Dynamic Supply Chains, Transportation Network Equilibria, and Evolutionary Variational Inequalities

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Outline

- Research Motivation
- Overview of the Supply Chain Network Equilibrium Model
- Overview of the Transportation Network Equilibrium Model
- The Supernetwork Equivalence of Supply Chain Networks and Transportation Networks
- Evolutionary Variational Inequalities and Projected Dynamical Systems; Applications to Transportation Network Equilibrium
- Supply Chain Network Model with Time Varying Demands

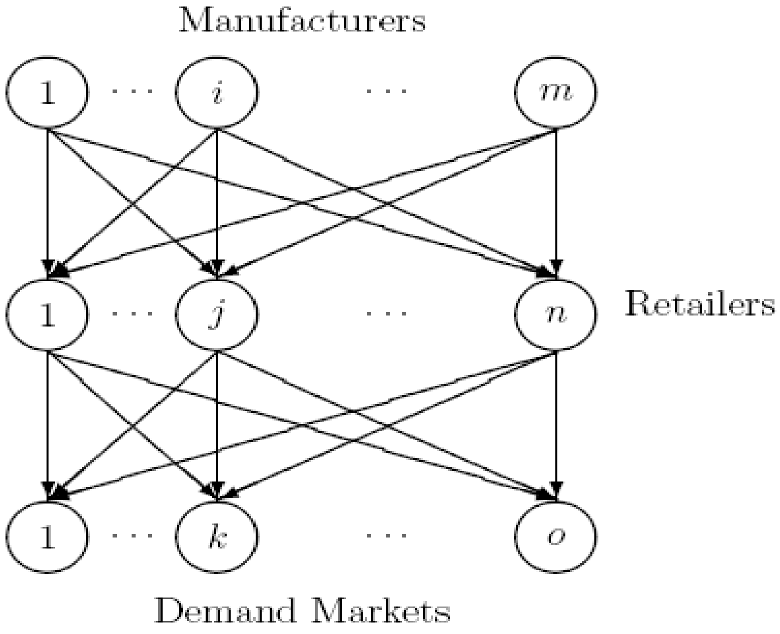
Motivation

- The theory that has originated from the study of transportation networks can inform a new kind of time-dependent equilibrium modeling framework for supply chain networks.

- The new dynamic supply chain network model that we developed in this research is motivated by the unification of projected dynamical systems theory and evolutionary (infinite-dimensional) variational inequalities.

Overview of the Supply Chain Network Equilibrium Model

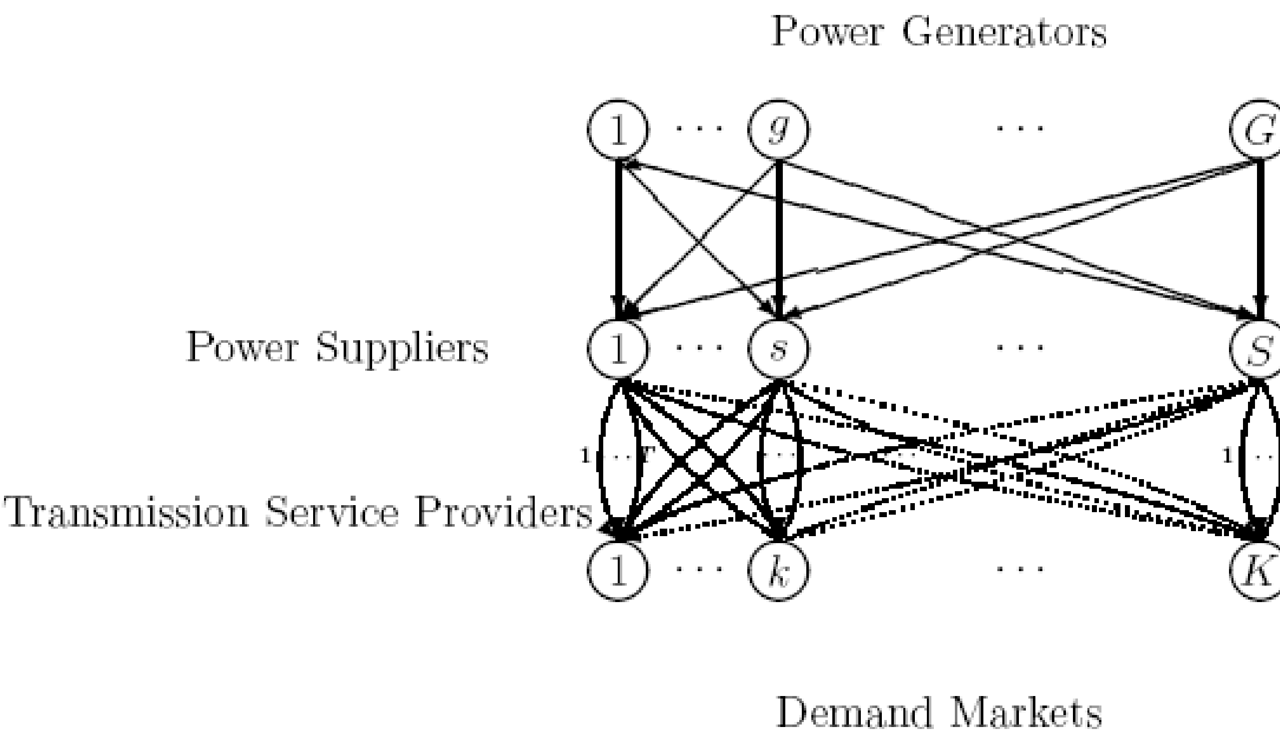
- Nagurney, A., Dong, J. and D. Zhang (2002),"A Supply Chain Network Equilibrium Model,"*Transportation Research E* 38, 281-303.



The Network Structure of the Supply Chain at Equilibrium

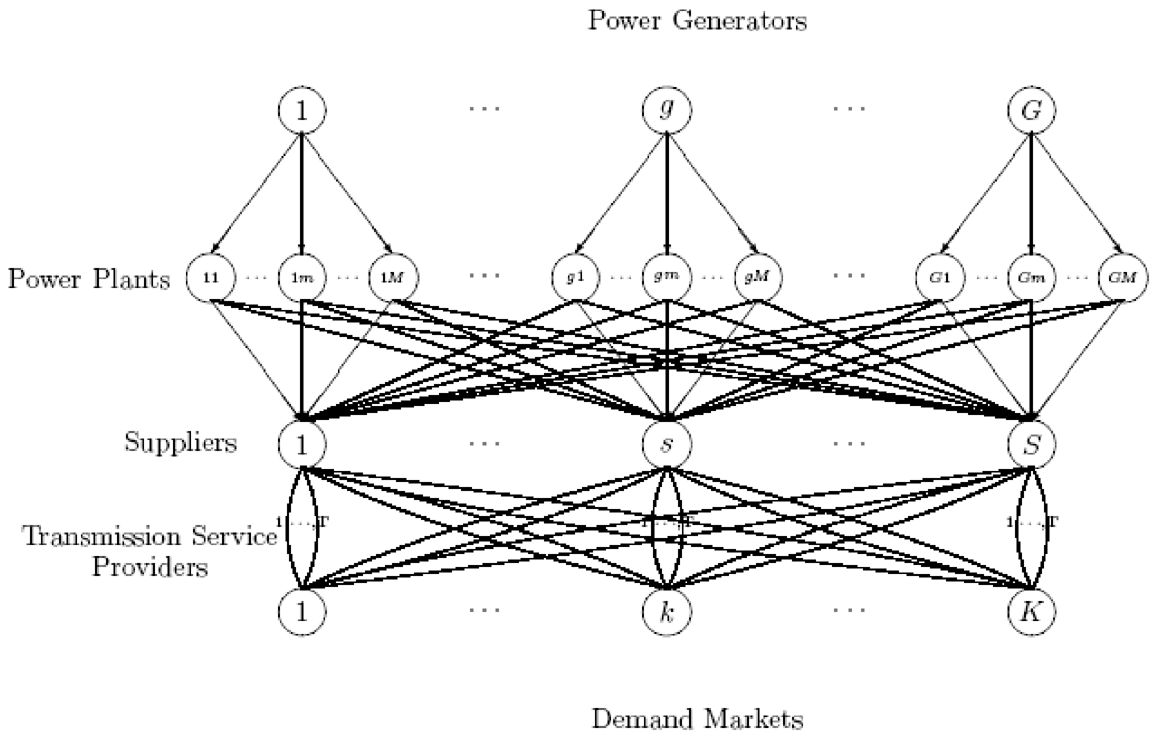
Overview of the Electric Power Supply Chain Network Equilibrium Model

- A. Nagurney and D. Matsypura "A Supply Chain Network Perspective for Electric Power Generation, Supply, Transmission, and Consumption" (*Proceedings of the International Conference on Computing, Communications and Control Technologies*, Austin, Texas, Volume VI: (2004) pp 127-134.))



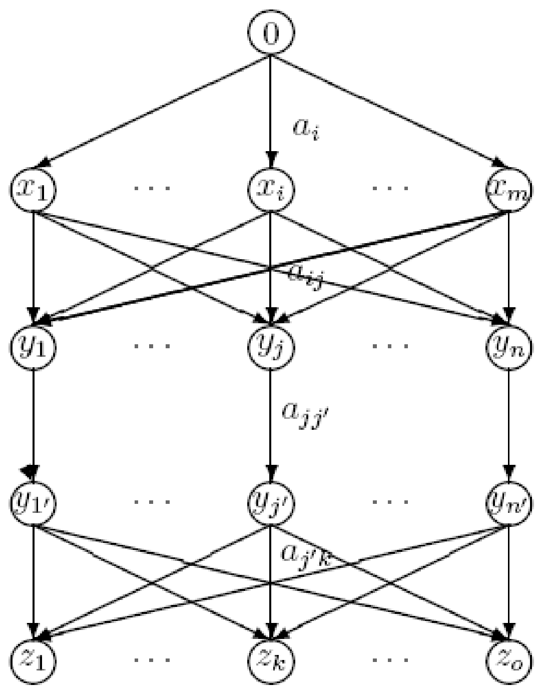
Overview of the Electric Power Supply Chain with Power Plants and Carbon Taxes

- K. Wu, A. Nagurney, Z. Liu, and J. Stranlund, "Modeling Generator Power Plant Portfolios and Pollution Taxes in Electric Power Supply Chain Networks: A Transportation Network Equilibrium Transformation" (To appear in *Transportation Research D* (2006).)



Overview of the Transportation Network Equilibrium Model with Elastic Demands

- Dafermos, S., Nagurney, A., 1984. Stability and sensitivity analysis for the general network equilibrium - travel choice model. In: Volmuller, J., Hamerslag, R. (Eds.), *Proceedings of the Ninth International Symposium on Transportation and Traffic Theory*. VNU Science Press, Utrecht, The Netherlands, pp. 217-232.



Transportation Network Equilibrium Conditions

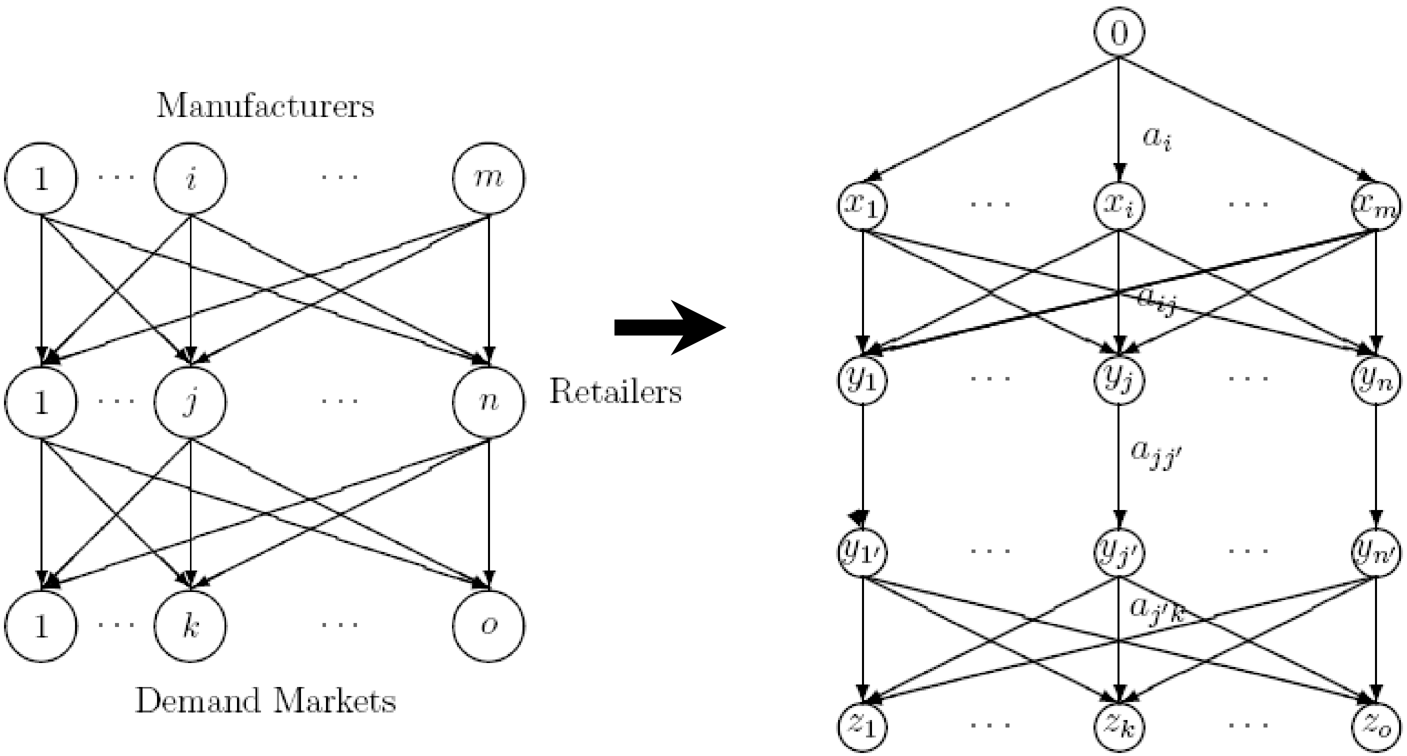
$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0 \\ \geq 0, & \text{if} \quad x_p^* = 0 \end{cases}$$

$$\sum_{p \in P_w} x_p^* \begin{cases} = d_w(\lambda^*), & \text{if} \quad \lambda_w^* > 0 \\ \geq d_w(\lambda^*), & \text{if} \quad \lambda_w^* = 0. \end{cases}$$

$$\sum_{w \in W} \sum_{p \in P_w} [C_p(x^*) - \lambda_w^*] \times \left[x_p - x_p^*\right] + \sum_{w \in W} \sum_{p \in P_w} \left[x_p^* - d_w(\lambda^*)\right] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (x, \lambda) \in R_+^{QZ}.$$

The Supernetwork Equivalence of Supply Chain Network Equilibrium and Transportation Network Equilibrium

- A. Nagurney, (2005) "On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations" (To appear in *Transportation Research E*)



Projected Dynamical Systems and Evolutionary Variational Inequalities

- M. Cojocaru, P. Daniele, and A. Nagurney, (2005) “Projected Dynamical Systems and Evolutionary Variational Inequalities via Hilbert Spaces with Applications” (Appears in the *Journal of Optimization Theory and Applications* 27, no.3, 1-15)

- Projected Dynamical Systems (PDSs) (Dupuis and Nagurney (1993))

  – PDS describes how the state of the network system approaches an equilibrium point on the curve of equilibria.

  – For almost every moment ‘t’ on the equilibria curve, there is a PDS_t associated with it.

Projected Dynamical Systems

- PDSs: $$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))).$$

In this formulation, $\mathcal{K}$ is a convex polyhedral set in R^n, $F : \mathcal{K} \rightarrow R^n$ is a Lipschitz continuous function with linear growth and $\Pi_{\mathcal{K}} : R \times \mathcal{K} \rightarrow R^n$ is the Gateaux directional derivative

$$\Pi_{\mathcal{K}}(x, -F(x)) = \lim_{\delta \rightarrow 0^+} \frac{P_{\mathcal{K}}(x - \delta F(x)) - x}{\delta}$$

of the projection operator $P_{\mathcal{K}} : R^n \rightarrow \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$

Evolutionary Variational Inequalities

- Evolutionary Variational Inequalities (EVIs)

  – EVI provides a curve of equilibria of the network system over a finite time interval [0,T]

  – EVIs have been applied to time-dependent equilibrium problems in transportation, and in economics and finance. (See JOTA paper.)

Evolutionary Variational Inequalities

Define $$\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt,$$

$$\text{find } v \in \mathcal{K} \text{ such that } \ll F(v), u - v \gg \geq 0, \ \forall u \in \mathcal{K}.$$

where

$$\mathcal{K} = \bigcup_{t \in [0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$$

Projected Dynamical Systems and Evolutionary Variational Inequalities

An EVI is usually used to model large scale time, i.e, [0, T]

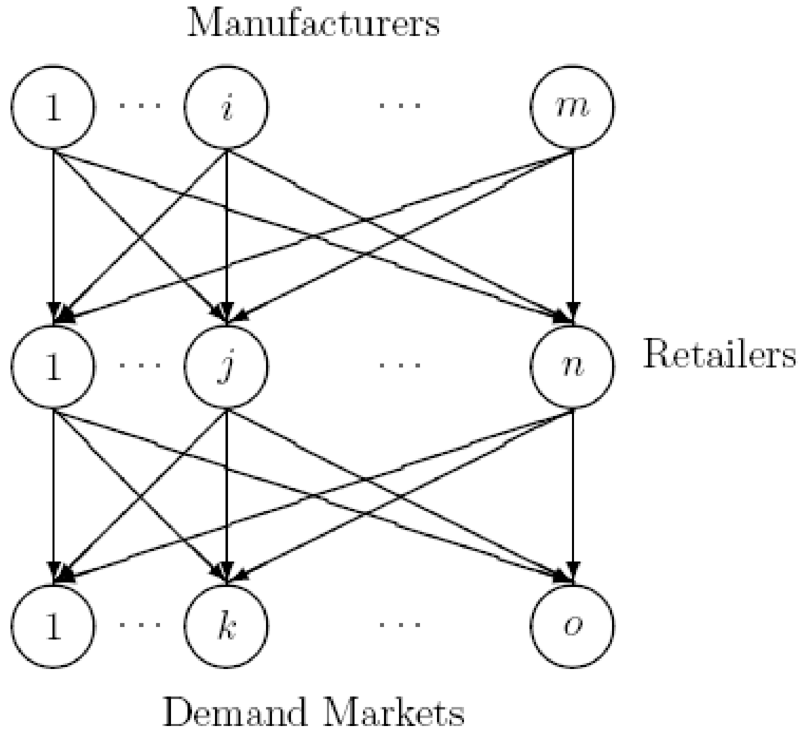
A PDS_t is usually applied to study small scale time dynamics, i.e [t, t+τ]

Given a time-dependent network equilibrium problem, the solutions to the EVI are the same as the critical points of a series of PDS_t , and vice versa.

Supply Chain Networks with Fixed Demands

Commodities with price-insensitive demand

– Electricity, gasoline, milk, etc.



The Behavior of Manufacturers and their Optimality Conditions

- Manufacturer's optimization problem

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho_{1ij}^{*} q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}),$$

- Optimality conditions of the manufacturers

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^{*})}{\partial q_{ij}} - \rho_{1ij}^{*} \right] \times \left[q_{ij} - q_{ij}^{*} \right] \geq 0, \quad \forall Q^1 \in R_{+}^{mn}.$$

The Behavior of Retailers and their Optimality Conditions

- Retailer's optimization problem

$$\text{Maximize} \quad \sum_{k=1}^{o} \rho_{2j}^* q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho_{1ij}^* q_{ij}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} q_{ij},$$

- Optimality conditions of the retailers

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ij}} + \rho_{1ij}^* - \gamma_j^*\right] \times \left[q_{ij} - q_{ij}^*\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho_{2j}^* + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0, \quad \forall (Q^1, Q^2, \gamma) \in R_+^{mn+no+n}.$$

The Equilibrium Conditions of the Demand Markets

- Conservation of flow equations must hold

$$d_k = \sum_{j=1}^{n} q_{jk}, \quad k = 1, \ldots, o,$$

- We say that vector (Q^2*, ρ_3*) is an equilibrium vector if for each *s, k pair*:

$$\rho_{2j}^* + c_{jk}(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0. \end{cases}$$

Supply Chain Network Equilibrium (For Fixed Demand at the Markets)

Definition: The equilibrium state of the supply chain network is one where the product flows between the tiers of the network coincide and the product flows satisfy the sum of optimality conditions of the manufacturers and the retailers, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

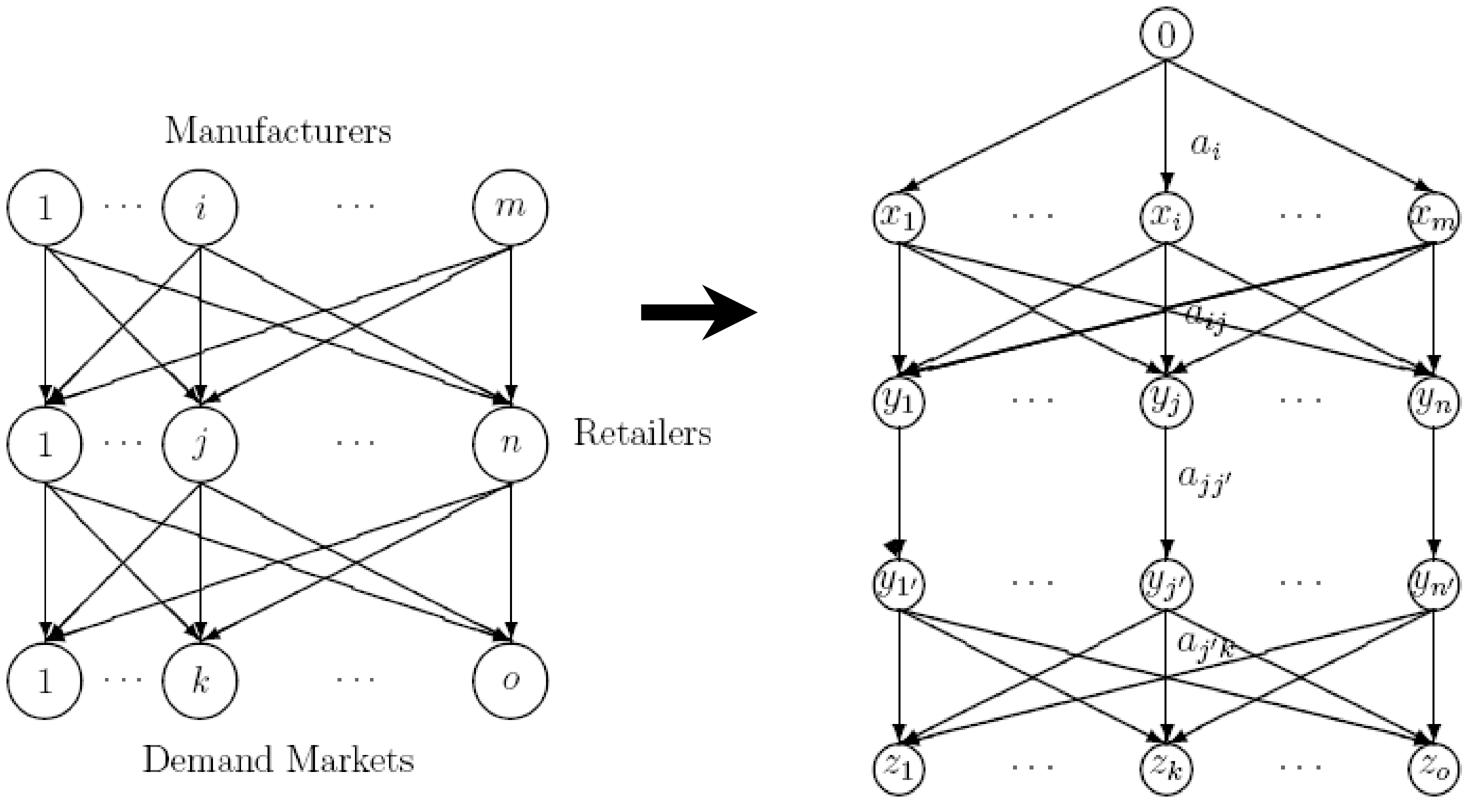
- Determine $(Q^{1*}, Q^{2*}, \gamma^*) \in \mathcal{K}^2$ satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ij}} - \gamma_j^*\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*}) + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right] + \sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma) \in \mathcal{K}^2.$$

Transportation Network Equilibrium Reformulation



EVI Transportation Model

Feasible set

$$\hat{\mathcal{K}} = \left\{ x \in L^2([0,T], R^Q) : 0 \le x(t) \le \mu \text{ a.e. in } [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0,T] \right\}.$$

Define

$$\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$$

EVI

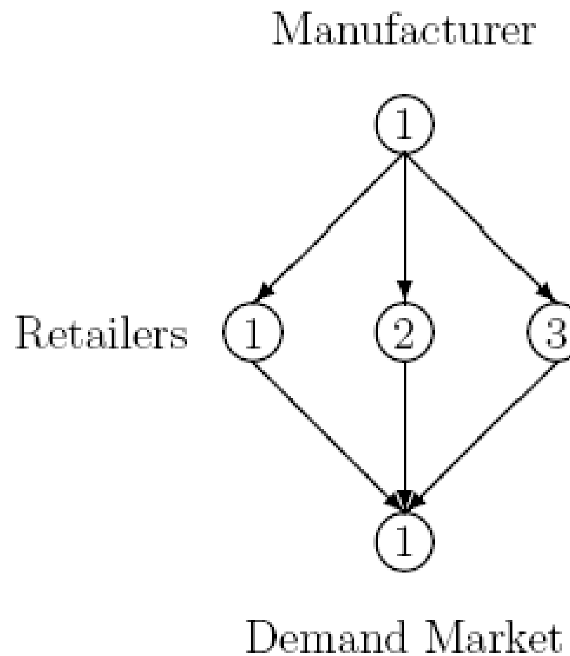
determine $x^* \in \hat{\mathcal{K}}$ such that:

$$\langle\langle F(x^*), x - x^* \rangle\rangle \ge 0, \quad \forall x \in \hat{\mathcal{K}}.$$

where *F(x)* denotes the vector of path costs as a function of flows.

Dynamic Supply Chain Network Examples with Computations

- Example 1



The time horizon $T = 1$. The time-varying demand function was given by:

$$d_1(t) = 100 + 10t.$$

Numerical Example 1

The production cost function for the manufacturer was given by:

$$f_1(q(t)) = 2.5q_1(t)^2 + 2q_1(t).$$

The transaction cost functions faced by the manufacturer and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}(t)) = .5q_{11}(t)^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5q_{12}(t)^2 + 2.5q_{12}(t), \quad c_{13}(q_{13}(t)) = .5q_{13}(t)^2 + 1.5q_{13}(t).$$

The operating costs of the retailers, in turn, were given by:

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

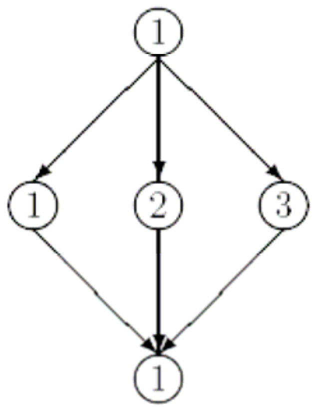
The unit transaction costs associated with transacting between the retailers and the demand market were:

$$c_{11}(Q^2(t)) = q_{11}(t) + 1, \quad c_{21}(Q^2(t)) = q_{21}(t) + 5, \quad c_{31}(Q^2(t)) = q_{31}(t) + 10.$$

Numerical Example 1

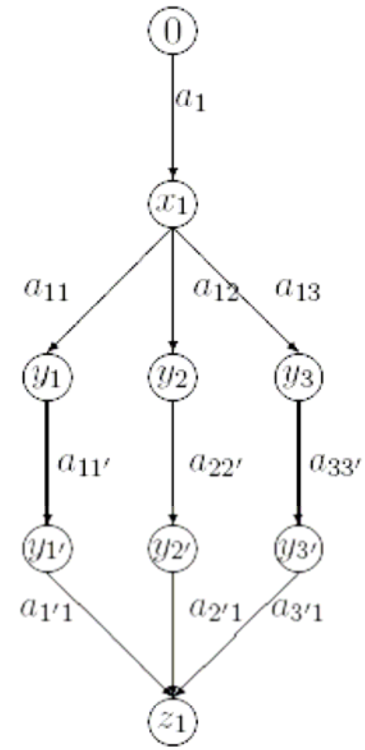
Manufacturers

Retailers



Demand Markets

$\Longrightarrow$



Corresponding Supernetwork

- Production cost functions

$$f_1(q_1(t)) = 2.5(q_1(t))^2 + 2q_1(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5(q_{12}(t))^2 + 2.5q_{12}(t),$$

$$c_{13}(q_{13}(t)) = .5(q_{13}(t))^2 + 1.5q_{13}(t).$$

- Handling cost functions of the retailers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

- Unit transaction cost between the retailers and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{21}(Q^2(t)) = q^1_{21}(t) + 5, \quad \hat{c}^1_{31}(Q^2(t)) = q^1_{31}(t) + 10.$$

- Three paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

- The time-varying demand function

$$d_{w_1}(t) = d_1(t) = 41 + 10t.$$

Numerical Example 1

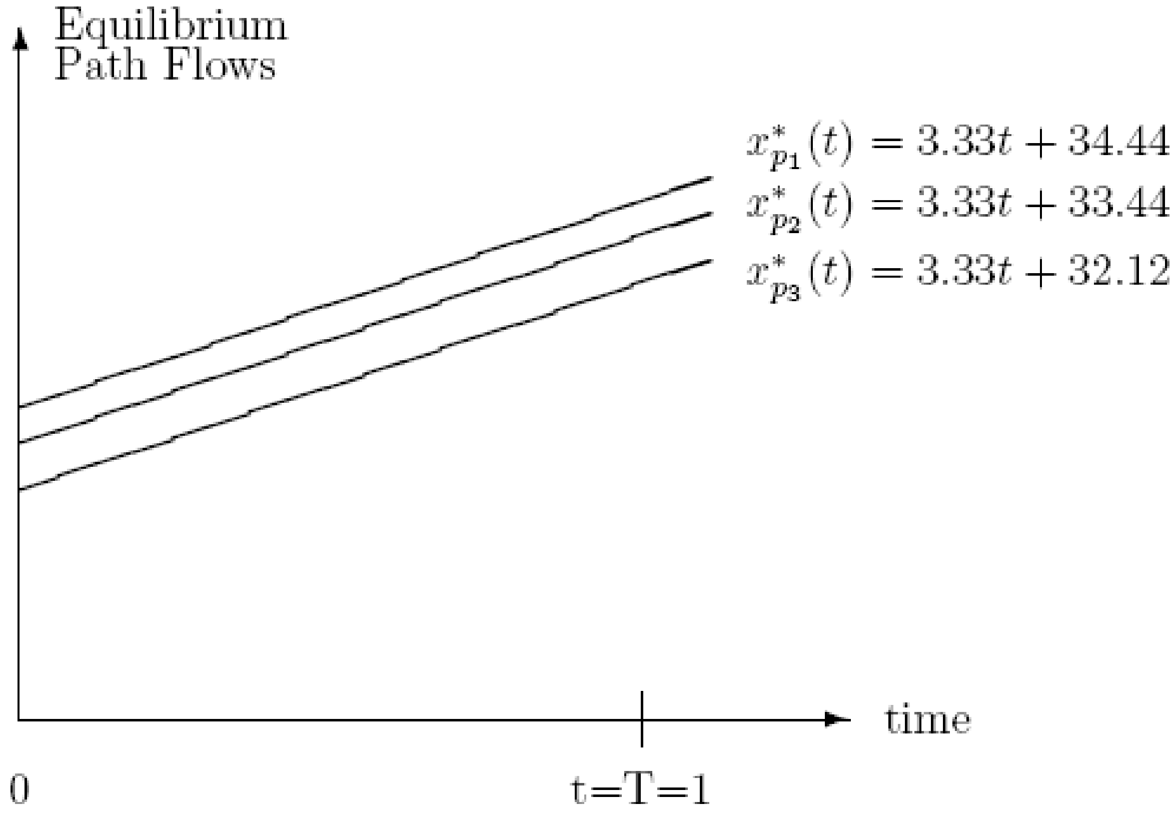
- Explicit Solution
  - Path flows

$$x^*_{p_1}(t) = 3.33t + 14.78,$$
$$x^*_{p_2}(t) = 3.33t + 13.78,$$
$$x^*_{p_3}(t) = 3.33t + 12.45,$$

  - Travel disutility

$$\lambda^*_{w_1}(t) = 60t + 255.83, \quad \text{for} \quad t \in [0, T].$$

Time-Dependent Equilibrium Path Flows for Numerical Example 1

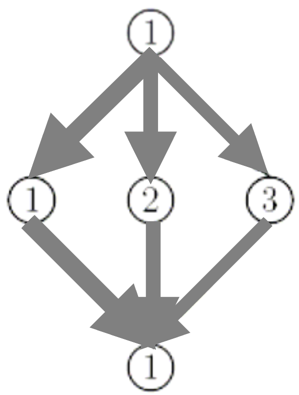


Numerical Example 1

t=1

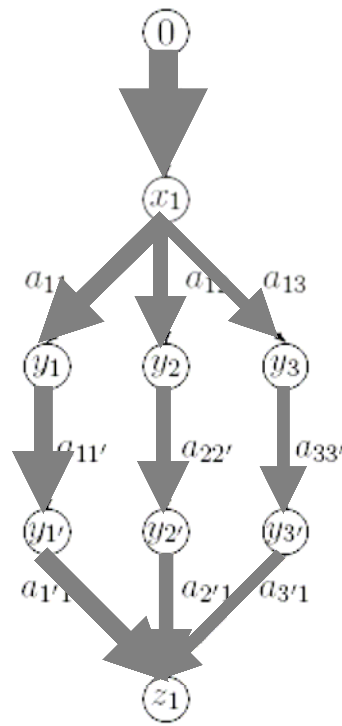
Manufacturers

Retailer



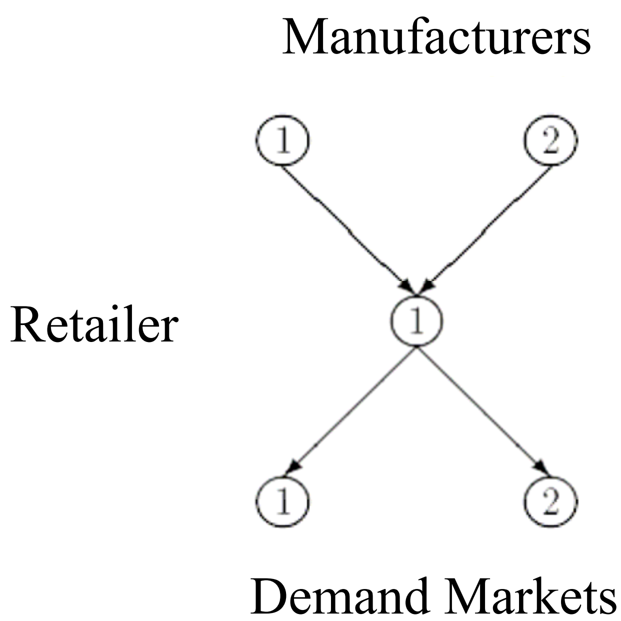
Demand Markets

$\Longrightarrow$



Corresponding Supernetwork

Numerical Example 2



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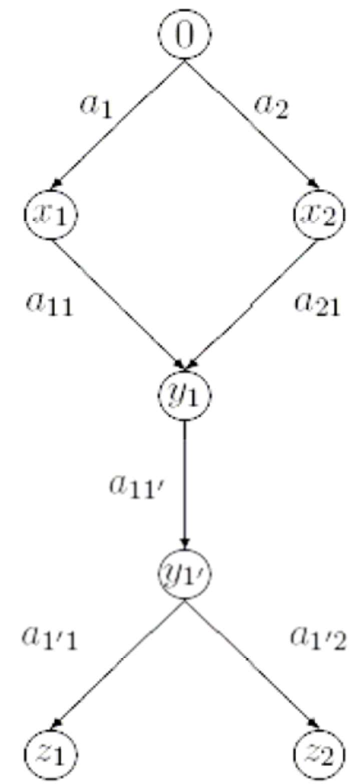


Corresponding Supernetwork

- Production cost functions

$$f_1(q(t)) = 2.5(q_1(t))^2 + q_1(t)q_2(t) + 2q_1(t), \quad f_2(q(t)) = 2.5(q_2(t))^2 + q_2(t)q_1(t) + 2q_2(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{21}(q_{21}(t)) = .5(q_{21}(t))^2 + 1.5q_{21}(t).$$

- Handling cost function of the retailer

$$c_1(Q^1(t)) = .5(q_{11}(t))^2.$$

- Unit transaction costs between the retailer and the demand markets

$$\hat{c}_{11}^1(Q^2(t)) = q_{11}^1(t) + 1, \quad \hat{c}_{12}^1(Q^2(t)) = q_{12}^1(t) + 1,$$

- Four paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}),\ p_2 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$

$$p_3 = (a_1, a_{11}, a_{11'}, a_{1'2}),\ p_4 = (a_2, a_{21}, a_{11'}, a_{1'2}).$$

- The time-varying demand functions

$$d_{w_1}(t) = d_1(t) = 100 + 5t, \quad d_{w_2}(t) = d_2(t) = 80 + 4t.$$

Numerical Example 2

- Numerical Solution
  - t=0

$$x^*_{p_1} = 49.90, \quad x^*_{p_2} = 50.10, \quad x^*_{p_3} = 39.90, \quad x^*_{p_4} = 40.10.$$

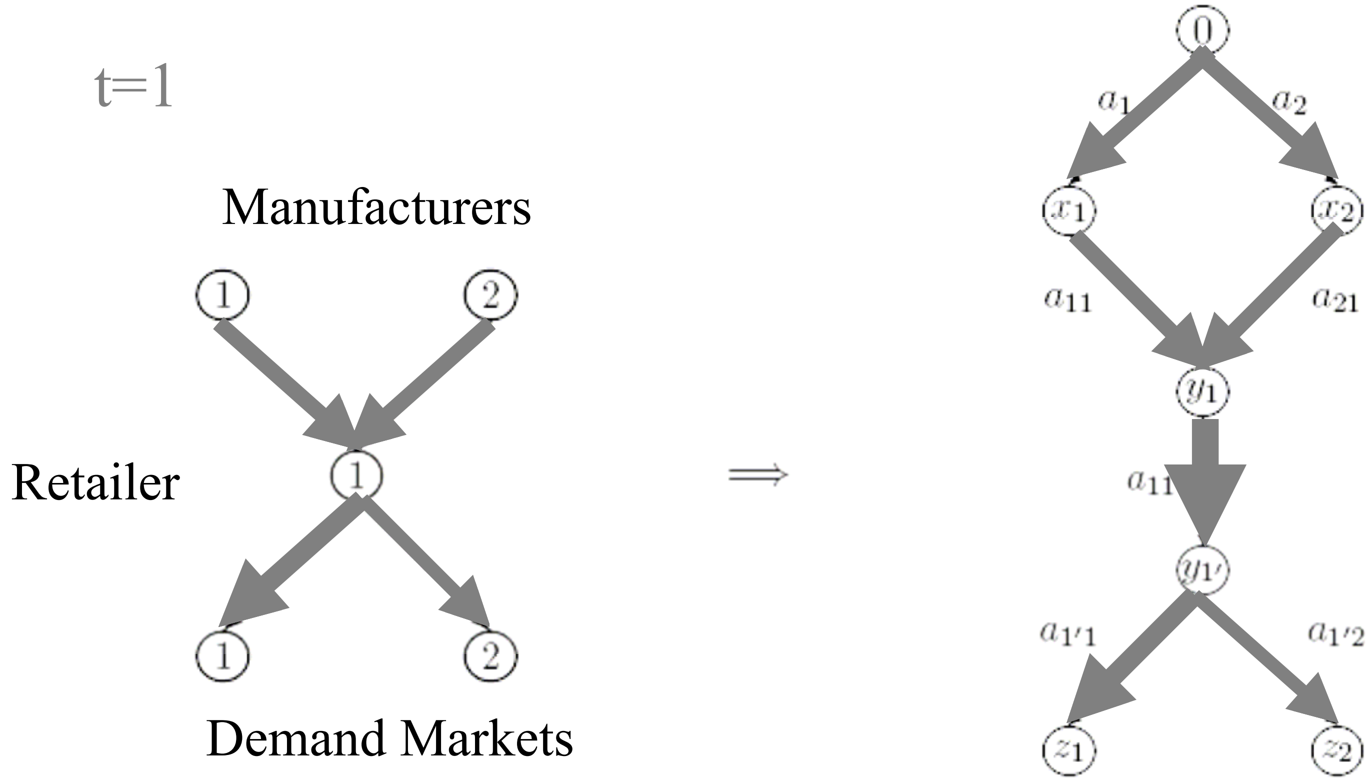
  - t=1/2

$$x^*_{p_1} = 51.15, \quad x^*_{p_2} = 51.35, \quad x^*_{p_3} = 40.90, \quad x^*_{p_4} = 41.10.$$

  - t=1

$$x^*_{p_1} = 52.40, \quad x^*_{p_2} = 52.60, \quad x^*_{p_3} = 41.90, \quad x^*_{p_4} = 42.10.$$

Numerical Example 2



Corresponding Supernetwork

Conclusions

We established the supernetwork equivalence of the supply chain networks with the transportation networks.

We utilized this isomorphism in the computation of the supply chain network equilibrium with time-varying demands.

We are also investigating applications to Electric Power.

Thank You!

For more information, see:

The Virtual Center for Supernetworks

http://supernet.som.umass.edu