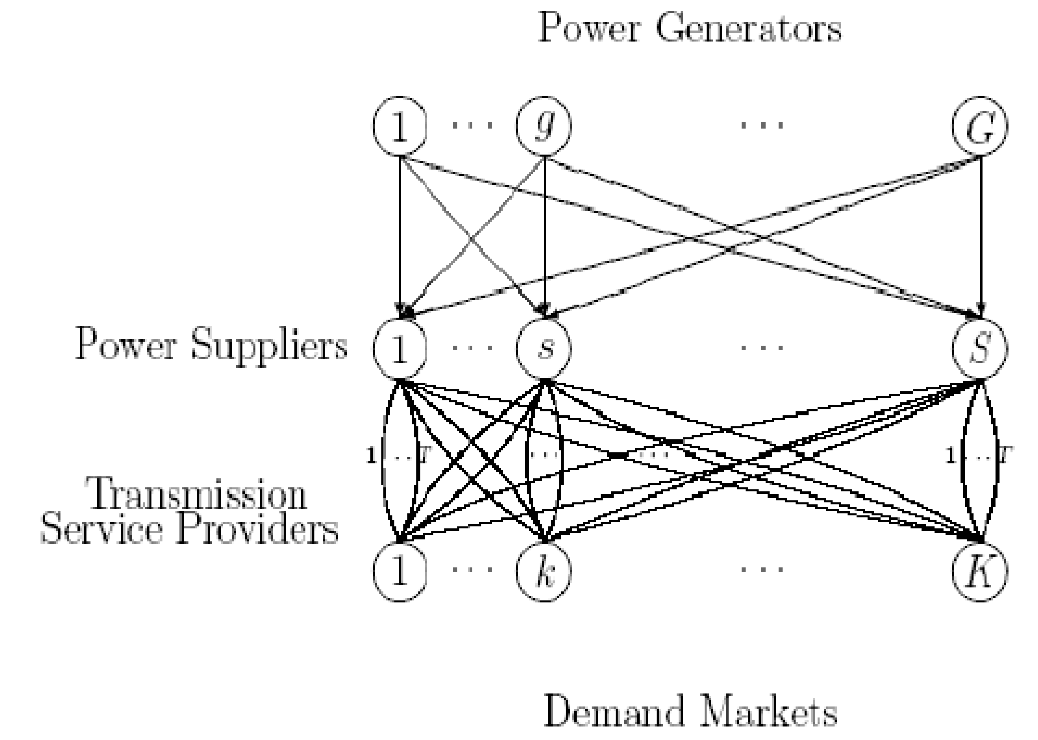
Specifically, they asked whether electric power generation and distribution networks can be reformulated as transportation network equilibrium problems.

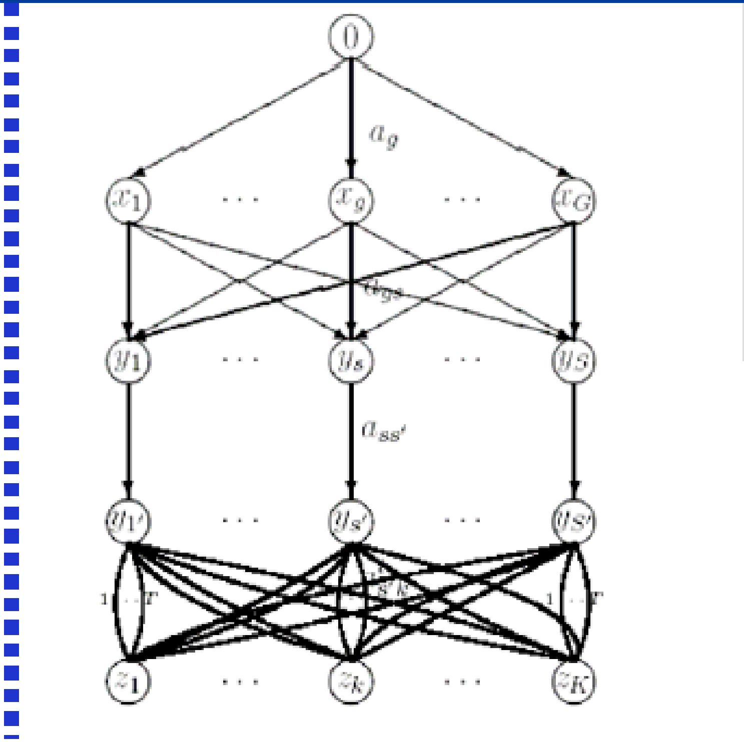
The Electric Power Supply Chain Network



Nagurney and Matsypura, *Proceedings of the CCCT* (2004).

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks





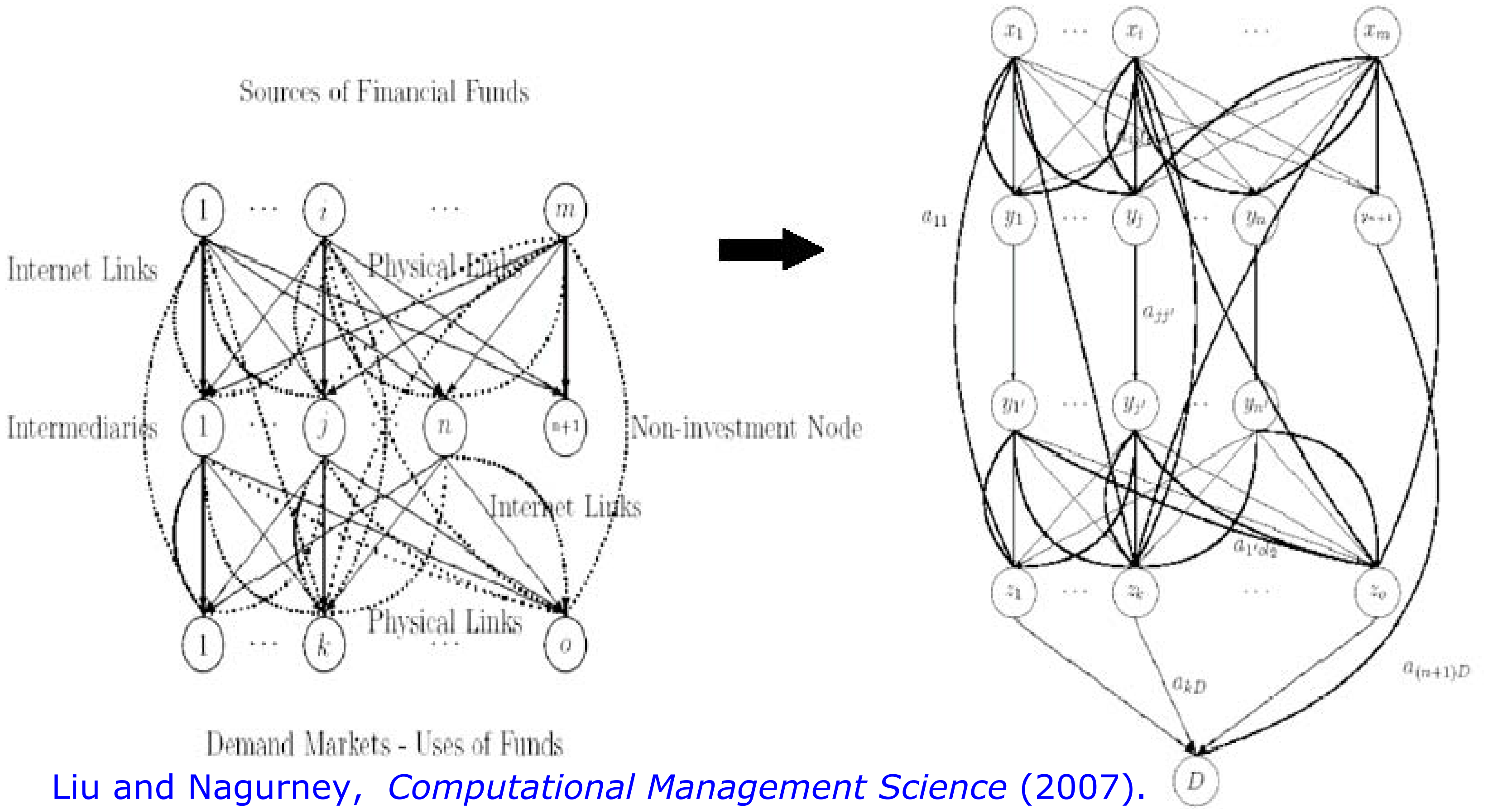
Electric Power Supply Network

Transportation Chain Network

Nagurney et al. T*ransportation Research E* (2007).

In 1952, Copeland wondered whether money flows like water or electricity.

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation



Liu and Nagurney, *Computational Management Science* (2007).

We have shown that *money* as well as *electricity* flow like *transportation* and have answered questions posed fifty years ago by Copeland and by Beckmann, McGuire, and Winsten!

The Tools that We are Using in Our Dynamic Network Research Include:

- network theory
- optimization theory
- game theory
- variational inequality theory
- evolutionary variational inequality theory
- projected dynamical systems theory
- double-layered dynamics theory
- network visualization tools.

PrDEs and PDSs

The most general mathematical context to date in which we can define a projected differential equation (PrDE) and, consequently, a projected dynamical system (PDS), is that of a Hilbert space *X* of arbitrary (finite or infinite) dimension.

Suppose that we have $K \subset X$, a nonempty, closed, convex subset in a Hilbert space X. Let $F : K \to X$ be a Lipschitz continuous mapping. It is well-known that the ODE:

$$\frac{\partial x(t)}{\partial t} = -F(x(t)), \quad x(0) \in K$$

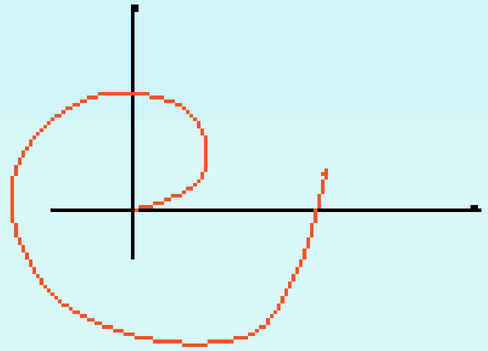
has solutions in a suitable class of functions; here that class will be that of absolutely continuous functions $AC([0, \infty), X)$.

Let us define a PrDE on an example, *with drawings*

Suppose $X := R^2$, $K := R^2_+$, and suppose that the image below represents a trajectory of the equation

$$\frac{\partial x(t)}{\partial t} = -F(x(t)),$$

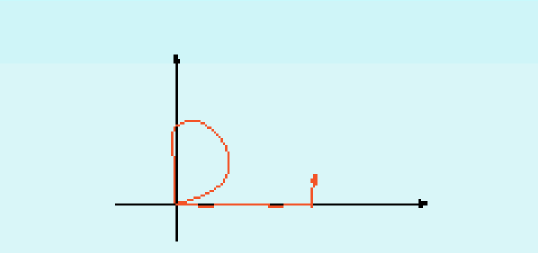
starting in R^2_+.



A PrDE describes the control problem:

$$\frac{\partial}{\partial t}(x(t)) = -F(x(t)), \quad x(0) \in R^2_+$$

such that $x(t) \in R^2_+$, as shown in the figure below:



In other words, a trajectory of a projected differential equation is always "trapped" in the constraint set $K = R^2_+$ and the velocity field along any such trajectory is not continuous.

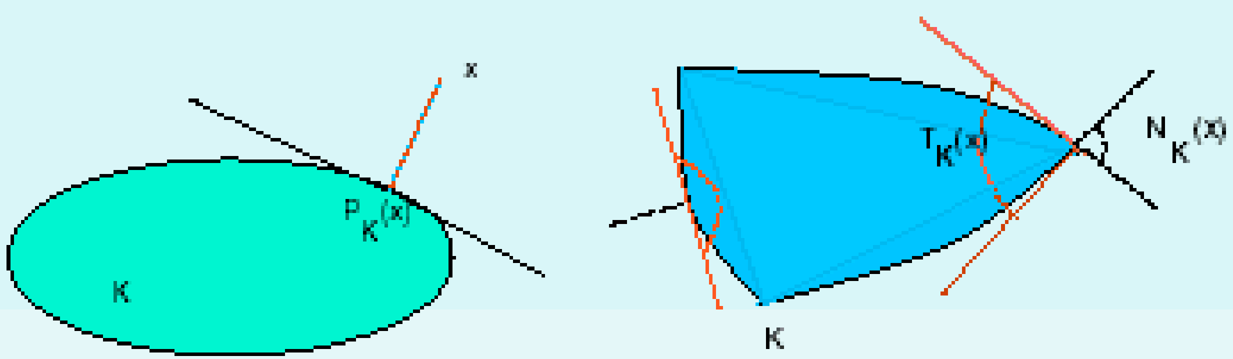
PrDEs and PDSs

To rigorously define the two notions, we recall the following:

1). the projection of X onto K by $P_K : X \to K$, with

$$\|P_K(x) - x\| = \inf_{z \in K} \|z - x\|, \quad \forall z \in X,$$

2. the tangent cone $T_K(x) = \overline{\cup_{h>0} \frac{1}{h}(K - x)}$.



PrDEs and PDSs

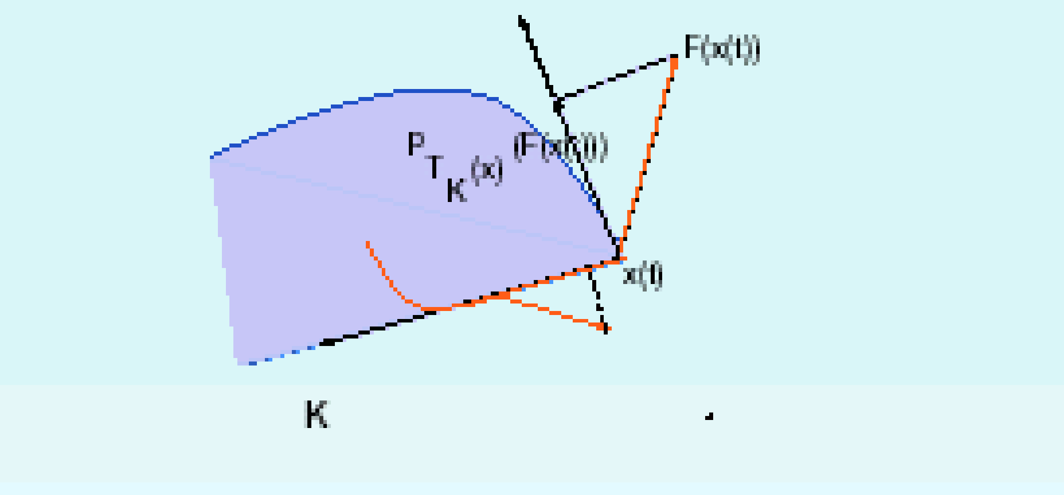
Let X, $K \subset X$, and $F : K \to X$ as before. Then a PrDE is defined by:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_K(x(t), -F(x(t))), \quad x(0) = x_0,$$

where

$$\Pi_K(x, -F(x)) = \lim_{\delta \to 0+} \frac{P_K(x - \delta F(x)) - x}{\delta} =: P_{T_K(x)}(-F(x)),$$

where $T_K(x)$ is the tangent cone to the set K at x and $N_K(x)$ is the normal cone to K at the same point x.



The right-hand side of any PrDE is nonlinear and discontinuous.

An existence result for such equations was obtained by Dupuis and Nagurney (1993) for $X:=R^n$, and by Cojocaru (2002) for general Hilbert spaces.

Theorem

Let X be a Hilbert space of arbitrary dimension and let $K \subset X$ be a non-empty, closed, and convex subset. Let $F : K \to X$ be a Lipschitz continuous vector field on K with $x_0 \in K$. Then the initial value problem

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))), x(0) = x_0$$

has a unique solution in $AC([0, \infty), K)$.

A projected dynamical system (PDS) is the dynamical system given by the set of trajectories of a PrDE.

EQUILIBRIA of PDSs and VARIATIONAL INEQUALITIES

An important feature of any PDS is that it is intimately related to a variational inequality problem (VI).

The starting point of VI theory: 1966 (Hartman and Stampacchia); 1967 (Lions and Stampacchia); it is now part of the calculus of variations; it has been used to show existence of equilibrium in a plethora of equilibrium problems and free boundary problems.

The following relation between a PDS and a VI was shown by Dupuis and Nagurney (1993) for $X := R^n$ and by Cojocaru (2002) for any Hilbert space. Here $F : K \to X$.

Theorem

The equilibria of a PDS:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_K(x(t), -F(x(t))),$$

that is, $x^* \in K$ *such that*

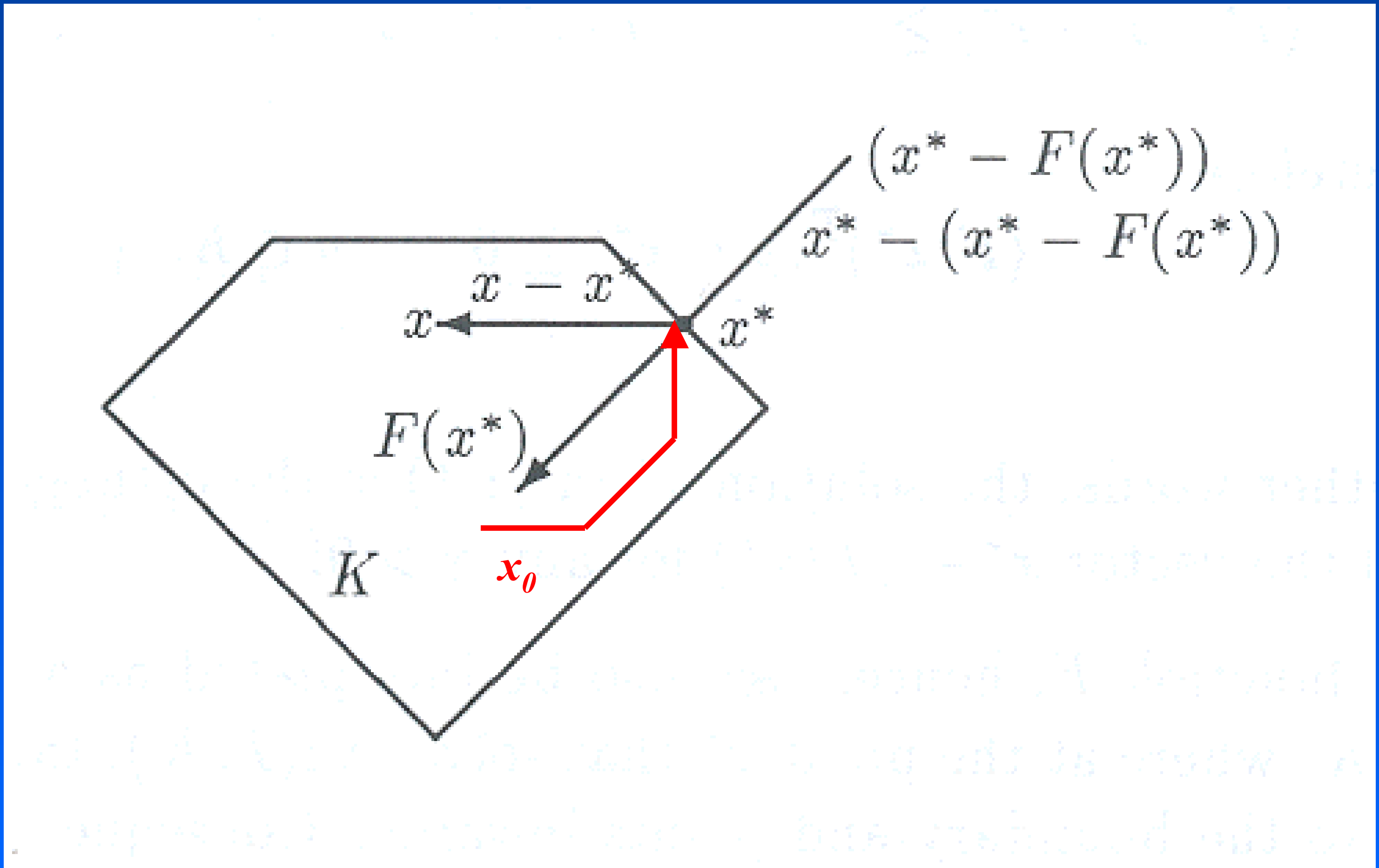
$$\Pi_K(x^*, -F(x^*)) = 0$$

are solutions to the $\mathrm{VI}(F, K)$*: find* $x^* \in K$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

and vice-versa, where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* X*.*

A Geometric Interpretation of a Variational Inequality and a Projected Dynamical System (Dupuis and Nagurney (1993), Nagurney and Zhang (1996))



We are using evolutionary variational inequalities to model dynamic networks with:

- *dynamic (time-dependent)* supplies and demands

- *dynamic (time-dependent)* capacities

- *structural changes* in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).

Evolutionary Variational Inequalities

Evolutionary variational inequalities, which are infinite dimensional, were originally introduced by Lions and Stampacchia (1967) and by Brezis (1967) in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems.

Steinbach (1998) studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity.

Daniele, Maugeri, and Oettli (1998, 1999), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others. See also Ran and Boyce(1996).

Evolutionary Variational Inequalities, Transportation, and the Internet

We model the Internet as a network $\mathcal{G} = [N, L]$, consisting of the set of nodes N and the set of directed links L. The set of origin/destination (O/D) pairs of nodes is denoted by W and consists of n_W elements. We denote the set of routes (with a route consisting of links) joining the origin/destination (O/D) pair w by P_w. We assume that the routes are acyclic. We let P with n_P elements denote the set of all routes connecting all the O/D pairs in the Internet. Links are denoted by a, b, etc; routes by r, q, etc., and O/D pairs by w_1, w_2, etc. We assume that the Internet is traversed by "jobs" or "classes" of traffic and that there are K "jobs" with a typical job denoted by k.

Let $d_w^k(t)$ denote the demand, that is, the traffic generated, between O/D pair w at time t by job class k. The flow on route r at time t of class k, which is assumed to be nonnegative, is denoted by $x_r^k(t)$ and the flow on link a of class k at time t by $f_a^k(t)$.

Since the demands over time are assumed known, the following conservation of flow equations must be satisfied at each t:

$$d_w^k(t) = \sum_{r \in P_w} x_r^k(t), \quad \forall w \in W, \forall k,$$

that is, the demand associated with an O/D pair and class must be equal to the sum of the flows of that class on the routes that connect that O/D pair. We assume that the traffic associated with each O/D pair is divisible and can be routed among multiple routes/paths. Also, we must have that

$$0 \leq x_r^k(t) \leq \mu_r^k(t), \quad \forall r \in P, \forall k,$$

where $\mu_r^k(t)$ denotes the capacity on route r of class k at time t.

We group the demands at time t of classes for all the O/D pairs into the Kn_W-dimensional vector $d(t)$. Similarly, we group all the class route flows at time t into the Kn_P-dimensional vector $x(t)$. The capacities on the routes at time t are grouped into the Kn_P-dimensional vector $\mu(t)$.

The link flows are related to the route flows, in turn, through the following conservation of flow equations:

$$f_a^k(t) = \sum_{r \in P} x_r^k(t)\delta_{ar}, \quad \forall a \in L, \forall k,$$

where $\delta_{ar} = 1$ if link a is contained in route r, and $\delta_{ar} = 0$, otherwise. Hence, the flow of a class on a link is equal to the sum of the flows of the class on routes that contain that link. All the link flows at time t are grouped into the vector $f(t)$, which is of dimension Kn_L.

The cost on route r at time t of class k is denoted by $C_r^k(t)$ and the cost on a link a of class k at time t by $c_a^k(t)$.

We allow the cost on a link to depend upon the entire vector of link flows at time t, so that

$$c_a^k(t) = c_a^k(f(t)), \quad \forall a \in L, \forall k.$$

We may write the link costs as a function of route flows, that is,

$$c_a^k(x(t)) \equiv c_a^k(f(t)), \quad \forall a \in L, \forall k.$$

The costs on routes are related to costs on links through the following equations:

$$C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t))\delta_{ar}, \quad \forall r \in P, \forall k.$$

We group the route costs at time t into the vector $C(t)$, which is of dimension Kn_P.

We now define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0,T], R^{Kn_P})$ (where $[0,T]$ denotes the time interval under consideration) given by

$$\mathcal{K} = \Big\{ x \in L^2([0,T], R^{Kn_P}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k \text{ a.e. in } [0,T] \Big\}.$$

We assume that the capacities $\mu_r^k(t)$, for all r and k, are in $\mathcal{L}$, and that the demands, $d_w^k \geq 0$, for all w and k, are also in $\mathcal{L}$. Further, we assume that

$$0 \leq d(t) \leq \Phi\mu(t), \text{ a.e. on } [0,T],$$

where Φ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element (kw, kr) equal to 1 if route r is contained in P_w, and 0, otherwise. The feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \rightarrow \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

The conditions below are a generalization of the Wardrop's (1952) first principle of traffic behavior.

Definition: Dynamic Multiclass Network Equilibrium

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle) if, for every O/D pair $w \in W$, every route $r \in P_w$, every class k; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if} \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if} \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if} \quad x_r^{k*}(t) = 0. \end{cases}$$

The standard form of the EVI that we work with is:

determine $x^* \in \mathcal{K}$ such that $\langle\langle F(x^*), x - x^* \rangle\rangle \geq 0, \ \forall x \in \mathcal{K}$.

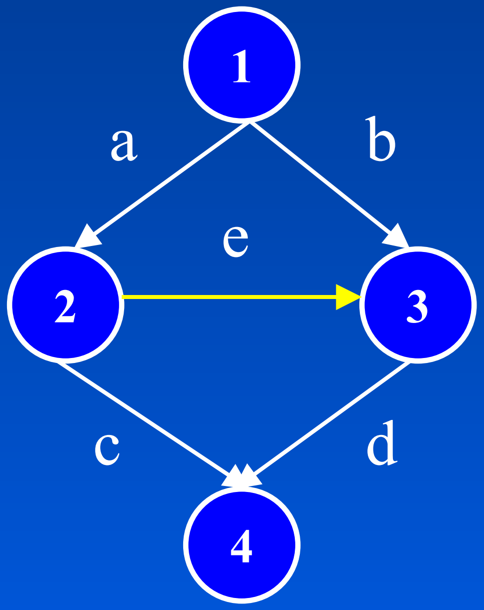
Theorem

$x^* \in \mathcal{K}$ is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:

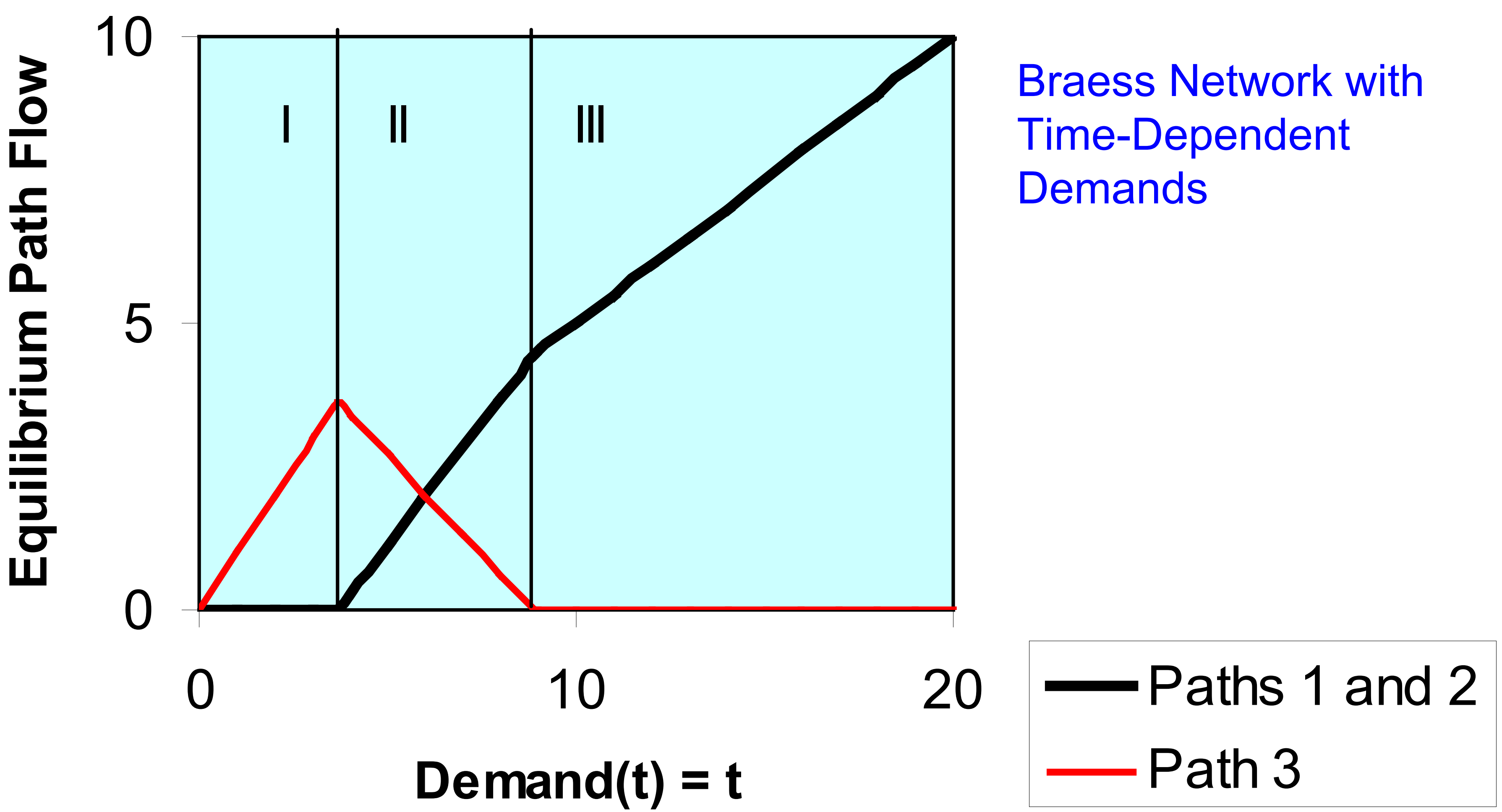
$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

Nagurney, Parkes, and Daniele, *Computational Management Science* (2007).

Recall the Braess Network
where we add the link e.

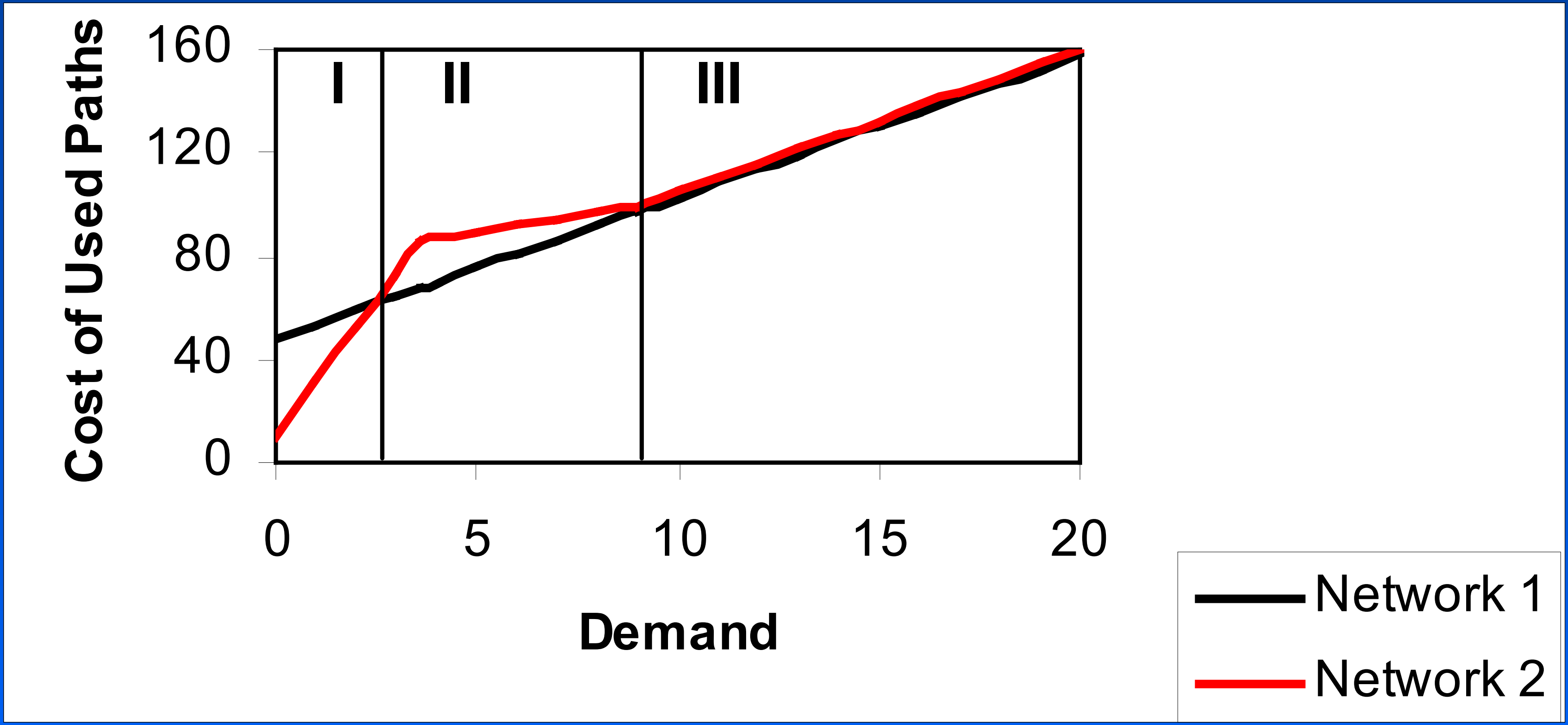


The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)



Braess Network with Time-Dependent Demands

In Demand Regime I, only the new path is used.
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off!
In Demand Regime III, only the original paths are used.



Network 1 is the Original Braess Network - Network 2 has the added link.

The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!

Double-Layered Dynamics

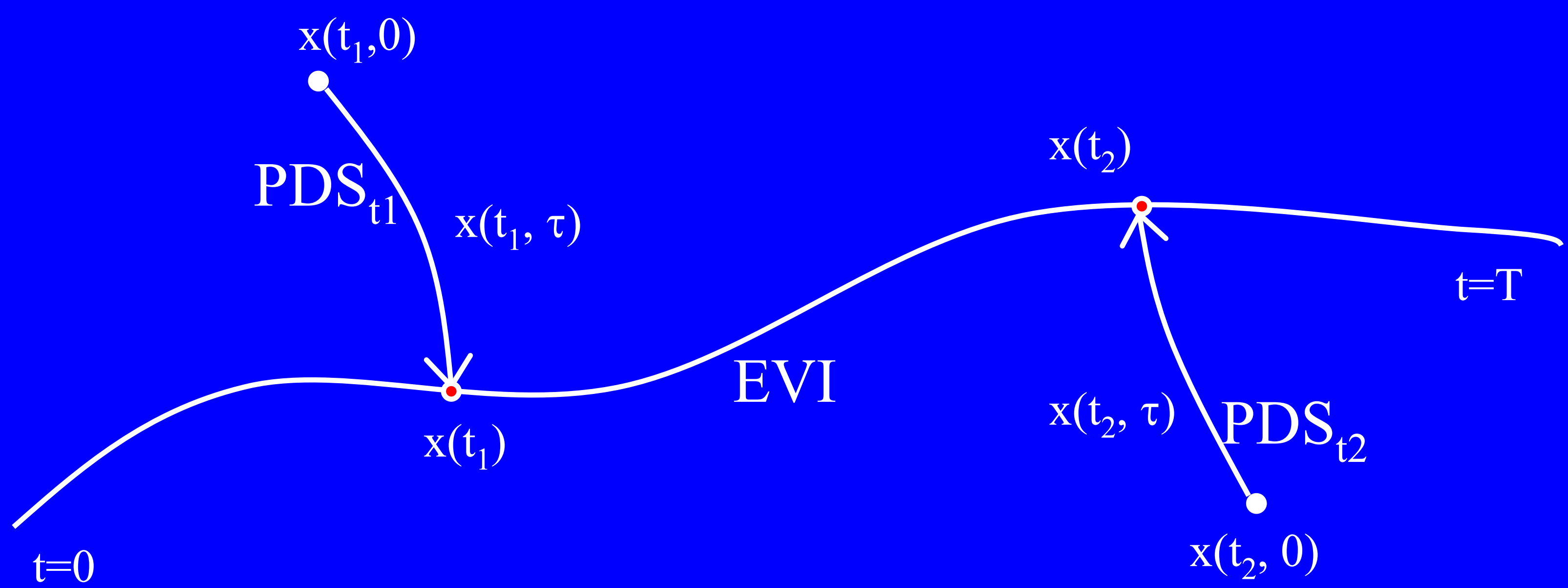
The unification of EVIs and PDSs allows the modeling of dynamic networks over *different time scales*.

Papers:

Projected Dynamical Systems and Evolutionary Variational Inequalities via Hilbert Spaces with Applications (Cojocaru, Daniele, and Nagurney), *Journal of Optimization Theory and Applications*, vol. 127, 2005.

Double-Layered Dynamics: A Unified Theory of Projected Dynamical Systems and Evolutionary Variational Inequalities (Cojocaru, Daniele, and Nagurney), *European Journal of Operational Research,* vol. 175, 2006.

A Pictorial of the Double-Layered Dynamics



There are new exciting questions, both theoretical and computational, arising from this multiple time structure.

In the course of answering these questions, a new theory is taking shape from the synthesis of PDS and EVI, and, as such, it deserves a name of its own; we call it **double-layered dynamics.**

We have also extended the Nagurney and Qiang network efficiency measure to *dynamic networks.*

Recent disasters have demonstrated the importance as well as the vulnerability of network systems.

For example:

- Minneapolis Bridge Collapse, August 1, 2007
- Hurricane Katrina, August 23, 2005
- The biggest blackout in North America, August 14, 2003
- 9/11 Terrorist Attacks, September 11, 2001.

Some Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Barrat, Barthélemy and Vespignani (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Chassin and Posse (2005)
- Holme, Kim, Yoon and Han (2002)
- Sheffi (2005)
- Taylor and D'este (2004)
- Jenelius, Petersen and Mattson (2006)
- Murray-Tuite and Mahmassani (2004)

A Network Efficiency Measure with Application to Critical Infrastructure

- A Network Efficiency Measure for Congested Networks (2007), Nagurney and Qiang, *Europhysics Letters*.
- Applications to Transportation Networks -- 2007 *Proceedings of the POMS Conference* in Dallas, Texas.
- Additional papers in press in *Journal of Global Optimization* and *Optimization Letters.*

The Nagurney and Qiang Network Efficiency Measure

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2006), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w\in W} \frac{d_w}{\lambda_w}}{n_W},$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

Europhysics Letters (2007).

Importance of a Network Component

Definition **Importance of a Network Component**

The importance, $I(g)$ of a network component $g \in G$, is measured by the relative network efficiency drop after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where $G - g$ is the resulting network after component g is removed from network G.

The Approach to Study the Importance of Network Components

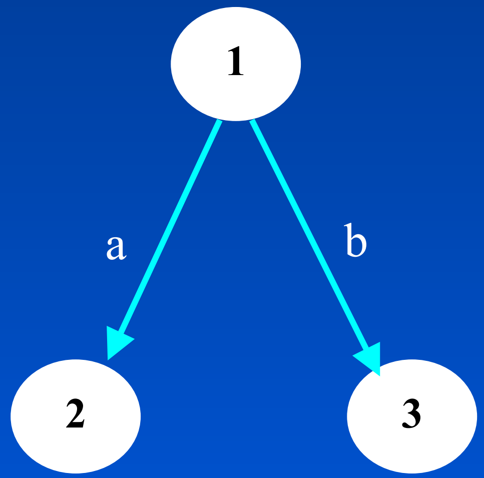
The elimination of a link is treated in the Nagurney and Qiang network efficiency measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks.

The measure generalizes the Latora and Marchiori network measure for complex networks.

Example 1

Assume a network with two O/D pairs:
$\boldsymbol{w_1}$=(1,2) and $\boldsymbol{w_2}$=(1,3) with demands:
$\boldsymbol{d_{w_1}}$=100 and $\boldsymbol{d_{w_2}}$=20.



The paths are:

for $\boldsymbol{w_1}$, $\boldsymbol{p_1 = a}$; for $\boldsymbol{w_2}$, $\boldsymbol{p_2 = b}$.

The equilibrium path flows are:

$\boldsymbol{x_{p_1}^* = 100}$, $\boldsymbol{x_{p_2}^* = 20}$.

$$c_a(f_a) = 0.01f_a + 19$$
$$c_b(f_b) = 0.05f_b + 19$$

The equilibrium path travel costs are:

$\boldsymbol{C_{p_1} = C_{p_2} = 20}$.

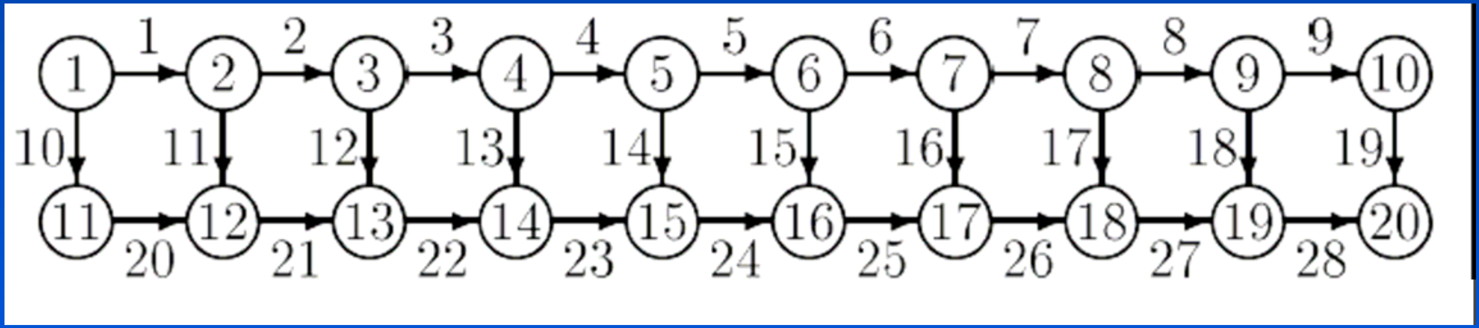
Importance and Ranking of Links and Nodes

Link	Importance Value from Our Measure	Importance Ranking from Our Measure
a	0.8333	1
b	0.1667	2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure
1	1	1
2	0.8333	2
3	0.1667	3

Example 2

The network is given by:



$w_1=(1,20)$ $w_2=(1,19)$

$d_{w_1} = 100$ $d_{w_2} = 100$

Link Cost Functions

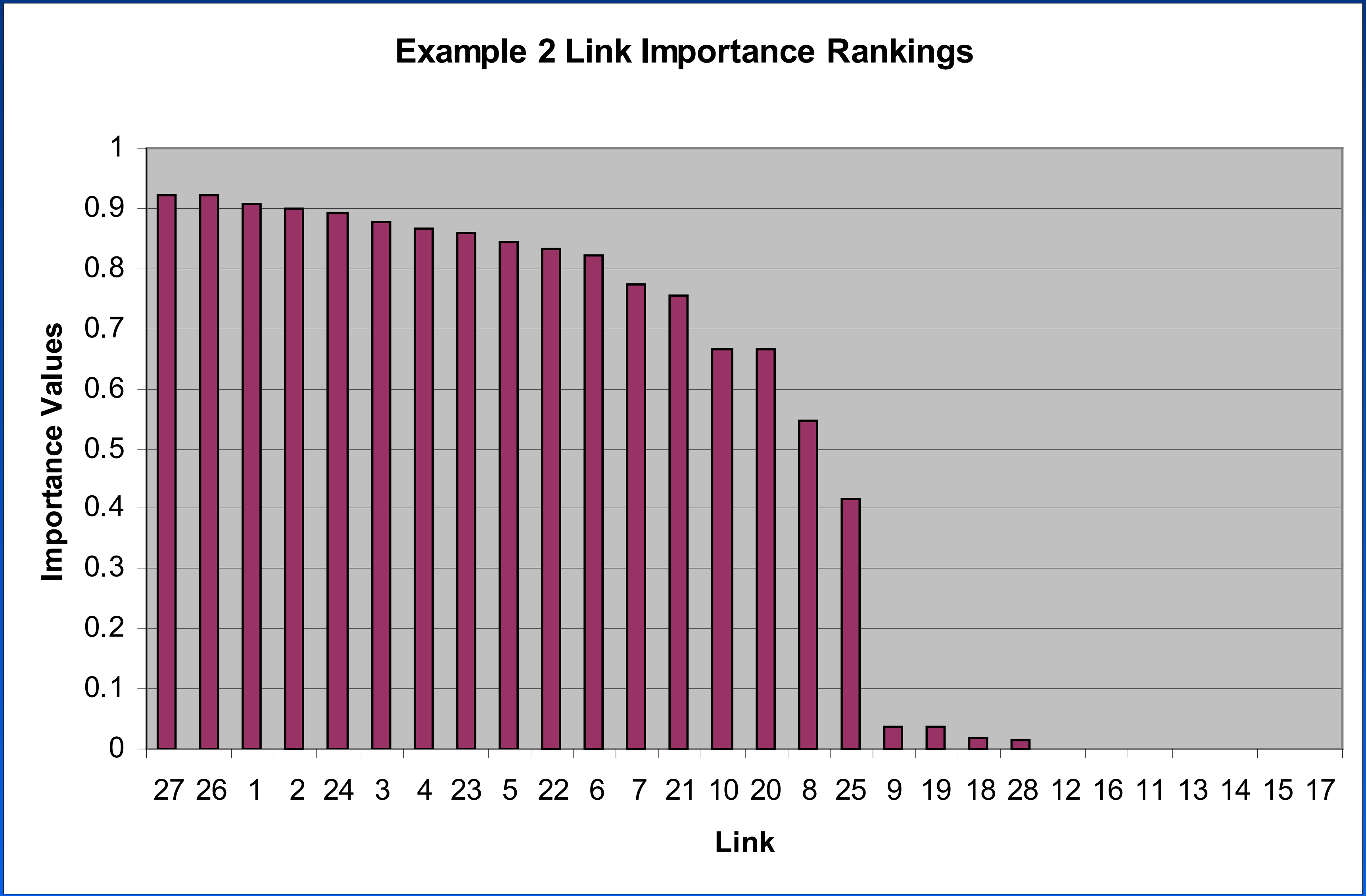
Link a	Link Cost Function $c_a(f_a)$
1	$.00005f_1^4 + 5f_1 + 500$
2	$.00003f_2^4 + 4f_2 + 200$
3	$.00005f_3^4 + 3f_3 + 350$
4	$.00003f_4^4 + 6f_4 + 400$
5	$.00006f_5^4 + 6f_5 + 600$
6	$7f_6 + 500$
7	$.00008f_7^4 + 8f_7 + 400$
8	$.00004f_8^4 + 5f_8 + 650$
9	$.00001f_9^4 + 6f_9 + 700$
10	$4f_{10} + 800$
11	$.00007f_{11}^4 + 7f_{11} + 650$
12	$8f_{12} + 700$
13	$.00001f_{13}^4 + 7f_{13} + 600$
14	$8f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003f_{15}^4 + 9f_{15} + 200$
16	$8f_{16} + 300$
17	$.00003f_{17}^4 + 7f_{17} + 450$
18	$5f_{18} + 300$
19	$8f_{19} + 600$
20	$.00003f_{20}^4 + 6f_{20} + 300$
21	$.00004f_{21}^4 + 4f_{21} + 400$
22	$.00002f_{22}^4 + 6f_{22} + 500$
23	$.00003f_{23}^4 + 9f_{23} + 350$
24	$.00002f_{24}^4 + 8f_{24} + 400$
25	$.00003f_{25}^4 + 9f_{25} + 450$
26	$.00006f_{26}^4 + 7f_{26} + 300$
27	$.00003f_{27}^4 + 8f_{27} + 500$
28	$.00003f_{28}^4 + 7f_{28} + 650$

Importance and Ranking of Links

Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19


Example 2 Link Importance Rankings
Importance Values
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
27 26 1 2 24 3 4 23 5 22 6 7 21 10 20 8 25 9 19 18 28 12 16 11 13 14 15 17
Link

The Advantages of the Nagurney and Qiang Network Efficiency Measure

- The measure captures demands, flows, costs, and behavior of users, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
- It can be applied to assess the efficiency/performance of a wide range of network systems.
- It is applicable also to elastic demand networks; (Qiang and Nagurney, *Optimization Letters*, in press).

Motivation for Research on Transportation Network Robustness

According to the ASCE:

Poor maintenance, natural disasters, deterioration over time, as well as unforeseen attacks now lead to estimates of \$94 billion in the US in terms of needed repairs for roads alone.

Poor road conditions in the United States cost US motorists \$54 billion in repairs and operating costs annually.

The focus of the robustness of networks (and complex networks) has been on the impact of different network measures when facing the removal of nodes on networks.

We focus on the *degradation of links through reductions in their capacities* and the effects on the induced travel costs in the presence of known travel demands and different functional forms for the links.

Robustness in Engineering and Computer Science

IEEE (1990) defined robustness as *the degree to which a system of component can function correctly in the presence of invalid inputs or stressful environmental conditions*.

Gribble (2001) defined system robustness as *the ability of a system to continue to operate correctly across a wide range of operational conditions, and to fail gracefully outside of that range*.

Schilllo et al. (2001) argued that robustness has to be studied *in relation to some definition of the performance measure*.

“Robustness” in Transportation

Sakakibara et al. (2004) proposed a topological index. The authors considered a transportation network to be robust if it is “dispersed” in terms of the number of links connected to each node.

Scott et al. (2005) examined transportation network robustness by analyzing the increase in the total network cost after removal of certain network components.

BPR Link Cost Functions

We use the Bureau of Public Roads (BPR) link cost functional form in our transportation network robustness study, which is given by:

$$c_a(f_a) = t_a^0 \left[1 + k \left(\frac{f_a}{u_a} \right)^{\beta} \right] \qquad \forall a \in L$$

where k and β are greater than zero and the u's are the practical capacities on the links.

The Transportation Network Robustness Measure of Nagurney and Qiang (2007)

The robustness measure $\mathcal{R}^\gamma$ for a transportation network G with the vector of demands d, the vector of user link cost functions c, and the vector of link capacities u is defined as the relative performance retained under a given uniform capacity retention ratio γ ($\gamma \in (0,1]$) so that the new capacities are given by γu. Its mathematical definition is given as:

$$\mathcal{R}^\gamma = \mathcal{R}(G, c, d, \gamma, u) = \frac{\mathcal{E}^\gamma}{\mathcal{E}} \times 100\%$$

where $\mathcal{E}$ and $\mathcal{E}^\gamma$ are the network performance measures with the original capacities and the remaining capacities, respectively.

Simple Example

Assume a network with one O/D pair: w_1=(1,2) with demand given by d_{w1}=10.

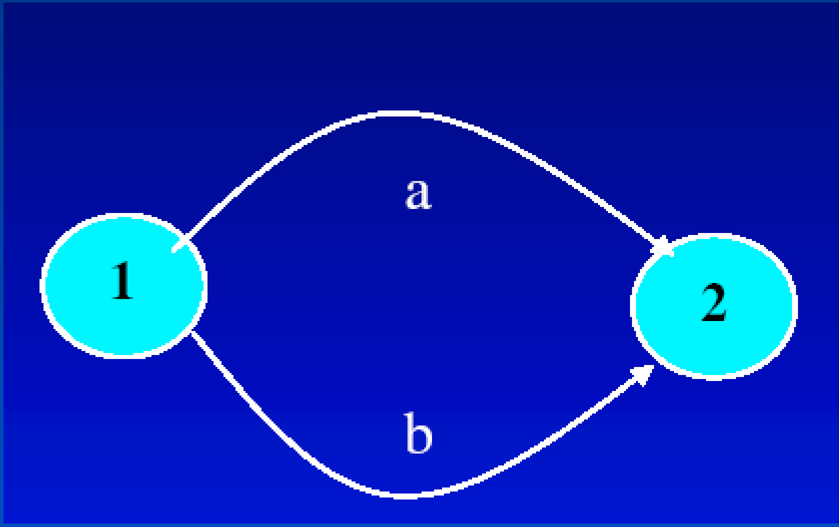
The paths are: $p_1=a$ and $p_2=b$.

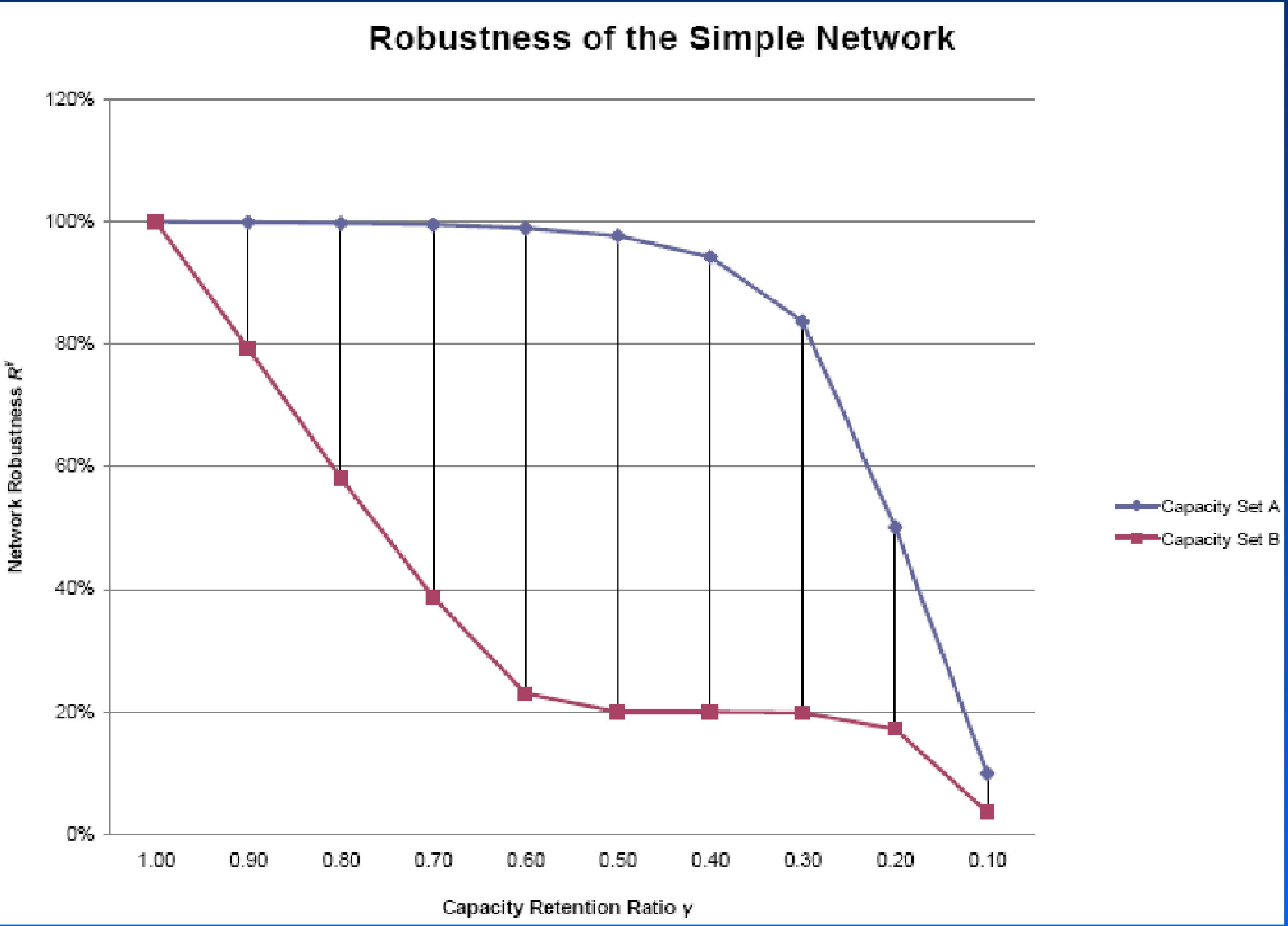
In the BPR link cost function, k=1 and β=4; t_a^0=10 and t_a^0=1.

Assume that there are two sets of capacities:

Capacity Set A, where $u_a=u_b$=50;

Capacity Set B, where u_a=50 and u_b=10.




Robustness of the Simple Network
120%
100%
80%
60%
40%
20%
0%
Network Robustness R^r
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
Capacity Retention Ratio γ
Capacity Set A
Capacity Set B

Example: Braess Network with Quadratic BPR Functions

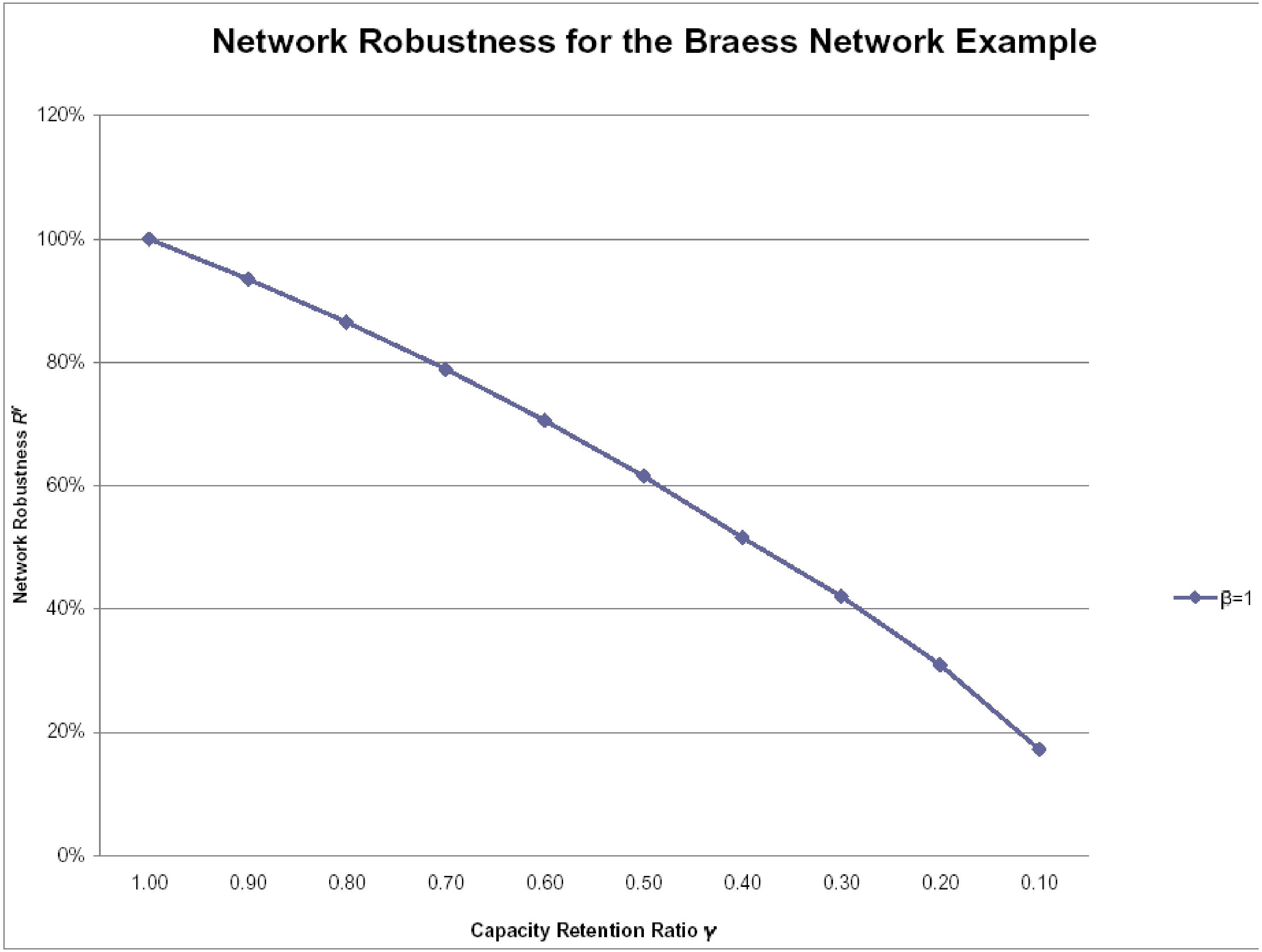
Instead of using the original cost functions, we construct a set of BPR functions as below under which the Braess Paradox still occurs. The new demand is 110.

$$c_a(f_a) = 1 + (\frac{f_a}{20})^\beta, \quad c_b(f_b) = 50(1 + (\frac{f_b}{50})^\beta),$$

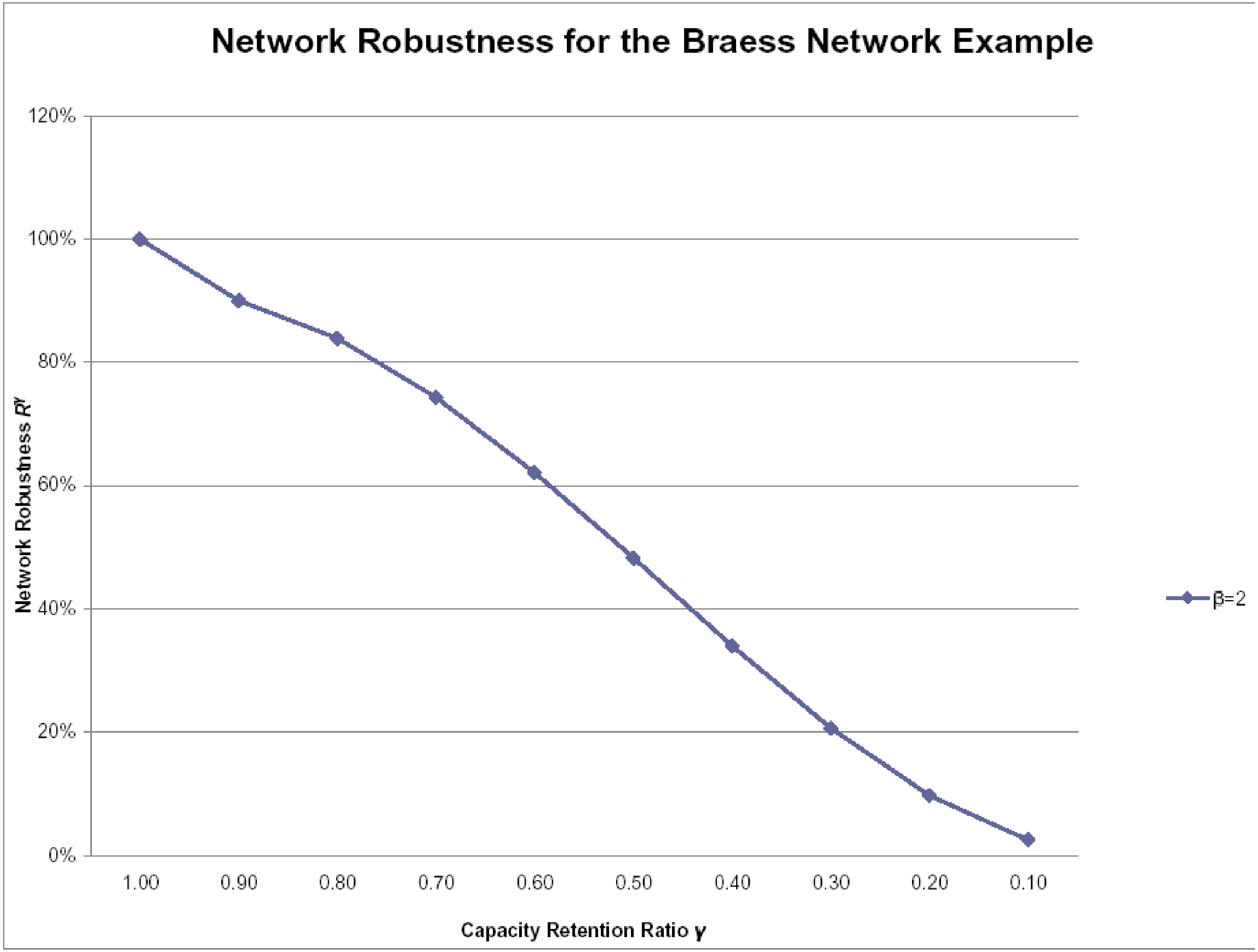
$$c_c(f_c) = 50(1 + (\frac{f_c}{50})^\beta), \quad c_d(f_d) = 1 + (\frac{f_d}{20})^\beta,$$

$$c_e(f_e) = 10(1 + (\frac{f_e}{100})^\beta).$$

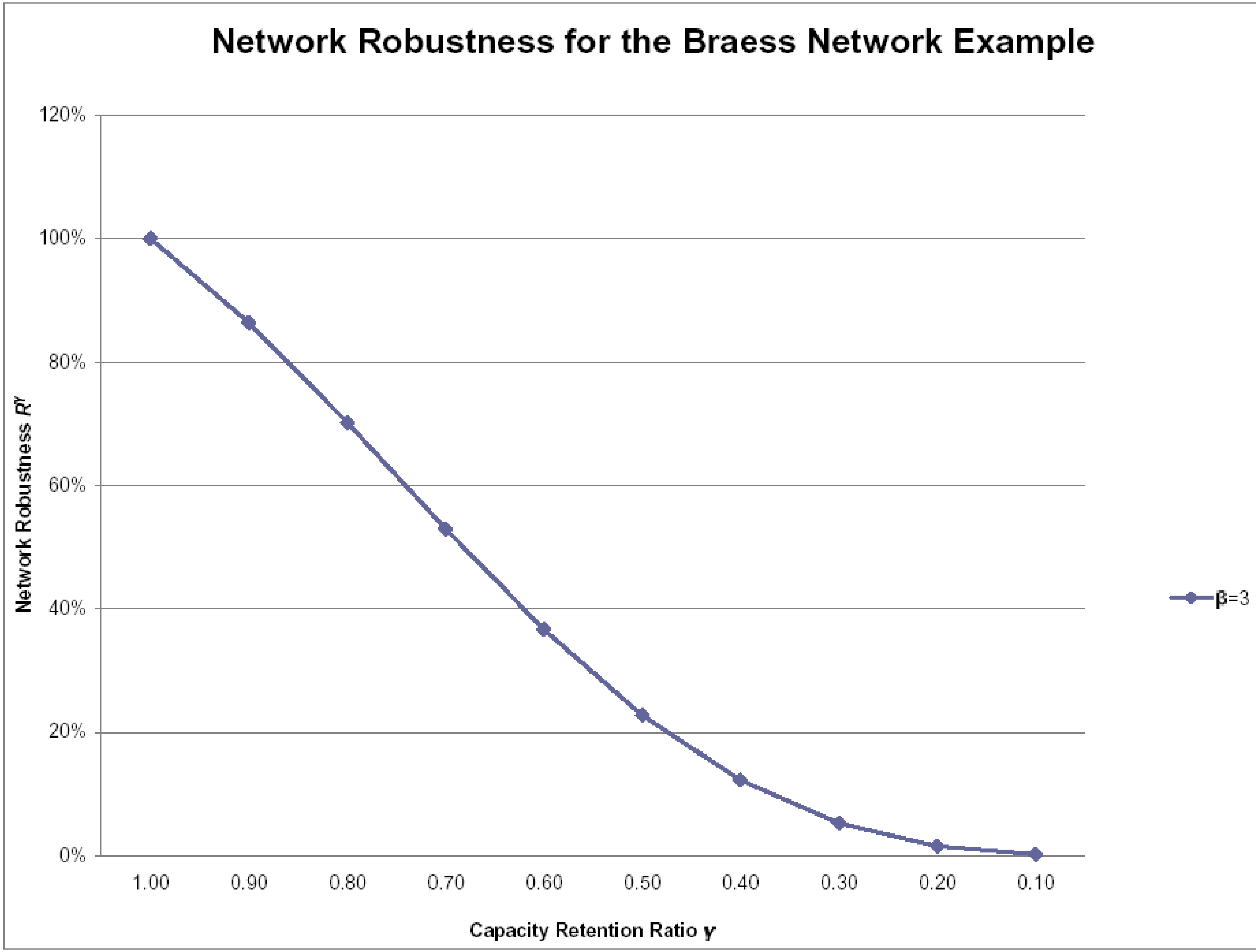
$\beta = 1$


Network Robustness for the Braess Network Example
120%
100%
80%
60%
40%
20%
0%
Network Robustness R^{γ}
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
Capacity Retention Ratio γ
β=1

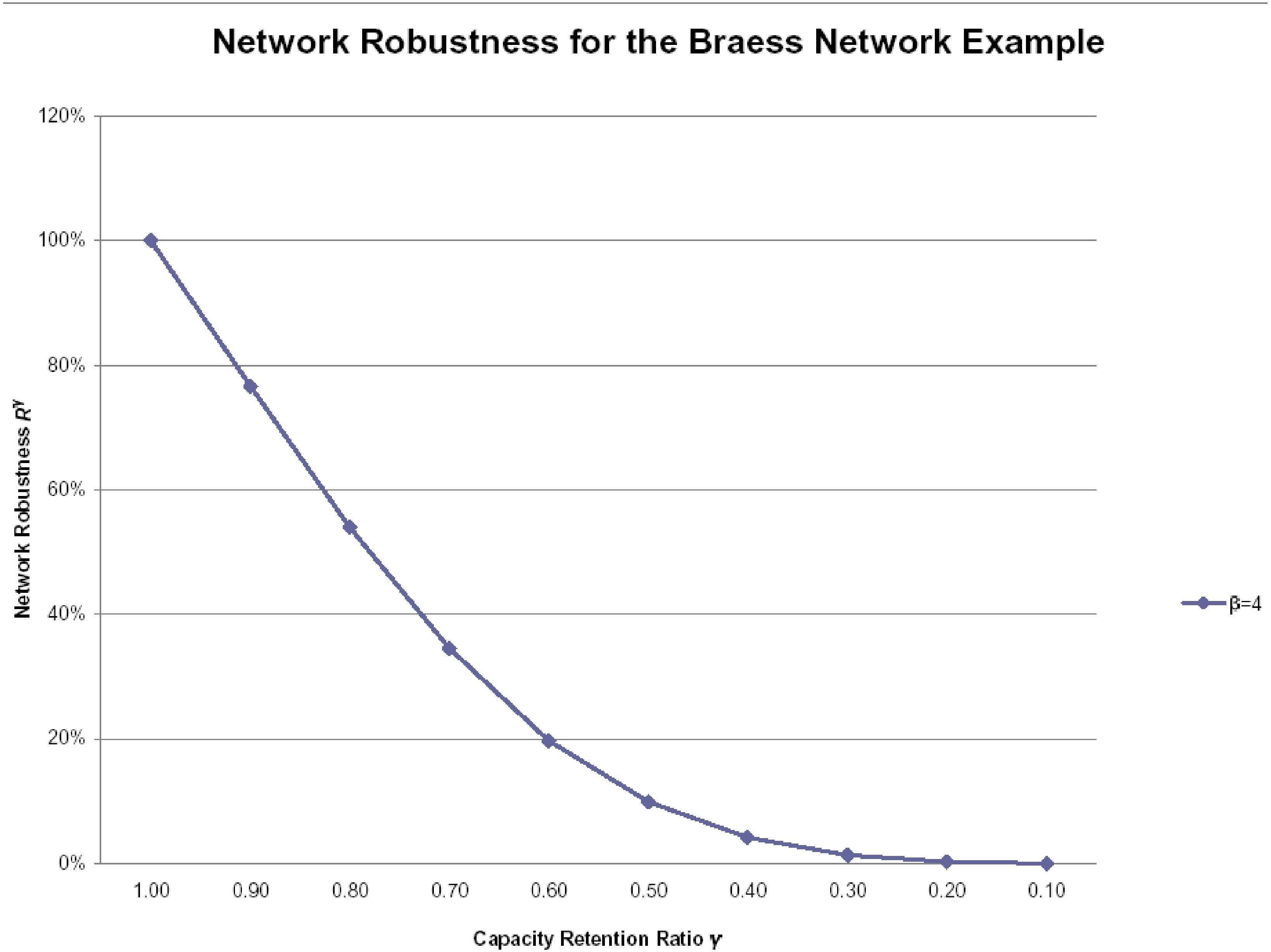
$\beta = 2$


Network Robustness for the Braess Network Example
120%
100%
80%
60%
40%
20%
0%
Network Robustness R^{γ}
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
Capacity Retention Ratio γ
β=2

$\beta = 3$


Network Robustness for the Braess Network Example
120%
100%
80%
60%
40%
20%
0%
Network Robustness R^γ
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
Capacity Retention Ratio γ
β=3

$\beta = 4$


Network Robustness for the Braess Network Example
120%
100%
80%
60%
40%
20%
0%
Network Robustness R^{γ}
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
Capacity Retention Ratio γ
β=4

Some Theoretical Results

Theorem

Consider a network consisting of two nodes 1 and 2, which are connected by a single link a and with a single O/D pair $w_1 = (1,2)$. Assume that the user link cost function associated with link a is of the BPR form. Then the network robustness given by the expression is given by the explicit formula:

$$\mathcal{R}^{\gamma} = \frac{\gamma^{\beta}[u_a^{\beta} + k d_{w_1}^{\beta}]}{[\gamma^{\beta} u_a^{\beta} + k d_{w_1}^{\beta}]} \times 100\%,$$

where d_{w_1} is the given demand for O/D pair $w_1 = (1,2)$.

Moreover, the network robustness $\mathcal{R}$ is bounded from below by $\gamma^{\beta} \times 100\%$.

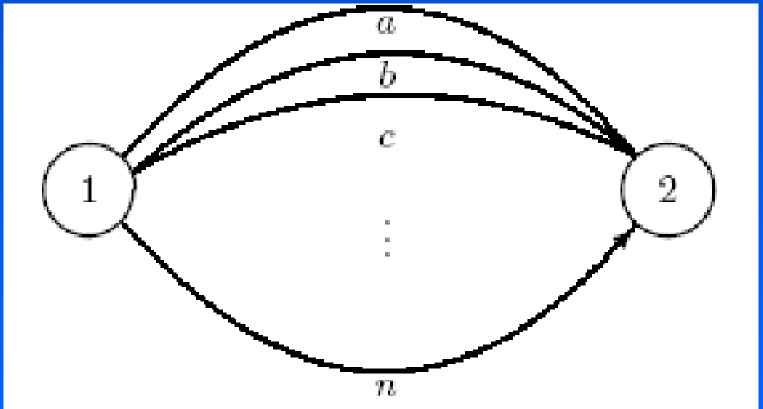
Theorem

Consider a network consisting of two nodes 1 and 2 as in the figure below, which are connected by a set of parallel links. Assume that the associated BPR link cost functions have $\beta = 1$. Furthermore, let's assume that there are positive flows on all the links at both the original and partially degraded capacity levels. Then the network robustness given by the expression is given by the explicit formula:

$$\mathcal{R}^{\gamma} = \frac{\gamma U + k\gamma d_{w_1}}{\gamma U + k d_{w_1}} \times 100\%,$$

where d_{w_1} is the given demand for O/D pair $w_1 = (1,2)$ and $U \equiv u_a + u_b + \cdots + u_n$.

Moreover, the network robustness $\mathcal{R}^{\gamma}$ is bounded from below by $\gamma \times 100\%$.



The Virtual Center for Supernetworks

Critical Infrastructure Networks · Transportation Networks · Communication Networks · Logistical Networks · Power/Energy Networks · Environmental Networks · Economic Networks · Financial Networks · Social Networks · Knowledge Networks

Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

Home | About | Background | Activities | Publications | Media | Links | What's New | Search

INFORMS Puerto Rico
July 8-11, 2007

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

Updated: September 29, 2007

Bloomberg.com · Imperial College London · DUSP MIT · In the Media · Web Week

FULBRIGHT Lectures Special Feature

Photos of Center Activities

MIND THE GAP

Professor Nagurney will be a Panelist at Mind the Gap! Career Summit for Women & Technology Sept. 27, 2007

Fall 2007 Operations Research/ Management Science Seminar Series

Knowing what comes next when things go wrong

NEW!

EUREKA!

The Supernetwork Sentinel
The Newsletter of the Virtual Center for Supernetworks
Fall 2007

You are visitor number

44,070

to the Virtual Center for Supernetworks.

NEW TALKS

Panel: Why Women are Turned Off by Technology and How to Overcome It

INFORMS Student Chapter

Google

Google Search

http://supernet.som.umass.edu

Thank you!

For more information, see
http://supernet.som.umass.edu

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks