Double-layered dynamics and transportation

Monica-Gabriela Cojocaru
Dept. of Mathematics & Statistics
University of Guelph, ON Canada

Part of this work is joint with:
A. Nagurney, UMASS Amherst USA
P. Daniele, U. Catania, Italy

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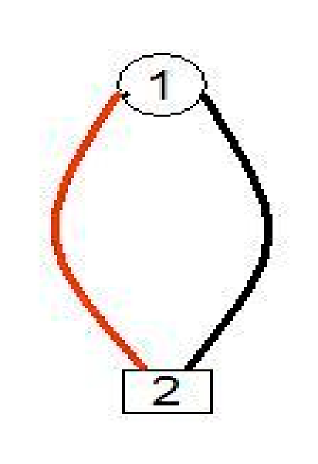
Outline

- Evolutionary variational inequalities (EVI)→ Projected dynamical systems (PDS)
- Double-layered dynamics (DLD)
- Interpretation of time scales
- Discontinuous DLD - Persistence and adjustment time
- Numerical Example
- Final remarks & References

Equilibrium problems

- Problems in economics, operations research, mathematical physics, transportation
- Transportation: 2 locations and 2 links, each link with an associated cost and a flow u_i

i e{1,2}.



Question #1

Can we model the time evolution of this problem towards an equilibrium state, respecting the constraints u_j ¤ 0?

Yes, with finite-dimensional VI+PDS theory. [Nagurney & al.]

Variational Inequalities

- 1964 part of calculus of variations; are used to show existence of equilibria in economic problems & free boundary problems, among others.
- Given a Hilbert space X, K closed, convex in X,

f : K $\rightarrow$ X, the variational inequality defined by f and K is:

find $x^* \in K$ s.t. $<f(x^*), y - x^*> \geq 0, \forall y \in K$.

Projected dynamical systems-I

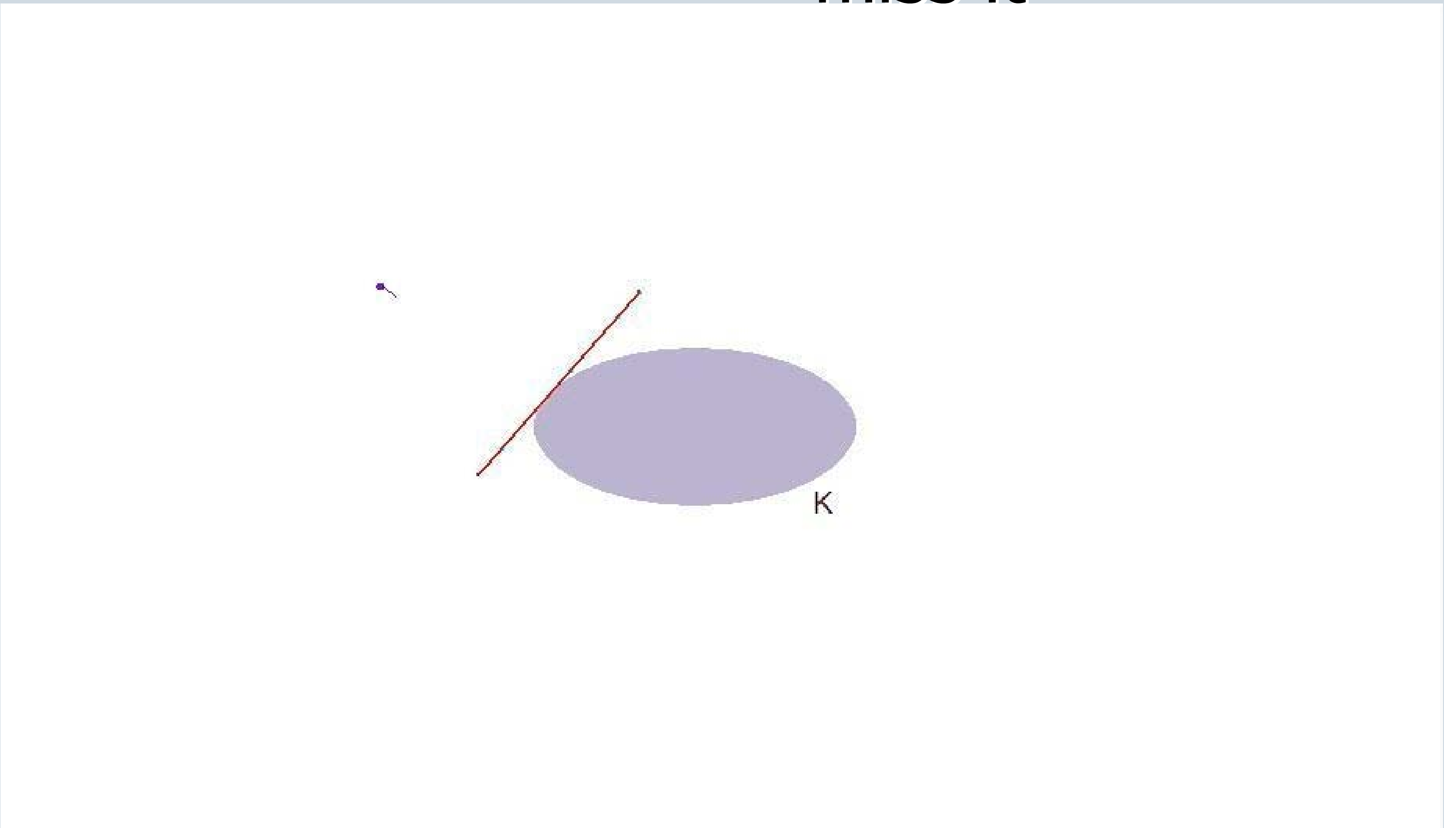
- Most generic context =a Hilbert space X. [Cojocaru, 2002]

We need:

- K a nonempty, closed, convex subset of X;
- $F : K \to X$ a Lipschitz continuous mapping;
- the tangent cone $T_K(x)$ to $x \in K$;
- the projection operator $P : X \to T_K(x)$;

Super short animation #1

- Do not blink, you may miss it



Projected dynamical systems - II

- A projected differential equation is a discontinuous ODE of the form

$$dx(t)/dt = P_{TK(x(t))}(-F(x(t))) \qquad (*)$$

the righthand side is the projection of the velocity field $-F(x(t))$ on the tangent cone to K at x(t)

- Theorem [Cojocaru, '02]: A PrDE (*) with $x(0) \in K$ and F Lipschitz, has a unique solution in $AC([0,\infty),K)$ through each $x(0) \in K$.

- A PDS is the dynamical system given by the set of trajectories of a PrDE.

Equilibrium analysis for PDS

- An equilibrium for PDS is a point x e K such that $P_{TK(x)}(-F(x)) = 0$.

VI ★ PDS:

Theorem [Nagurney, '93 – Cojocaru & Isac '02]:

The equilibria of an infinite-dim PDS are solutions to the VI(F,K):

find x e K such that <F(x), y – x> ¤ 0, ✂ y e K

and vice versa.

[0, ∞) = adjustment scale

Question #2

- Can we model the occurrence of equilibrium states for the transportation problem over a time scale [0,T], assuming the constraints of the problem are time-dependent?

Yes, with a class of time-dependent VI (also called EVI)

Time-dependent VI (EVI)

- EVI: back to ’67-’68 works of Stampacchia, Lyons, Brezis;
- Let *K* in $L^2([0, T], R^q)$ and the EVI

find u e *K* s. t. << F(u), v – u>> ¤ 0, ✂ v e *K*,
where

K = { u e $L^2([0,T],R^q)$ | ●(t) ⦿ u(t) ⦿ μ(t) a.e. [0, T];   Σ $⊠_{ji} u_i(t)$ = $❒_j(t)$ a.e. [0, T], $⊠_{ji}$ e { 0, 1},

i e {1,..,q}, j e {1,..,l}},

F : *K* → $L^2([0, T],R^q)$ and <<,>>:= $⇩_0^T$

Solutions to EVI

- Theorem [Daniele, Maugeri & al. ’98-’03]: If F is pseudo-monotone and hemicontinuous along line segments, then the EVI admits a solution over the constraint set *K*.

- Theorem [Cojocaru-Daniele-Nagurney ‘04]: If F is in addition strictly pseudo-monotone, then the solution to the EVI is unique.

Solutions to EVI –cont'd

- Interpretation:

• at each fixed t in [0, T], the solution(s) of the EVI represent one or more equilibrium states of the problem with the time-dependent constraint set K_t;

• as t varies over [0, T], these equilibria describe one (or more) curve(s);

[0,T] = prediction scale

Question #3

What is then the theoretical interaction of the two infinite-dimensional theories (PDS+EVI) and their associated time scales, and what does it mean for applications?

Double-layered dynamics [Cojocaru-Daniele-Nagurney, ‘04]

Double-layered dynamics (DLD)

Reminder:

K = { u eL^2([0,T],R^q) | ●(t) ⊙u(t) ⊙ μ(t) a.e. [0, T]; Σ ⊠$_{ji}$u$_i$(t) = ❒$_j$(t) a.e. [0, T], ⊠$_{ji}$ e{ 0, 1},

i e{1,..,q}, j e{1,..,l}

- Let an EVI as before and consider the PDS on *K* of L^2([0,T],R^q), given by:

du(.,♦)/d♦ = P$_{TK(u(.,♦))}$(−F(u(.,♦))), u(.,0) e*K* (**)

Time scales interpretation-I

Theorem [C-D-N,'04]: The solutions to EVI are the same as the critical points of PDS (**) and vice versa.

For each fixed t in [0,T], ♦ = evolution time of the problem towards one of the equilibria u*(t), given an initial state in the underlying PDS_t.

Consequence: stability analysis of the curve of equilibria done via infinite-dimensional PDS.

Time scales interpretation-II

- Theorem [C-D-N]: If F is strongly pseudo-monotone with degree ♋< 2 and Lipschitz on *K*, for a.a. fixed t in [0, T], then:

  👓 l_t > 0 finite, s.t. the unique equilibrium u*(t) of the PDS_t is reached by the solution u(t, l_t) of the PDS_t, starting at the initial point u_t^0 e K_t.

l_t = the adjustment time to equilibrium u*(t).

Continuous and Discontinuous curves of equilibria

- Theorem [C-D-N,'05]: Whenever the equilibrium constraints depend continuously on t, then the solutions of the EVI are continuous.
- Additional assumptions: We consider

K = { u eL^{2}([0,T],R^q) | *const*$_1$ ⦿u(t) ⦿ *const*$_2$ a.e. [0, T]; Σ u_i(t) = ❒$_1$(t) a.e. [0, T], ⊠$_{1i}$ =1,

i e{1,..,q}}, where

❒$_1$(t)= is a step function on [0,T]=[0,t_1]U$_p$ (t_{p-1},t_p]

Discontinuous curves of equilibria

Theorem [C-’05]: If the EVI has a unique solution u* on such *K*, then it is also a step function, i.e. $u^*(t)=u^*_p$, if t is in $(t_{p-1},t_p]$, etc.

This follows from the fact that all K_t, with t in $[t_{p-1},t_p]$ are the same, and so are all the PDS_t

Persistence and adjustment times

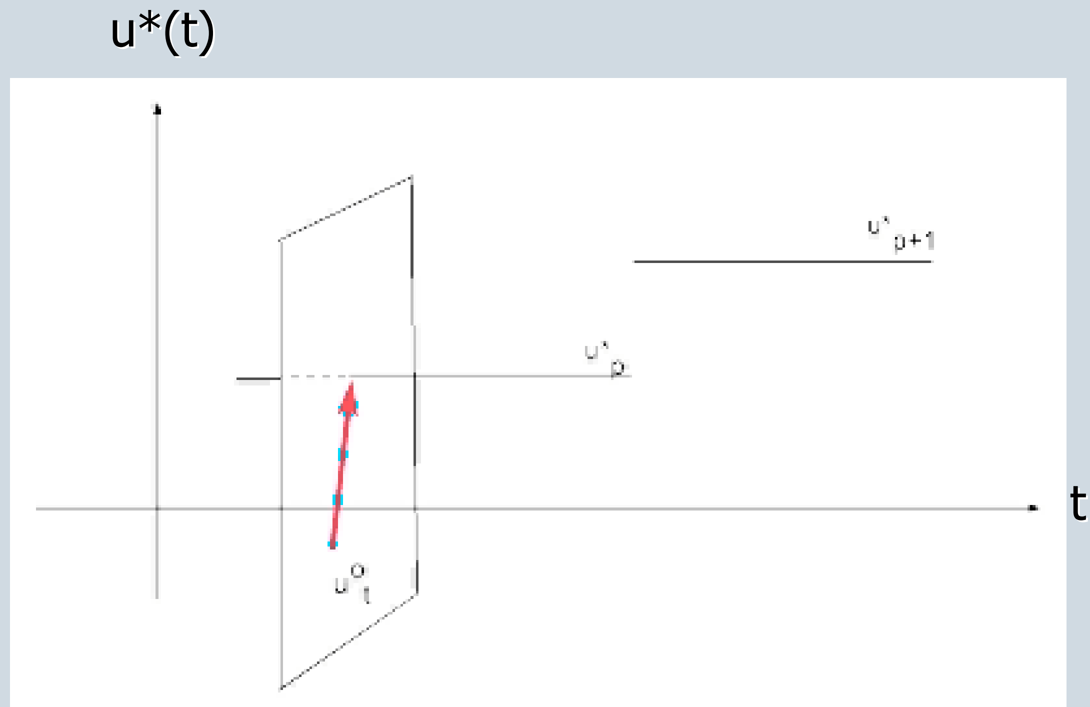
- Fix t in [0,T] w.l.o.g. so that t e $(t_{p-1},t_p]$;
- Then t_p-t is called the persistence time of the equilibrium u^*_p.

Time scales interpretation: if we start with a disequilibrium state u_t^0 for a PDS_t, then:

- $u^*(t)=u^*_p$ is reached if l_t ⊙ t_p-t
- $u^*(t)=u^*_p$ is not reached otherwise.

Super-short animation #2

- The process can be visualized here:



DLD model of transportation

- Let T = 90 min and ❒$_1$(t)= 90, 80, resp. 60 cars/min over intervals of 30 min.
- K = { u e L^2([0,T],R^2) | (1,0)⦿u(t) ⦿(140,140) a.e. [0, T]; Σ u_i(t) = ❒$_1$(t) a.e. [0, T]}
- $F(u_1(t),u_2(t))=(2u_1(t)-u_2(t)-2,u_1(t)+u_2(t)-2)$ are the cost vector functions;
- F is strongly pseudo-monotone with degree ♋=1 and ♒=$(1/2)^{1/2}$.

DLD transportation-cont'd

- In this scenario the predicted equilibrium curve is $u^*(t)$ = (41,49), (41,39), respectively (41,19).
- Suppose we start studying the traffic at t=10min with u_{10}^{0} =(43,47).
- Since u^*(10min)=(41,49) will persist for another 20min, then since l_{10} =4min $\odot$ 20min, the problem will reach $u^*(10)$.
- If in turn at t=10min, the traffic is (73,27), then u^*(10min) will not be reached since now $l_{10}>20$.

References

- Online source:

  http://www.uoguelph.ca/~mcojocar/

- DLD articles are in JOTA, EJOR, AML, JEDC, Kluwer NOIA series (P. Pardalos & al Edts.)