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Dynamic Electric Power Supply Chains and Transportation Networks: an Evolutionary Variational Inequality Formulation (To appear in Transportation Research E)

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Electricity is Modernity



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Motivation

- In US: half a trillion dollars worth of net assets
- Consumes almost 40% of domestic primary energy
- Electric power supply chains, provide the foundations for the functioning of our modern economies and societies.
 - Communication, transportation, heating, lighting, cooling, computers and electronics.
 - August 14, 2003, blackout in the Midwest, the Northeastern United States, and Ontario, Canada.
 - Two significant power outages during the month of September 2003 – one in England and one in Switzerland and Italy.
- Deregulation: from vertically integrated to competitive markets
 - ◆ In US, Europe and many other countries
- Inelastic, seasonal demand.



Literature

- Kahn (1998)
- Day et al. (2002)
- Schweppe et al. (1988)
- Hogan (1992)

- Chao and Peck (1996)
- Wu et al. (1996)
- Willems (2002)
- Casazza and Delea (2003)
- Zaccour (1998) and Singh (1999).



Objectives

- The objective of this research was to develop a dynamic electric power supply chain network equilibrium model with exogenous time-varying demand
- The theory that has originated from the study of transportation networks was utilized to construct this time-dependent equilibrium modeling framework for electric power supply chain networks
- The new dynamic electric power supply chain network model that we developed in this research is also motivated by the unification of projected dynamical systems theory and evolutionary (infinite-dimensional) variational inequalities



Outline

- The static electric power network model with fixed demands
- The supernetwork equivalence of the electric power supply chain networks and the transportation networks
 - Overview of the transportation network equilibrium models
 - The supernetwork equivalence of the transportation networks and the electric power supply chain networks with fixed demands
- The electric power supply chain network model with time-varying demands
 - Evolutionary variational inequalities and projected dynamical systems; Applications to transportation network equilibrium
 - The computation of the electric power supply chain network equilibrium model with time-varying demands.



Some of the Related Literature

- Beckmann, M. J., McGuire, C. B., and Winsten, C. B. (1956), Studies in the Economics of Transportation. Yale University Press, New Haven, Connecticut.
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- Nagurney, A., Dong, J., and Zhang, D. (2002), A Supply Chain Network Equilibrium Model, *Transportation Research E* 38, 281-303.
- Nagurney, A (2005), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, Appears in *Transportation Research E* 42: (2006) pp 293-316.



Some of the Related Literature (Cont'd)

- Nagurney, A. and Matsypura, D. (2004), A Supply Chain Network Perspective for Electric Power Generation, Supply, Transmission, and Consumption, *Proceedings of the International Conference on Computing, Communications and Control Technologies*, Austin, Texas, Volume VI: (2004) pp 127-134.
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The Electric Power Supply Chain Network Equilibrium Model with Fixed Demands



Demand Markets



The Behavior of Power Generator and Their Optimality Conditions

Conservation of flow equations must hold for each generator

$$\sum_{s=1}^{S} q_{gs} = q_g, \quad g = 1, \dots, G.$$

Power generator's optimization problem

Maximize
$$\sum_{s=1}^{S} \rho_{1gs}^* q_{gs} - f_g(Q^1) - \sum_{s=1}^{S} c_{gs}(q_{gs})$$
subject to:
$$q_{gs} \ge 0, \qquad s = 1, \dots, S.$$

The optimality conditions of the generators

$$\sum_{g=1}^{G}\sum_{s=1}^{S} \left[\frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(q_{gs}^*)}{\partial q_{gs}} - \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \ge 0, \quad \forall Q^1 \in R_+^{GS}.$$



The Behavior of Power Suppliers

Supplier's optimization problem

Maximize
$$\sum_{k=1}^{K} \sum_{v=1}^{V} \rho_{2sk}^{v*} q_{sk}^{v} - c_s(Q^1) - \sum_{g=1}^{G} \rho_{1gs}^* q_{gs} - \sum_{g=1}^{G} \hat{c}_{gs}(q_{gs}) - \sum_{k=1}^{K} \sum_{v=1}^{V} c_{sk}^v(q_{sk}^v)$$

subject to:

$$\sum_{k=1}^{K} \sum_{v=1}^{V} q_{sk}^{v} = \sum_{g=1}^{G} q_{gs}$$
$$q_{gs} \ge 0, \quad g = 1, \dots, G,$$
$$q_{sk}^{v} \ge 0, \quad k = 1, \dots, K; v = 1, \dots, V$$

For notational convenience, we let

$$h_s \equiv \sum_{g=1}^G q_{gs}, \quad s = 1, \dots, S.$$



The Optimality Conditions of the Power Suppliers

The optimality conditions of the suppliers

$$\sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[\frac{\partial c_{sk}^v(q_{sk}^{v*})}{\partial q_{sk}^v} - \rho_{2sk}^{v*} \right] \times [q_{sk}^v - q_{sk}^{v*}]$$
$$+ \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} + \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \ge 0, \qquad \forall (Q^1, Q^2, h) \in \mathcal{K}^3,$$

where $\mathcal{K}^3 \equiv \{(h, Q^2, Q^1) | (h, Q^2, Q^1) \in \mathbb{R}^{S(1+VK+G)}_+$ and conservation of flow equations hold $\}$.



The Equilibrium Conditions at the Demand Markets

Conservation of flow equations must hold

$$d_k = \sum_{s=1}^{S} \sum_{v=1}^{V} q_{sk}^v, \quad k = 1, \dots, K.$$

The vector (Q^{2*}, ρ_{3}^{*}) is an equilibrium vector if for each *s*, *k*, *v* combination:

$$\rho_{2sk}^{v*} + \hat{c}_{sk}^{v}(Q^{2*}) \begin{cases} = \rho_{3k}^{*}, & \text{if} \quad q_{sk}^{v*} > 0, \\ \ge \rho_{3k}^{*}, & \text{if} \quad q_{sk}^{v*} = 0, \end{cases}$$



Electric Power Supply Chain Network Equilibrium (For Fixed Demands at the Markets)

Definition 1: The equilibrium state of the electric power supply chain network is one where the electric power flows between the tiers of the network coincide and the electric power flows satisfy the sum of the optimality conditions of the power generators and the suppliers, and the equilibrium conditions at the demand markets.



Variational Inequality Formulation

Theorem 1: Variational Inequality Formulation of the Electric Power Supply Chain Network Equilibrium

The equilibrium conditions governing the electric power supply chain network according to Definition 1 coincide with the solution of the variational inequality given by: determine $(q^*, h^*, Q^{1*}, Q^{2*}) \in \mathcal{K}^5$ satisfying:

$$\begin{split} \sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial c_{gs}(q_{gs}^*)}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} \right] \times [q_{gs} - q_{gs}^*] \\ + \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[\frac{\partial c_{sk}^v(q_{sk}^{v*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}] \ge 0, \quad \forall (q, h, Q^1, Q^2) \in \mathcal{K}^5, \quad (20) \\ where \ \mathcal{K}^5 \equiv \{(q, h, Q^1, Q^2) | (q, h, Q^1, Q^2) \in \mathbb{R}_+^{G+S+GS+VSK} \\ and \ conservation \ of \ flow \ equations \ hold\}. \end{split}$$



The Supernetwork Equivalence of Supply Chain Network Equilibrium and Transportation Network Equilibrium

Nagurney, A. (2006), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, *Transportation Research E (2006)* 42: (2006) pp 293-316





- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.
- Dafermos, S. (1980), Traffic equilibrium and variational inequalities. *Transportation Science* 14, 42-54.
- In equilibrium, the following conditions must hold for each O/D pair and each path. $f = 0, \text{ if } x_p^* > 0,$

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if } x_p^* > 0, \\ \ge 0, & \text{if } x_p^* = 0. \end{cases}$$

• A path flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality:

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^* \right] \ge 0, \quad \forall x \in \mathcal{K}^6.$$



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Demand Markets



Figure 2



Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands The following conservation of flow equations must hold on the equivalent transportation network: $f_{a_g} = \sum_{s=1}^{\infty} \sum_{s'=1}^{\infty} \sum_{k=1}^{\infty} \sum_{v=1}^{\infty} x_{p_{gss'k}^v}, \quad g = 1, \dots, G,$ $f_{a_{gs}} = \sum_{s'=1}^{-} \sum_{k=1}^{-} \sum_{v=1}^{-} x_{p_{gss'k}^v}, \quad g = 1, \dots, G; s = 1, \dots, S,$

$$f_{a_{ss'}} = \sum_{g=1}^{G} \sum_{k=1}^{K} \sum_{v=1}^{V} x_{p_{gss'k}^{v}}, \quad ss' = 11', \dots, SS',$$

$$f_{a_{s'k}^{v}} = \sum_{g=1}^{G} \sum_{s=1}^{S} x_{p_{gss'k}^{v}}, \quad s' = 1, \dots, S'; v = 1, \dots, V; k = 1, \dots, K.$$
$$d_{w_{k}} = \sum_{g=1}^{G} \sum_{s=1}^{S} \sum_{s'=1}^{S'} \sum_{v=1}^{V} x_{p_{gss'k}^{v}}, \quad k = 1, \dots, K.$$



We can construct a feasible link flow pattern for the equivalent transportation network based on the corresponding feasible electric power flow pattern in the electric power supply chain network model in the following way:

$$q_{g} \equiv f_{a_{g}}, \quad g = 1, \dots, G,$$

$$q_{gs} \equiv f_{a_{gs}}, \quad g = 1, \dots, G; s = 1, \dots, S,$$

$$h_{s} \equiv f_{a_{ss'}}, \quad ss' = 11', \dots, SS',$$

$$q_{sk}^{v} = f_{a_{s'k}^{v}}, \quad s = s' = 1', \dots, S'; v = 1, \dots, V; k = 1, \dots, K,$$

$$d_{k} = \sum_{s=1}^{S} \sum_{v=1}^{V} q_{sk}^{v}, \quad k = 1, \dots, K.$$



We assign user (travel) costs on the links of the transportation network as follows:

$$c_{a_g} \equiv \frac{\partial f_g}{\partial q_g}, \quad g = 1, \dots, G,$$

$$c_{a_{gs}} \equiv \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}}, \quad g = 1, \dots, G; s = 1, \dots, S,$$

$$c_{a_{ss'}} \equiv \frac{\partial c_s}{\partial h_s}, \quad ss' = 11', \dots, SS'.$$

$$c_{a_{s'k}^v} \equiv \frac{\partial c_{sk}^v}{\partial q_{sk}^v} + \hat{c}_{sk}^v, \quad s' = s = 1, \dots, S; v = 1, \dots, V; k = 1, \dots, K$$



Path cost

$$C_{p_{gss'k}^{v}} = c_{a_g} + c_{a_{gs}} + c_{a_{ss'}} + c_{a_{ss'}^{v}} = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c_{sk}^{v}}{\partial q_{sk}^{v}} + \hat{c}_{sk}^{v}.$$
(44)

We assign the (travel) demands associated with the O/D pairs as follows:

$$d_{w_k} \equiv d_k, \quad k = 1, \dots, K,$$

The (travel) disutilities:

$$\lambda_{w_k} \equiv \rho_{3k}, \quad k = 1, \dots, K.$$

The equilibrium conditions:

$$C_{p_{gss'k}^v} - \lambda_{w_k}^* = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c_{sk}^v}{\partial q_{sk}^v} + \hat{c}_{sk}^v - \rho_{3k}^* \begin{cases} = 0, & \text{if } x_{p_{gss'k}}^* > 0, \\ \ge 0, & \text{if } x_{p_{gss'k}}^* = 0. \end{cases}$$



Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

The variational inequality in link flow form

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial c_{gs}(q_{gs}^*)}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} \right] \times [q_{gs} - q_{gs}^*]$$
$$+ \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[\frac{\partial c_{sk}^v(q_{sk}^{v*})}{\partial q_{sk}} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}] \ge 0, \quad \forall (q, h, Q^1, Q^2) \in \mathcal{K}^5.$$
(49)



Theorem

A solution $(q^*, h^*, Q^{1^*}, Q^{2^*}) \in \mathcal{K}^5$ of the variational inequality (20) governing the electric power supply chain network equilibrium coincides with the feasible link flow pattern for the supernetwork $\mathcal{G}_{\mathcal{S}}$ constructed above and satisfies variational inequality (49). Hence, it is a transportation network equilibrium

Theorem

A path flow pattern on the supernetwork in Figure 2 is a transportation network equilibrium if and only if it satisfies the variational inequality problem: determine $x_{p_{gss'k}}^* \ge 0$, for all g, s, s', v, k and satisfying conservation of flow equations such that

$$\sum_{g=1}^{G} \sum_{s=1}^{S} \sum_{s'=1'}^{S'} \sum_{v=1}^{V} \sum_{k=1}^{K} C_{p_{gss'k}^{v}}(x^{*}) \times \left[x_{p_{gss'k}^{v}} - x_{p_{gss'k}^{v}}^{*} \right] \ge 0,$$

 $\forall x_{p_{gss'k}^v} \geq 0, \forall g, s, s', v, k \text{ and satisfying conservation of flow equations}$



Finite-Dimentional Variational Inequalities and Projected Dynamical Systems Literature

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More Finite-Dimentional Variational Inequalities Literature

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More Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Cojocaru, M.-G., Daniele, P., Nagurney, A. (2005c). Projected dynamical systems, evolutionary variational inequalities, applications, and a computational procedure. *Pareto Optimality, Game Theory and Equilibria*. A. Migdalas, P. M. Pardalos, and L. Pitsoulis, editors, Springer Verlag.
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Finite-Dimensional Projected Dynamical Systems

- Finite-Dimensional Projected Dynamical Systems (PDSs) (Dupuis and Nagurney (1993))
 - PDS_t describes how the state of the network system approaches an equilibrium point on the curve of equilibria at time t.
 - For almost every moment 't' on the equilibria curve, there is a PDS_t associated with it.
 - A PDS_t is usually applied to study small scale time dynamics, i.e [t, t+τ]



Finite-Dimensional Projected Dynamical Systems

Definition:

$$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))).$$

In this formulation, \mathcal{K} is a convex polyhedral set in \mathbb{R}^n , $F: \mathcal{K} \to \mathbb{R}^n$ is a Lipschitz con-

tinuous function with linear growth and $\Pi_{\mathcal{K}}: R \times \mathcal{K} \to R^n$ is the Gateaux directional

derivative

$$\Pi_{\mathcal{K}}(x,-F(x)) = \lim_{\delta \to 0^+} \frac{P_{\mathcal{K}}(x-\delta F(x))-x}{\delta}$$

of the projection operator $P_{\mathcal{K}}: \mathbb{R}^n \to \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$



Projected Dynamical Systems and Finite-Dimensional Variational Inequalities

Theorem

The equilibria of a PDS:

$$\frac{\partial x(t)}{\partial t} = \Pi_{\mathcal{K}}(x(t), -F(x(t)))$$

that is, $x^* \in \mathcal{K}$ such that

 $\Pi_{\mathcal{K}}(x^*, -F(x^*)) = 0$

are solutions to the VI(F, K): find $x^* \in \mathcal{K}$ such that

$$\langle F(x^*), x - x^* \rangle \ge 0, \quad \forall x \in \mathcal{K}$$

and vice versa.



Infinite-Dimensional Projected Dynamical Systems

Definition:
$$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))), \quad x(t,0) \in \hat{\mathcal{K}},$$

where
$$\Pi_{\hat{\mathcal{K}}}(y, -F(y)) = \lim_{\delta \to 0^+} \frac{P_{\hat{\mathcal{K}}}((y - \delta F(y)) - y)}{\delta}, \quad \forall y \in \hat{\mathcal{K}}$$

with the projection operator $P_{\hat{\mathcal{K}}} : H \to \hat{\mathcal{K}}$ given by $\|P_{\hat{\mathcal{K}}}(z) - z\| = \inf_{y \in \hat{\mathcal{K}}} \|y - z\|,$

The feasible set $\hat{\mathcal{K}}$ defined as follows

$$\hat{\mathcal{K}} = \bigcup_{t \in [0,T]} \left\{ u \in L^p([0,T], R^q) \, | \, \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \right.$$

 $\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,...,q\}, j \in \{1,...,l\} \Big\}.$



Evolutionary Variational Inequalities

Evolutionary Variational Inequalities (EVIs)

- EVI provides a curve of equilibria of the network system over a finite time interval [0,T]
- An EVI is usually used to model large scale time, i.e, [0, T]
- EVIs have been applied to time-dependent equilibrium problems in transportation, and in economics and finance.



Evolutionary Variational Inequalities

Define

$$\langle \langle \Phi, x \rangle \rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$$

EVI:

determine
$$x \in \hat{\mathcal{K}} : \langle \langle F(x), z - x \rangle \rangle \ge 0, \quad \forall z \in \hat{\mathcal{K}}.$$

where

$$\hat{\mathcal{K}} = \bigcup_{t \in [0,T]} \Big\{ u \in L^p([0,T], \mathbb{R}^q) \, | \, \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \\ \sum_{i=1}^q \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,...,q\}, j \in \{1,...,l\} \Big\}.$$



Projected Dynamical Systems and Evolutionary Variational Inequalities

Cojocaru, Daniele, and Nagurney (2005b) showed the following:

Theorem

Assume that $\hat{\mathcal{K}} \subseteq H$ is non-empty, closed and convex and $F : \hat{\mathcal{K}} \to H$ is a pseudo-monotone Lipschitz continuous vector field, where H is a Hilbert space. Then the solutions of EVI are the same as the critical points of the projected differential equation, that is, they are the functions $x \in \hat{\mathcal{K}}$ such that

 $\Pi_{\hat{\mathcal{K}}}(x(t), -F(x(t))) = 0,$

and vice versa.



Projected Dynamical Systems and Evolutionary Variational Inequalities

The solutions to the evolutionary variational inequality:

determine
$$x \in \hat{\mathcal{K}}$$
: $\int_0^T \langle F(x(t)), z(t) - x(t) \rangle dt \ge 0, \quad \forall z \in \hat{\mathcal{K}},$

are the same as the critical points of the equation:

$$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))),$$

that is, the points such that

 $\Pi_{\hat{\mathcal{K}}}(x(t,\tau),-F(x(t,\tau)))\equiv 0 \text{ a.e. in } [0,T],$



A Pictorial of EVIs and PDSs





The EVI Formulation of the Transportation Network Model with Time-Varying Demands

Define
$$\langle \langle \Phi, x \rangle \rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$$

EVI Formulation:

determine
$$x \in \hat{\mathcal{K}} : \langle \langle C(x), z - x \rangle \rangle \ge 0, \quad \forall z \in \hat{\mathcal{K}},$$

where C is the vector of path costs.

Feasible set

$$\hat{\mathcal{K}} = \left\{ x \in L^2([0,T], R^Q) : 0 \le x(t) \le \mu \text{ a.e. in } [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0,T] \right\}$$



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The Numerical Solution of Evolutionary Variational Inequalities (Cojocaru, Daniele, and Nagurney (2005 a, b, c))

- The vector field *F* satisfies the requirement in the preceding Theorem.
- We first discretize time horizon T. (Barbagallo, A., (2005))
- At each fixed time point, we solve the associated finite dimensional projected dynamical system PDS_t
- We use the Euler method to solve the finite dimensional projected dynamical system PDS_t.



The Euler Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$ and set T = 0. T is an iteration counter which may also be interpreted as a time period.

Step 1: Computation

Compute X^{T+1} by solving the variational inequality problem:

$$X^{T+1} = P_{\mathcal{K}}(X^T - \alpha_T F(X^T)),$$

where $\{\alpha_T\}$ is a sequence of positive scalars satisfying: $\sum_{T=0}^{\infty} \alpha_T = \infty, \ \alpha_T \to 0 \text{ as } T \to \infty.$ and $P_{\mathcal{K}}$ is the projection of X on the set \mathcal{K} defined as:

$$y = P_{\mathcal{K}}X = \arg\min_{z\in\mathcal{K}} ||X - z||.$$

Step 2: Convergence Verification

If $||X^{T+1} - X^T|| \le \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; else, set T = T + 1, and go to Step 1.



The EVI Formulation of the Electric Power Network Model with Time-Varying Demands

- We know that the electric power supply chain network equilibrium problem with fixed demands can be reformulated as a fixed demand transportation network equilibrium problem in path flows over the equivalent transportation network.
- Evolutionary variational inequality provides us with a dynamic version of the electric power supply chain network problem in which the demands and path costs vary over time.
- Evolutionary variational inequality is a dynamic (and infinitedimensional) version of variational inequality with the path costs defined in (44).



Solving Electric Power Supply Chain Network Model with Time-Varying Demands

First, construct the equivalent transportation network equilibrium model

- Solve the transportation network equilibrium model with time-varying demands
- Convert the solution of the transportation network into the timedependent electric power supply chain network equilibrium model



Dynamic Electric Power Supply Chain Network Examples with Computations

Example 1









Generating cost functions

 $f_1(q_1(t)) = 2.5(q_1(t))^2 + 2q_1(t).$

Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5(q_{12}(t))^2 + 2.5q_{12}(t),$$





Operating cost functions of the suppliers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

Unit transaction cost between the suppliers and the demand markets

$$\hat{c}_{11}^1(Q^2(t)) = q_{11}^1(t) + 1, \quad \hat{c}_{21}^1(Q^2(t)) = q_{21}^1(t) + 5, \quad \hat{c}_{31}^1(Q^2(t)) = q_{31}^1(t) + 10.$$



Three paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

The time-varying demand function

$$d_{w_1}(t) = d_1(t) = 41 + 10t.$$



Explicit Solution

• Path flows

$$\begin{aligned} x_{p_1}^*(t) &= 3.33t + 14.78, \\ x_{p_2}^*(t) &= 3.33t + 13.78, \\ x_{p_3}^*(t) &= 3.33t + 12.45, \end{aligned}$$

Travel disutility

$$\lambda_{w_1}^*(t) = 60t + 255.83, \quad \text{for} \quad t \in [0, T].$$



Time-Dependent Equilibrium Path Flows for Numerical Example 1







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Corresponding Supernetwork





Corresponding Supernetwork





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Corresponding Supernetwork



- The network structure and the cost functions are the same as the first example.
- The demand function is the step function:

$$d_1(t) = \begin{cases} 100, & \text{if } 0 < t \le t_1 = \frac{1}{2}, \\ 110, & \text{if } t_1 < t \le t_2 = T = 1. \end{cases}$$

The explicit solution:

$$x^*(t) = (x^*_{p_1}(t), x^*_{p_2}(t), x^*_{p_3}(t)) = \begin{cases} (34.44, 33.44, 32.12), & \text{if} \quad 0 < t \le t_1 = \frac{1}{2}, \\ (37.77, 36.77, 35.45), & \text{if} \quad t_1 < t \le t_2 = 1 = T. \end{cases}$$



Time-Dependent Equilibrium Path Flows for Numerical Example 2







Corresponding Supernetwork



Generating cost functions

 $f_1(q(t)) = 2.5(q_1(t))^2 + q_1(t)q_2(t) + 2q_1(t), \quad f_2(q(t)) = 2.5(q_2(t))^2 + q_2(t)q_1(t) + 2q_2(t).$

Transaction cost functions of the products

 $c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{21}(q_{21}(t)) = .5(q_{21}(t))^2 + 1.5q_{21}(t).$



Operating cost function of the suppliers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2.$$

Unit transaction costs between the suppliers and the demand markets

$$\hat{c}_{11}^1(Q^2(t)) = q_{11}^1(t) + 1, \quad \hat{c}_{12}^1(Q^2(t)) = q_{12}^1(t) + 1,$$



Four paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), p_2 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$
$$p_3 = (a_1, a_{11}, a_{11'}, a_{1'2}), p_4 = (a_2, a_{21}, a_{11'}, a_{1'2})$$

The time-varying demand functions

$$d_{w_1}(t) = d_1(t) = 100 + 5t, \quad d_{w_2}(t) = d_2(t) = 80 + 4t.$$



Numerical Solution

◆ t=0

$$x_{p_1}^* = 49.90, \quad x_{p_2}^* = 50.10, \quad x_{p_3}^* = 39.90, \quad x_{p_4}^* = 40.10,$$

 $\lambda_{w_1}^*(t_0) = 915.50 \quad \lambda_{w_2}^*(t_0) = 895.50.$

◆ t=1/2

$$x_{p_1}^* = 51.15, \quad x_{p_2}^* = 51.35, \quad x_{p_2}^* = 40.90, \quad x_{p_4}^* = 41.10.$$

 $\lambda_{w_1}^*(t_1) = 938.25, \quad \lambda_{w_2}^*(t_1) = 917.75$

$$x_{p_1}^* = 52.40, \quad x_{p_2}^* = 52.60, \quad x_{p_3}^* = 41.90, \quad x_{p_4}^* = 42.10.$$

 $\lambda_{w_1}^*(T) = 961.00, \quad \lambda_{w_2}^*(T) = 940.00.$





Corresponding Supernetwork





Corresponding Supernetwork



Supernetworks http://supernet.som.umass.edu



Corresponding Supernetwork



Conclusions

- We established the supernetwork equivalence of the electric power supply chain networks with transportation networks with fixed demands.
- This identification provided a new interpretation of equilibrium in electric power supply chain networks in terms of path flows.
- We utilized this isomorphism in the computation of the electric power supply chain network equilibrium with time-varying demands.



Thank You!

For more information, please see: The Virtual Center for Supernetworks <u>http://supernet.som.umass.edu</u>

