Financial Networks with Intermediation and Transportation Network Equilibria: A Supernetwork Equivalence and Reinterpretation of the Equilibrium Conditions

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Related Literature

- Quesnay (1758)
- Thore (1969, 1970)
- Ferguson and Dantzig (1956)
- Dantzig and Madansky (1961)
- Storoy, Thore, and Boyer (1975)
- Thore (1980)
- Nagurney, Dong, and Hughes (1992)
- Nagurney and Siokos (1997)
- Nagurney and Ke (2001, 2003),
- Nagurney and Cruz (2003).

Related Literature

- Nagurney (2006) demonstrated that supply chain networks could be reformulated as transportation networks.

- Nagurney and Liu (2006) proved the supernetwork equivalence of the electric power supply chain networks with traffic networks.

Outline

- In this paper, we ask the question as to whether financial network problems with intermediation can be reformulated as transportation network equilibrium problems?
- We first extend the model of Nagurney and Ke (2003) to the case where the inverse demand (price) functions are assumed known.
- We recall the transportation network equilibrium model with fixed demands due to Smith (1979) and Dafermos (1980).
- We establish the supernetwork equivalence of the financial network model with a special configuration of the fixed demand transportation/traffic network equilibrium model.

The Financial Network with Intermediation

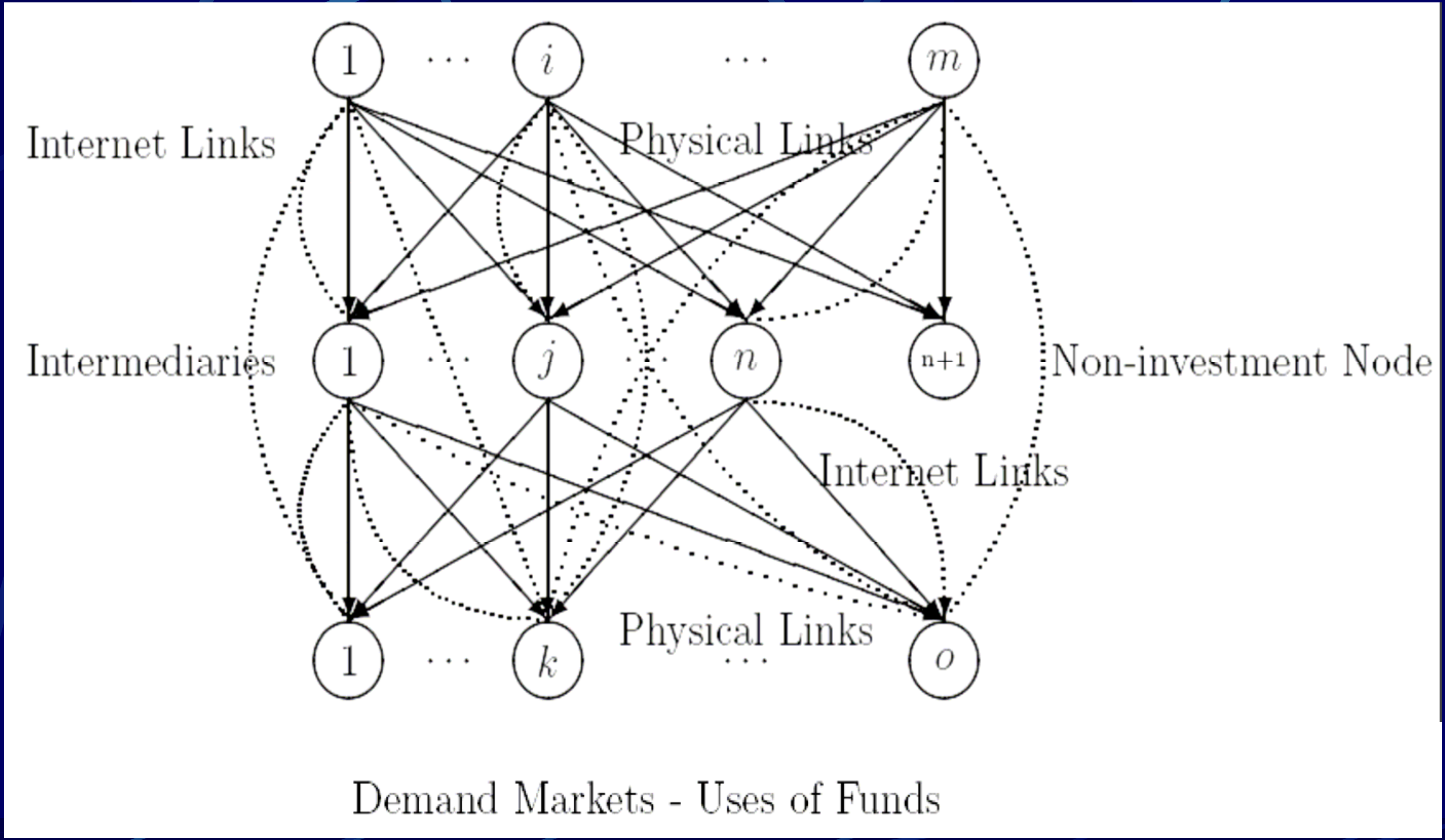


Table 1: Notation for the Financial Network Model

Notation	Definition
S	m-dimensional vector of the amount of funds held by the source agents with components i denoted by S^i
Q^0	m-dimensional vector of uninvested portion of funds held by source agents with component i denoted by $q_{i(n+1)}$
Q^1	$2mn$-dimensional vector of all financial flows for all source agents/intermediaries/modes with component ijl denoted by q_{ijl}
Q^2	mo-dimensional vector of all direct electronic financial transaction flows between sources of funds and demand markets with component ik denoted by q_{ik}
Q^3	$2no$-dimensional vector of all financial flows for all intermediaries/demand markets/modes with component jkl denoted by q_{jkl}
g	n-dimensional vector of total financial flows received by intermediaries with component j denoted by g_j, $g_j \equiv \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}$
γ	n-dimensional vector of shadow prices associated with the intermediaries with component j denoted by γ_j
d	o-dimensional vector of market demand with component k denoted by d_k
$\rho_{3k}(d)$	inverse demand function at market k
V^i	the variance-covariance matrix associated with source agent i, which is of dimension $(2n+o) \times (2n+o)$
V^j	the variance-covariance matrix associated with intermediary j, which is of dimension $(2m+o) \times (2m+o)$

Table 1 (Continue)

Notation	Definition
$c_{ijl}(q_{ijl})$	the trading cost incurred at source agent i in the transaction between source agent i and intermediary j using mode l with marginal transaction cost denoted by $\frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$c_{ik}(q_{ik})$	the trading cost incurred at source agent i in the electronic transaction between source agent i and demand market k with marginal transaction cost denoted by $\frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}}$
$c_{jkl}(q_{jkl})$	the trading cost incurred at intermediary j in the transaction between intermediary j and demand market k via mode l with marginal transaction cost denoted by $\frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}}$
$c_j(Q^1) \equiv c_j(g_j)$	converting/handling cost of intermediary j with marginal handling cost with respect to g_j denoted by $\frac{\partial c_j}{\partial g_j}$ and the marginal handling cost with respect to q_{ijl} denoted by $\frac{\partial c_j(Q^1)}{\partial q_{ijl}}$
$\hat{c}_{ijl}(q_{ijl})$	the trading cost incurred at intermediary j in the transaction between source agent i and intermediary j via mode l with the marginal transaction cost denoted by $\frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$\hat{c}_{jkl}(Q^2, Q^3)$	the unit trading cost incurred at demand market k in the transaction between intermediary j and demand market k
$\hat{c}_{ik}(Q^2, Q^3)$	the unit trading cost incurred at demand market k in the transaction between source agent i and demand market k

The Behavior of Source Agents

- Limited source of funds

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} + q_{i(n+1)} = S^i, \quad i = 1, \ldots, m.$$

- The source agent's optimization problem

Maximize $\quad U^i(q_i) = \sum_{j=1}^{n}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n}\sum_{l=1}^{2} c_{ijl}(q_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}) - q_i^T V^i q_i,$

subject to

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} + q_{i(n+1)} = S^i$$

$$q_{ijl} \geq 0, \forall i, l; \quad q_{ik} \geq 0, \forall k; \quad q_{i(n+1)} \geq 0$$

The Optimization Conditions of the Source Agents

- The Optimization Conditions of the Source Agents
  - Determine $(Q^{1*}, Q^{2*}) \in \mathcal{K}^0$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}] \geq 0, \quad \forall (Q^1, Q^2) \in \mathcal{K}^0,$$

where $V^i_{z_{jl}}$ denotes the z_{jl}-th row of V^i and z_{jl} is defined as the indicator: $z_{jl} = (l-1)n + j$. Similarly, $V^i_{z_{2n+k}}$ denotes the z_{2n+k}-th row of V^i but with z_{2n+k} defined as the $2n+k$-th row, and the feasible set $\mathcal{K}^0 \equiv \{(Q^1, Q^2) | (Q^1, Q^2) \in R^{2mn+mo}_+$ and conservation of flow equations hold$\}$.

The Behavior of Intermediaries

- The intermediary's optimization problem

$$\text{Maximize} \quad U^j(q_j) = \sum_{k=1}^{o}\sum_{l=1}^{2} \rho^*_{2jkl} q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2} \hat{c}_{ijl}(q_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2} c_{jkl}(q_{jkl})$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} - q_j^T V^j q_j$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} \leq \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl},$$

The Optimization Conditions of the Intermediaries

- The Optimization Conditions of the Intermediaries
  - Determine $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{2mn+2no+n}$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^j_{z_il}\cdot q_j^* + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma_j^*\right]\times\left[q_{ijl}-q^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{zkl}\cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} - \rho^*_{2jkl} + \gamma_j^*\right]\times\left[q_{jkl}-q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{o}\sum_{l=1}^{2} q^*_{jkl}\right]\times\left[\gamma_j-\gamma_j^*\right]\geq 0, \quad \forall (Q^1, Q^3, \gamma)\in R_+^{2mn+2no+o}.$$

where $V^j_{z_{il}}$ denotes the z_{il}-th row of V^j where z_{il} is defined as the indictor: $z_{il} = (l-1)m+i$. Similarly, $V^j_{z_{kl}}$ denotes the z_{kl}-th row of V^j and z_{kl} is the indicator: $z_{kl} = 2m+(l-1)o+k$.

The Equilibrium Conditions at the Demand Markets

- Flow conservation at demand market k

$$d_k = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} + \sum_{i=1}^{m} q_{ik}, \quad k = 1, \ldots, o.$$

- Equilibrium conditions at demand market k

The equilibrium condition for consumers at demand market k are as follows: For intermediary j; $j = 1, \ldots, n$:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} = 0, \end{cases}$$

For source of funds i; $i = 1, \ldots, m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} = 0, \end{cases}$$

The Variational Inequality for the Demand Markets

- The Equilibrium Conditions at the Demand Markets

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\rho^{*}_{2jkl}+\hat{c}_{jkl}(Q^{2*},Q^{3*})\right]\times\left[q_{jkl}-q^{*}_{jkl}\right]+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho^{*}_{1ik}+\hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times\left[q_{ik}-q^{*}_{ik}\right]$$

$$-\sum_{k=1}^{o}\rho_{3k}(d^{*})\times[d_k-d^{*}_{k}]\geq 0,\quad \forall (Q^2,Q^3,d)\in\mathcal{K}^1,$$

where $\mathcal{K}^1\equiv\{(Q^2,Q^3,d)|(Q^2,Q^3,d)\in R^{2no+mo+o}_{+}$ and conservation of flow equations hold.$\}$

Financial Network Equilibrium

- Financial Network Equilibrium

Definition 1: Financial Network Equilibrium with Intermediation and with Electronic Transactions

The equilibrium state of the financial network with intermediation is one where: all source agents have achieved optimality; all intermediaries have achieved optimality, and, finally, the equilibrium conditions at the demand markets hold.

- Define the feasible set

$$\mathcal{K}^2 \equiv \{(Q^1, Q^2, Q^3, \gamma, d) | (Q^1, Q^2, Q^3, \gamma, d) \in R_+^{m+2mn+2no+mo+o}$$

$$\text{and conservation of flow equations hold}\}$$

Variational Inequality Formulation

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the financial network model with intermediation are equivalent to the solution to the variational inequality problem given by: determine $(Q^{1*}, Q^{2*}, Q^{3*}, \gamma^*, d^*) \in \mathcal{K}^2$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}}\cdot q_i^* + 2V^j_{z_{il}}\cdot q_j^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma_j^*\right]\times\left[q_{ijl}-q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}}\cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times\left[q_{ik}-q^*_{ik}\right]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*},Q^{3*}) + \gamma_j^*\right]\times\left[q_{jkl}-q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl} - \sum_{k=1}^{n}\sum_{l=1}^{2}q^*_{jkl}\right]\times\left[\gamma_j-\gamma_j^*\right] - \sum_{k=1}^{o}\rho_{3k}(d^*)\times\left[d_k-d_k^*\right]\geq 0,$$

$$\forall (Q^1,Q^2,Q^3,\gamma,d)\in\mathcal{K}^2. \qquad (11)$$

Corollary 1

Corollary 1

The market for the financial flows clears for each intermediary at the financial network equilibrium.

$$\sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} = \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl},$$

- For notational convenience

$$g_j \equiv \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}, \quad i = 1, \ldots, m; \quad l = 1, 2.$$

$$c_j(Q^1) \equiv c_j\left(\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}\right) \equiv c_j(g_j)$$

$$\frac{\partial c_j(Q^1)}{\partial q_{ijl}} \equiv \frac{\partial c_j(g_j)}{\partial g_j},$$

Corollary 2

Corollary 2

A solution $(Q^{1*}, Q^{2*}, Q^{3*}, g^*, d^*) \in \mathcal{K}^4$ *to the variational inequality problem:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\frac{\partial c_j(g^*)}{\partial g_j} \times \left[g_j - g^*_j\right] - \sum_{k=1}^{o}\rho_{3k}(d^*) \times [d_k - d^*_k] \geq 0, \quad \forall (Q^1, Q^2, Q^3, g, d) \in \mathcal{K}^4, \qquad (18b)$$

where

$$\mathcal{K}^4 \equiv \{(Q^1, Q^2, Q^3, g, d) | (Q^1, Q^2, Q^3, g, d) \in R_+^{m+2mn+mo+2no+n+o}$$

and conservation of flow equations hold$\}$

satisfies variational inequality (11).

Overview of the Transportation Network Equilibrium Model with Fixed Demands

- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.
- Dafermos, S. (1980), Traffic equilibrium and variational inequalities. *Transportation Science* 14, 42-54.
- In equilibrium, the following conditions must hold for each O/D pair and each path.

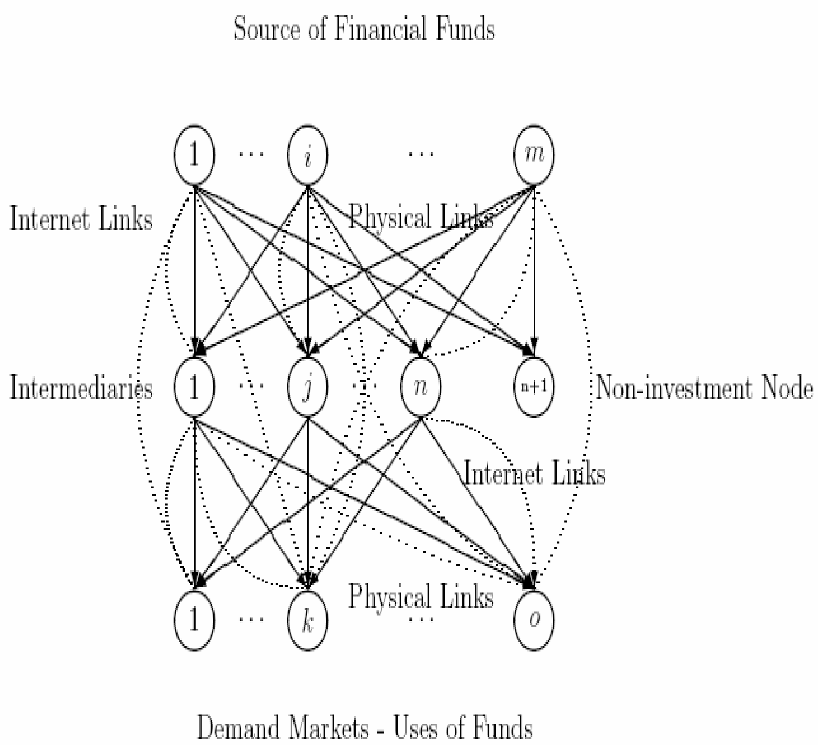
$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases}$$

- A path flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality:

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^* \right] \geq 0, \quad \forall x \in \mathcal{K}^5.$$

where $\mathcal{K}^5 \equiv \{x | x \geq 0$, and the conservation of flow equations holds$\}$.

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation



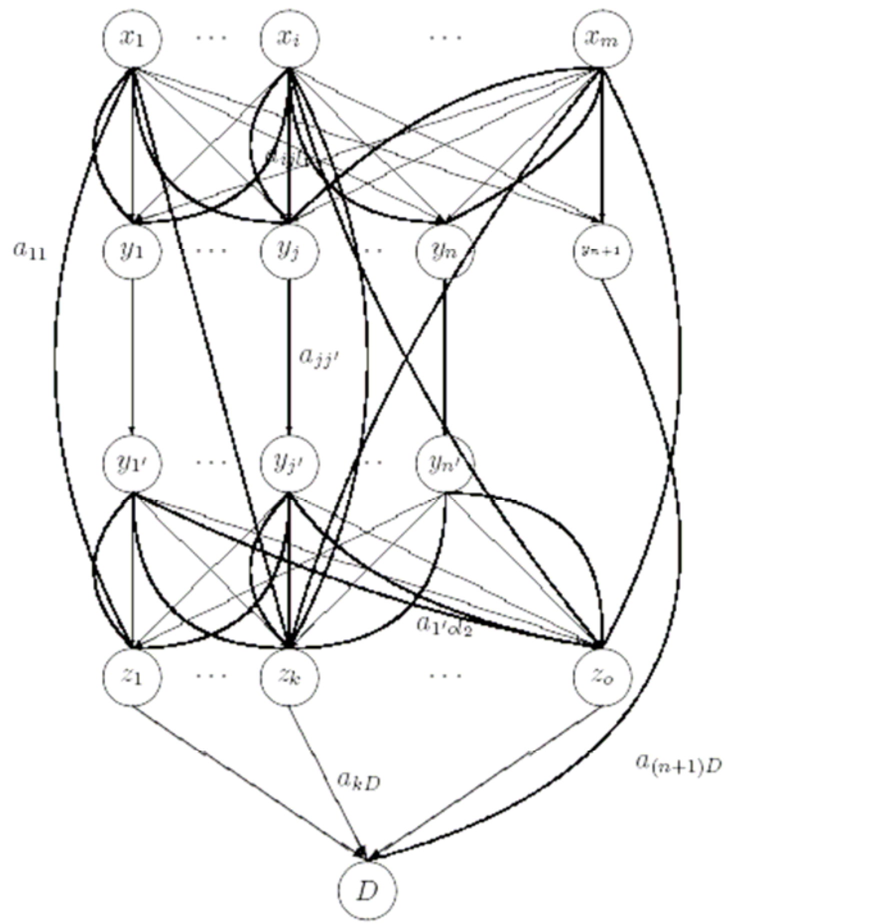


Figure 2: The G_S Supernetwork Representation of Financial Network Equilibrium

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- The following conservation of flow equations must hold on the equivalent transportation network:

$$f_{a_{ijl_1}} = \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad i=1,\ldots,m;\ j=1,\ldots,n;\ l_1=1,2,$$

$$f_{a_{jj'}} = \sum_{i=1}^{m} \sum_{l_1=1}^{2} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad j=1,\ldots,n;\ j'=1',\ldots,n',$$

$$f_{a_{j'kl_2}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l_1=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad j'=1',\ldots,n';\ k=1,\ldots,o;\ l_2=1,2,$$

$$f_{a_{kD}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{j'=1'}^{n'} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}} + \sum_{i=1}^{m} x_{p_{ik}}, \quad k=1,\ldots,o,$$

$$f_{a_{ik}} = x_{p_{ik}}, \quad i=1,\ldots,m;\ k=1,\ldots,o,$$

$$f_{a_{i(n+1)}} = x_{p_{i(n+1)}}, \quad i=1,\ldots,m,$$

$$f_{a_{(n+1)D}} = \sum_{i=1}^{m} x_{p_{i(n+1)}}.$$

$$d_{w_i} = \sum_{j=1}^{n} \sum_{l_1=1}^{2} \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}} + \sum_{k=1}^{o} x_{p_{ik}} + x_{p_{i(n+1)}}, \quad i=1,\ldots,m.$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- We can construct a feasible link flow pattern for the equivalent transportation network based on the corresponding feasible financial flow pattern in the financial network model in the following way:

$$q_{ijl} \equiv f_{a_{ijl_1}}, \quad i = 1, \ldots, m; \; j = 1, \ldots, n; \; l = 1, 2; \; l_1 = l,$$

$$q_{i(n+1)} \equiv f_{i(n+1)}, \quad i = 1, \ldots, m,$$

$$g_j \equiv f_{a_{jj'}}, \quad j = 1, \ldots, n; \; j' = 1', \ldots, n',$$

$$q_{jkl} \equiv f_{a_{j'kl_2}}, \quad j = 1, \ldots, n; j' = 1', \ldots, n'; \; k = 1, \ldots, o; \; l = 1, 2; \; l_2 = l,$$

$$q_{ik} \equiv f_{a_{ik}}, \quad i = 1, \ldots, m; \; k = 1, \ldots, o,$$

$$d_k \equiv f_{a_{kD}}, \quad k = 1, \ldots, o.$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- We assign the travel cost on the links of the transportation network as follows:

We now assign costs on the links of the supernetwork G_S as follows: to each link a_{ijl_1} assign a cost $c_{a_{ijl_1}}$ given by:

$$c_{a_{ijl_1}} \equiv 2V^i_{z_{jl}} \cdot q_i + 2V^j_{z_{il}} \cdot q_j + \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; l_1 = l = 1, 2;$$

to each link $a_{i(n+1)}$ assign a cost $c_{a_{i(n+1)}}$:

$$c_{a_{i(n+1)}} \equiv 0, \quad i = 1, \ldots, m;$$

to each link a_{ik} assign a cost $c_{a_{ik}}$:

$$c_{a_{ik}} \equiv 2V^i_{z_{2n+k}} \cdot q_i + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2, Q^3), \quad i = 1, \ldots, m; k = 1, \ldots, o;$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

to each link $a_{j'kl_2}$ assign a cost $c_{a_{j'kl_2}}$:

$$c_{a_{j'kl_2}} \equiv 2V^j_{z_{kl}} \cdot q_j + \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^2, Q^3), \quad j = 1, \ldots, n; k = 1, \ldots, o; l_2 = l = 1, 2;$$

and to the link $a_{jj'}$ assign a cost: $c_{a_{jj'}}$

$$c_{a_{jj'}} \equiv \frac{\partial c_j(g)}{\partial g_j}, \quad j = 1, \ldots, n; j' = 1', \ldots, n'.$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

In addition, to the link $a_{(n+1)D}$ assign a cost $c_{a_{(n+1)D}}$:

$$c_{a_{(n+1)D}} \equiv M.$$

Finally, for each link a_{kD} assign a cost:

$$c_{a_{kD}} \equiv M - \rho_{3k}(d), \quad k = 1, \ldots, o,$$

where M is a scalar and defined by:

$$M \equiv \max_{k=1,\ldots,o} \sup_{d \in \mathcal{D}} (\rho_{3k}(d))$$

where $\mathcal{D} \equiv \{d | d \in R^o_+ \text{ and } d_k \leq \sum_{i=1}^m S^i, \quad \forall k = 1, \ldots, o\}$.

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- Path Flow Cost

A user ("traveler") of path $p_{ijl_1j'kl_2}$, for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $j' = j$; $k = 1, \ldots, o$; $l_1 = 1, 2$; $l_2 = 1, 2$, on the supernetwork G_S in Figure 2, incurs a path cost $C_{p_{ijl_1j'kl_2}}$ given by

$$C_{p_{ijl_1j'kl_2}} = 2V^i_{z_{jl_1}} \cdot q_i + 2V^j_{z_{il_1}} \cdot q_j + \frac{\partial c_{ijl_1}(q_{ijl_1})}{\partial q_{ijl_1}} + \frac{\partial \hat{c}_{ijl_1}(q_{ijl_1})}{\partial q_{ijl_1}} + \frac{\partial c_j(g_j)}{\partial g_j}$$

$$+2V^j_{z_{kl_2}} \cdot q_j + \frac{\partial c_{jkl_2}(q_{jkl_2})}{\partial q_{jkl_2}} + \hat{c}_{jkl_2}(Q^2, Q^3) + M - \rho_{3k}(d).$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- Path Flow Cost

A user of path p_{ik}, for $i = 1, \ldots, m$; $k = 1, \ldots, o$; on G_S in incurs a path cost $C_{p_{ik}}$ given by

$$C_{p_{ik}} = 2V^i_{z_{2n+k}} \cdot q_i + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2, Q^3) + M - \rho_{3k}(d).$$

A user of path $p_{i(n+1)}$, for $i = 1, \ldots, m$, on G_S incurs a path cost $C_{p_{i(n+1)}}$ given by

$$C_{p_{i(n+1)}} = M.$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- We assign the fixed travel demands associated with the O/D pairs as follows:

$$d_{w_i} = S^i, \quad i = 1, \ldots, m.$$

- For each O/D pair w_i and each path connecting this O/D pair:

$$C_{p_{ijl_1j'kl_2}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{ijl_1j'kl_2}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{ijl_1j'kl_2}} = 0, \end{cases}$$

$$C_{p_{ik}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{ik}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{ik}} = 0, \end{cases}$$

$$C_{p_{i(n+1)}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{i(n+1)}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{i(n+1)}} = 0. \end{cases}$$

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation

- The variational inequality in link flow form

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^{i}_{z_{jl}}\cdot q_i^* + 2V^{j}_{z_{il}}\cdot q_j^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right]\times\left[q_{ijl}-q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^{i}_{z_{2n+k}}\cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times\left[q_{ik}-q^*_{ik}\right]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^{j}_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*},Q^{3*})\right]\times\left[q_{jkl}-q^*_{jkl}\right] + \sum_{j=1}^{n}\frac{\partial c_j(g^*)}{\partial g_j}\times\left[g_j-g_j^*\right]$$

$$-\sum_{k=1}^{o}\rho_{3k}(d^*)\times\left[d_k-d_k^*\right]\geq 0, \quad \forall (Q^1,Q^2,Q^3,g,d)\in\mathcal{K}^4. \qquad (60)$$

- VI (60) is precisely VI(18b) governing the Financial Network Equilibrium

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Theorem

A solution $(Q^{1^*}, Q^{2^*}, Q^{3^*}, d^*) \in \mathcal{K}^2$ *of the variational inequality (18b) governing a financial network equilibrium with intermediation and electronic transactions coincides with the feasible link flow pattern for the supernetwork* G_S *constructed above and satisfies variational inequality (60). Hence, it is a transportation network equilibrium pattern on the supernetwork* G_S.

The Euler Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$ and set $T = 0$. T is an iteration counter which may also be interpreted as a time period.

Step 1: Computation

Compute X^{T+1} by solving the variational inequality problem:

$$X^{\tau+1} = P_{\mathcal{K}^5}(X^\tau - a_\tau C(X^\tau)),$$

where $\{\alpha_T\}$ is a sequence of positive scalers satisfying: $\sum_{T=0}^{\infty} \alpha_T = \infty$, $\alpha_T \to 0$ as $T \to \infty$. and $P_{\mathcal{K}}$ is the projection of X on the set $\mathcal{K}$ defined as:

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \parallel X - z \parallel .$$

Step 2: Convergence Verification

If $\parallel X^{T+1} - X^T \parallel \leq \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; else, set $T = T + 1$, and go to Step 1.

The Euler Method

- The subproblem can be rewritten as:

$$X^{\tau+1} = \arg\min_{X \in \mathcal{K}^5} \frac{1}{2} X^T \cdot X - (X^\tau - a_\tau C(X^\tau))^T \cdot X.$$

- The solution of the above subproblem is equivalent to the solution of: for each O/D pair w, compute:

$$\min \frac{1}{2} \sum_{p \in P_w} X_p^2 + \sum_{p \in P_w} h_p^\tau X_p$$

subject to:

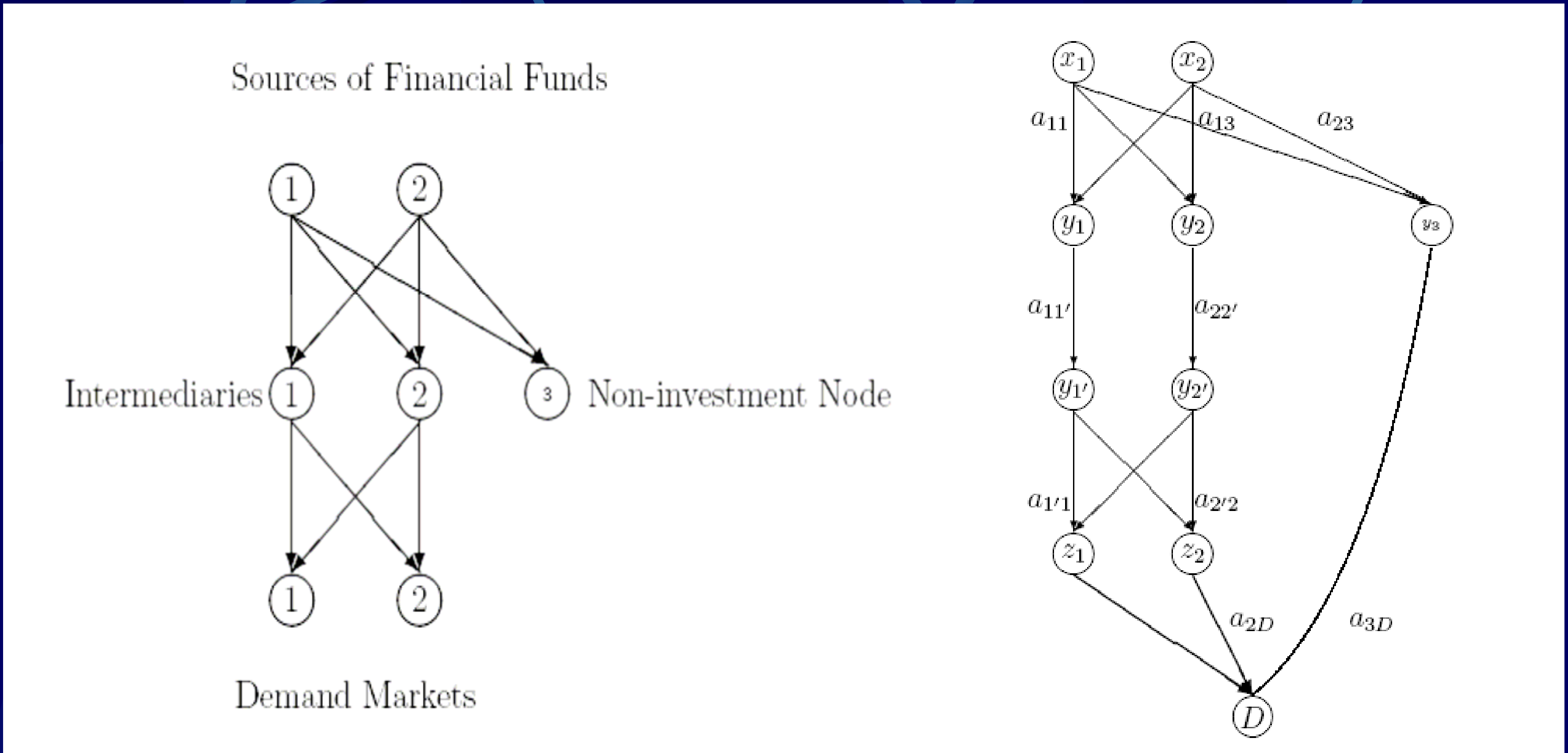
$$\sum_{p \in P_w} X_p = d_w$$

$$X_p \geq 0, \quad \forall p \in P_w,$$

where

$$h_p^\tau = a_\tau C_p(X^\tau) - X_p^\tau.$$

Numerical Examples



Numerical Examples

- Example 1

The financial holdings for the two source agents in the first example were: $S^1 = 10$ and $S^2 = 10$. The variance-covariance matrices V^i were identity matrices for all the source agents $i = 1, 2$. The variance-covariance matrices V^j, in turn, for intermediaries $j = 1, 2$ consisted of an identity submatrix associated with the q_{jk}; $k = 1, 2$ variables, with all other terms being equal to zero. We have suppressed the subscript l associated with the transaction cost functions since we have assumed a single (physical) mode of transaction only being available.

The transaction cost functions of the source agents associated with their transactions with the intermediaries were given by:

$$c_{ij}(q_{ij}) = 2q_{ij}^2 + q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2$$

Numerical Examples

- Example 1

The transaction cost functions of the intermediaries associated with transacting with the sources agents were given by:

$$\hat{c}_{ij}(q_{ij}) = 3q_{ij}^2 + 2q_{ij} + 1, \quad \text{for } i = 1,2; j = 1,2,$$

The handling costs of the intermediaries were:

$$c_1(Q^1) = 0.5(q_{11} + q_{21})^2, \quad c_2(Q^1) = 0.5(q_{12} + q_{22})^2,$$

The inverse demand (demand market price) functions at the demand markets were:

$$\rho_{3k}(d) = -2d_k + 100, \quad \text{for } k = 1,2.$$

The transaction costs between the intermediaries and the consumers at the demand markets, in turn, were given by:

$$\hat{c}_{jk} = q_{jk} + 2, \quad \text{for } j = 1,2; k = 1,2.$$

Numerical Examples

- Example 2
  - the data were identical to those in Example 1, except that the financial holdings of the source agents S_i; $i = 1, 2$, were now both equal to 6.
- Example 3
  - The third numerical example had the same data as Example 1 except that the handling cost of the *second intermediary* was given by:

$$c_2(Q^1) = (q_{12} + q_{22})^2,$$

Numerical Examples

- Example 4
  - Example 4 was constructed from Example 1 and had the identical data except that the demand market price (inverse demand) functions were now changed to:

$$\rho_{31}(d) = -1.14d_1 + .858d_2 + 281.71, \quad \rho_{32}(d) = -1.14d_2 + .858d_1 + 281.71,$$

- Example 5
  - Example 5, in turn, had the same data as Example 2 (S_i=6; i = 1, 2), but with the inverse demand functions as in Example 4,

Numerical Examples

- Example 6
  - Example 6 had the same data as Example 3 (S_i=6; i = 1, 2) except that the demand market price functions were as in Examples 4 and 5, i.e.

$$\rho_{31}(d) = -1.14d_1 + .858d_2 + 281.71, \quad \rho_{32}(d) = -1.14d_2 + .858d_1 + 281.71,$$

Solutions of the Numerical Examples

Table 2: Path Flow Solutions to Examples 1, 2, 3, 4, 5, and 6

Path Flows	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6
$x^*_{p_1}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_2}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_3}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_4}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_5}$	1.74	0.00	2.07	0.00	0.00	0.00
$x^*_{p_6}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_7}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_8}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_9}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_{10}}$	1.74	0.00	2.07	0.00	0.00	0.00

Solutions of the Numerical Example

Table 3: Link Flow Solutions to Examples 1, 2, 3, 4, 5, and 6

Link Flows	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6
$f^*_{a_{11}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{12}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{13}}$	1.74	0.00	2.07	0.00	0.00	0.00
$f^*_{a_{21}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{22}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{23}}$	1.74	0.00	2.07	0.00	0.00	0.00
$f^*_{a_{11'}}$	8.26	6.00	8.33	10.00	6.00	10.50
$f^*_{a_{22'}}$	8.26	6.00	7.54	10.00	6.00	9.50
$f^*_{a_{1'1}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{1'2}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{2'1}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{2'2}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{1D}}$	8.26	6.00	7.93	10.00	6.00	10.00
$f^*_{a_{2D}}$	8.26	6.00	7.93	10.00	6.00	10.00
$f^*_{a_{3D}}$	3.48	0.00	4.14	0.00	0.00	0.00

Conclusions

- We first presented a new model of financial network equilibrium with intermediation and electronic transactions in which the inverse demand functions were assumed known.
- We constructed the supernetwork representation of the financial network problem, which corresponds to an isomorphic transportation network equilibrium problem with fixed demands.
- This equivalence allowed us to provide a novel interpretation of the equilibrium conditions of the financial network problems with intermediation in terms of paths and path flows.

Conclusions

- We then proposed and applied an algorithm for the computation of solutions to fixed demand transportation network equilibrium problems by Nagurney and Zhang (1997) to compute solutions to six numerical financial network problems.

- The computational results yielded information that was not previously available since both the equilibrium path flows as well as the equilibrium link flows were now obtained.

Thank You Very Much!

For more information, please see:

The Virtual Center for Supernetworks

http://supernet.som.umass.edu