A Network Equilibrium Framework for Internet Advertising: Models, Quantitative Analysis, and Algorithms

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Highlights of this Research

- It is the first attempt to formulate the competitive Internet marketing strategies as a network equilibrium problem, which allows us to take advantage of network theory in terms of qualitative analysis and computations.

Highlights of this Research

- A Variational Inequality model is also established for the equilibrium of competitive Internet marketing strategies.

Highlights of this Research

- The size of a firm's online budget should not be a pre-fixed number, but rather, the size should be elastically adjusted with the online marginal responses which is

  --affected by the online advertising efforts;

  --affected by the inherent nature of the internet medium.

Highlights of this Research

- A numerical example is demonstrated, which shows that online budget is an increasing function of online marginal response (to online advertising efforts).

Highlights of this Research

- The existence and uniqueness conditions of the online marketing equilibrium are established.

Highlights of this Research

- An algorithm that takes the advantage of the network structure is proposed for the equilibrium solution.

  --size of the Internet marketing budget is calculated

  --allocation of the total budget to each of Internet websites is calculated

Highlights of this Research

- A numerical example is provided to test the algorithm

Network Modeling has been Used in Numerous Applications and Disciplines:

- Transportation Science and Logistics
- Telecommunications and Computer Science
- Regional Science and Economics
- Engineering
- Finance
- Operations Research and Management Science

Literature Referred

Advertising Competition Under Consumers Inertia, Banerjee, B and S. Bandyopadhyay (2003), Marketing Science 22, 131-144.

Modeling the Clickstream: Implications for Web-based Advertising Efforts, P. Chatterjee D.L. Hoffman, and T. P. Novak (2003), Marketing Science 22, 520-541.

Literature Referred

- *The General Multimodal Network Equilibrium Problem with Elastic Demand,* S. Dafermos, 1982, *Networks 12, 57-72.*
- *An Iterative Scheme for Variational Inequalities,* S. Dafermos, 1983, *Mathematics Programming 28, 174-185*.

Literature Referred

- *Network Economics: A Variational Inequalities and Their Applications*, A. Nagurney, 1993, Kluwer Academic Publishers.
- *A Network Modeling Approach for the Optimization of Internet-Based Advertising Strategies and Pricing with a Quantitative Explanation of Two Paradoxes, L. Zhao and A. Nagurney, 2005, Netnomics, in press.*

Assumptions

- There are N firms advertising in all mediums: one off-line medium and M online mediums.

- The off-line response is an increasing, concave function of off-line advertising spending.

$$r_{no} = r_{no}(f_o)$$

Assumptions

- Online response (Amount of click-through) is an increasing, concave function of the online advertisement spending.
- Amount of click-through in website i is also impacted by advertisement spending on other websites.

$$r_{nw} = r_{nw}(f_w)$$

Online Advertising Budget

to decide online/offline budget allocation, a firm needs to solve ,

$$\underset{f_{no}, f_{nw}}{Max} (r_{no} + r_{nw})$$

$$s.t.: \; f_{no} + f_{nw} \leq C_n$$

$$f_{no}, f_{nw} \geq 0$$

where C_n is firm's total budget:

Online Advertising Budget

- Solving the Max problem, we mathematically prove that budget is an increasing function of marginal response (to marketing investment):

$$b_n = b_n(\eta_{nw})$$

-- if additional online investment would yield more response than offline investment, then the firm is willing to increase online investment and reduce offline investment.

Example

$$r_w = -\frac{2}{100000} f_w^{\,2} + \frac{4}{100} f_w + 2$$

$$r_o = -\frac{4}{150000} f_o^{\,2} + \frac{2}{150} f_o + 1$$

Example

- The firm is to

$$\underset{f_o, f_w}{Max}\ (r_o + r_w)$$

$$s.t.:\ \ f_o + f_w \leq 900$$

$$f_o, f_w \geq 0$$

Example -- Result

Initial on line exp. f_w	Initial off line exp. f_o	Online margin η_w	Off line margin η_o	adjustment
$500	$400	0.020	-0.008000	$300
$600	$300	0.016	-0.002667	$200
$700	$200	0.012	0.002667	$100
$800	$100	0.008	0.008000	$0

The Network Equilibrium Model

- Basic Assumptions

  -- There are n=1,2,…,N firms compete in m=1,2,…,M internet websites.

  -- each firm is to maximize its own aggregate ad results (amount of total click-through).

  --amount of click-through in website m for firm n is a function of f, where f is a vector of ad expenditure of all firms on all websites.

  --firm's internet ad budget is an increasing function of marginal click-through.

The Network Equilibrium Model

Each firm is to:

$$\underset{f_{1n},\ldots,f_{MN}}{Max} \sum_{m=1}^{M} r_{mn}(f)$$

$$s.t.: \sum_{m=1}^{M} f_{mn} \leq b_n(\eta_n)$$

$$f_{mn} \geq 0,\ m = 1,2,\ldots,M$$

n=1,2,…,N

The Network Equilibrium Model

After applying Kuhn-Tucker conditions equilibrium conditions are obtained:

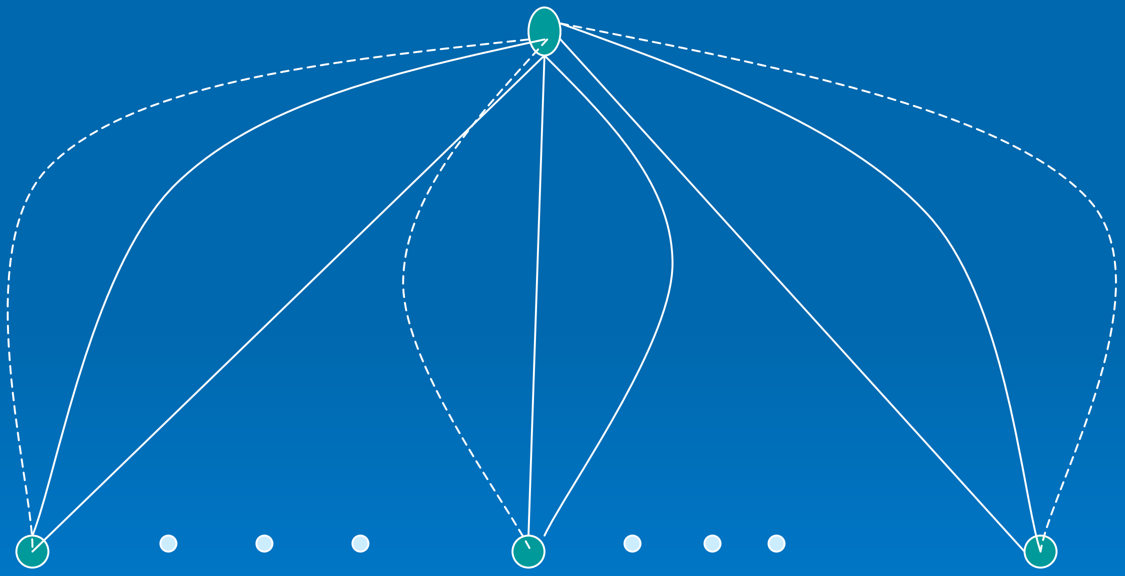
$$\frac{\partial r_n(f^*)}{\partial f_{mn}} \begin{cases} = \lambda_n(b_n^*), & if \quad f_{mn}^* > 0, \\ \leq \lambda_n(b_n^*), & if \quad f_{mn}^* = 0, \end{cases}$$

$$0 \begin{cases} = \lambda_n(b_n^*), & f_{ns}^* > 0, \\ \leq \lambda_n(b_n^*), & f_{ns}^* > 0, \end{cases}$$

$$\sum_{m=1}^{M} f_{mn}^* + f_{ns}^* = b_n^*$$

The Equivalent Network Model

The dotted link is the dummy link that absorbs the budget surplus f_{ns}.



The Variational Inequality Formulation

$$\nabla B(f^*) \bullet (f - f^*) \leq 0$$

$$\forall f \in S = \{f \mid f \geq 0, \sum_{i=1}^{n+1} f_i = C\}$$

The Variational Inequality Formulation

Let

$$u(f) = \left(\frac{\partial r_n(f)}{\partial f_{mn}},\ m = 1,\ldots,M; n = 1,\ldots,N\right)$$

$$\lambda(b) = (-\lambda_n(b_n),\ n = 1,\ldots,N)$$

$$\mathrm{K} = \{(f,b) \mid (f,b) \in R_+^{MN+N},\ \sum f_{mn} + f_{ns} = b_n\}$$

The Variational Inequality Formulation

Then, equilibrium online budget b* and its allocation f* is a solution of

$$u(f^*)(f - f^*) - \lambda(b^*)(b - b^*) \leq 0$$

$$\forall (f, b) \in \mathrm{K}$$

This variational inequality can be solved by an iterative scheme where the function in each of the sub problems is separable and quadratic (see Dafermos and Sparrow (1969), Dafermos (1980), Zhao and Dafermos (1991), Nagurney (1999)).

The solution (f^*,b^*) determines the size of the online budget and its allocation.

Existence and Uniqueness of the Solution

- If vector function ($-u(f)$, $\lambda(b)$) is strongly monotone on K, then
  - Equilibrium of the competition exists.
  - The equilibrium is unique.
  - The algorithm for finding the equilibrium is convergent.

Sufficient Conditions for Monotonicity

- The matrices of second derivatives of -r(f) and the first derivatives of λ(b) are positive definite.

Example

There are two firms competing over three websites:

$$u_{11} = -2f_{11} - f_{12} + 100$$

$$u_{21} = -4f_{21} - 1.5f_{22} + 80$$

$$u_{31} = -2f_{21} + f_{32} + 45;$$

Example

$$u_{12} = -f_{12} - 0.5f_{11} + 90$$
$$u_{22} = -3f_{22} - f_{21} + 80$$
$$u_{32} = -5f_{32} + 2f_{31} + 90;$$

$$\lambda_1 = 5b_1 - 10,\ \lambda_2 = 8b_2 - 10$$

Solution

n	f_n	b_n
0	(14.00,12.00,13.00,12.00,20.00,3.00)	(39,00,35.00)
1	(10.54,6.21,0.00,4.25,7.60,1.68)	(16.76,13.53)
2	(12.30,3.23,0.00,5.93,3.89,2.21)	(15.53,12.03)
…	……	……
23	(12.31,3.08,0.00,8.48,0.52,2.93)	(15.38,11.92)

Future Research

Modeling the impact of asymmetric information on optimal marketing strategies.

For more information see:

http://supernet.som.umass.edu

Thank you!!!