Dynamic Supply Chains, Transportation Network Equilibria, and Evolutionary Variational Inequalities

Anna Nagurney
Radcliffe Institute for Advanced Study
Harvard University
and
Isenberg School of Management
University of Massachusetts
Amherst

Zugang Liu
Department of Finance
and
Operations Management
Isenberg School of Management
University of Massachusetts
Amherst

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Motivation

- Many researchers have described the various networks that underlie supply chain analysis and management with the goal being primarily that of optimization.
- In 2002, Nagurney, Dong and Zhang in Transportation Research E presented apparently the first supply chain network equilibrium model.
- The objective of this research was to develop a general dynamic supply chain network equilibrium model with exogenous time-varying demand.

Motivation

- The theory that has originated from the study of transportation networks was utilized to construct this time-dependent equilibrium modeling framework for supply chain networks.

- The new dynamic supply chain network model that we developed in this research is also motivated by the unification of projected dynamical systems theory and evolutionary (infinite-dimensional) variational inequalities.

Outline

- The supply chain network model with fixed demands
  – Overview of the supply chain network equilibrium models
  – The development of the supply chain network equilibrium model with fixed demands
- The supernetwork equivalence of the supply chain networks and the transportation networks
  – Overview of the transportation network equilibrium models
  – The supernetwork equivalence of the transportation networks and the supply chain networks with fixed demand
- The supply chain network model with time-varying demands
  – Evolutionary variational inequalities and projected dynamical systems; Applications to transportation network equilibrium
  – The computation of the supply chain network equilibrium model with time-varying demands.

Some of the Related Network Economics Literature

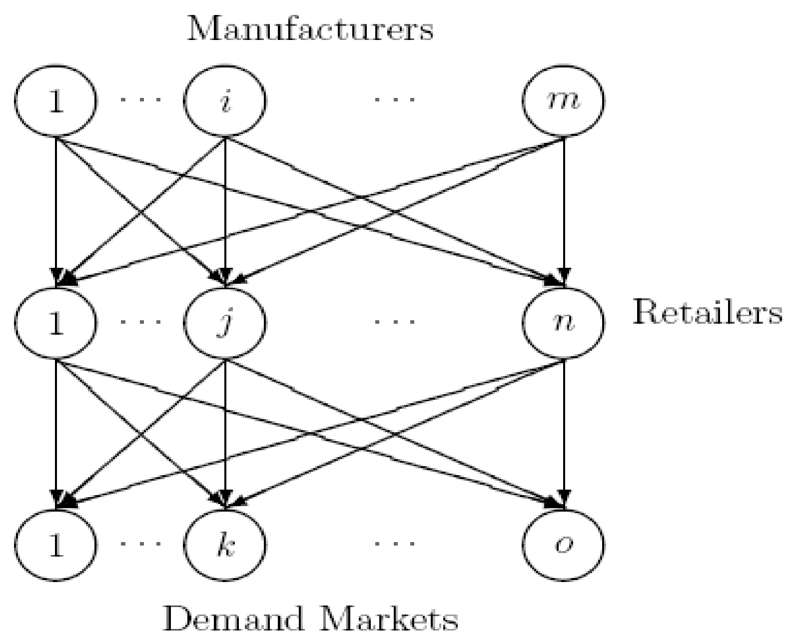
- Beckmann, M. J., McGuire, C. B., and Winsten, C. B. (1956), Studies in the Economics of Transportation. Yale University Press, New Haven, Connecticut.

- Nagurney, A (1999), Network Economics: A Variational Inequality Approach, Second and Revised Edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

- Nagurney, A., Dong, J., and Zhang, D. (2002), A Supply Chain Network Equilibrium Model, *Transportation Research E* 38, 281-303.

Overview of the Supply Chain Network Equilibrium Model

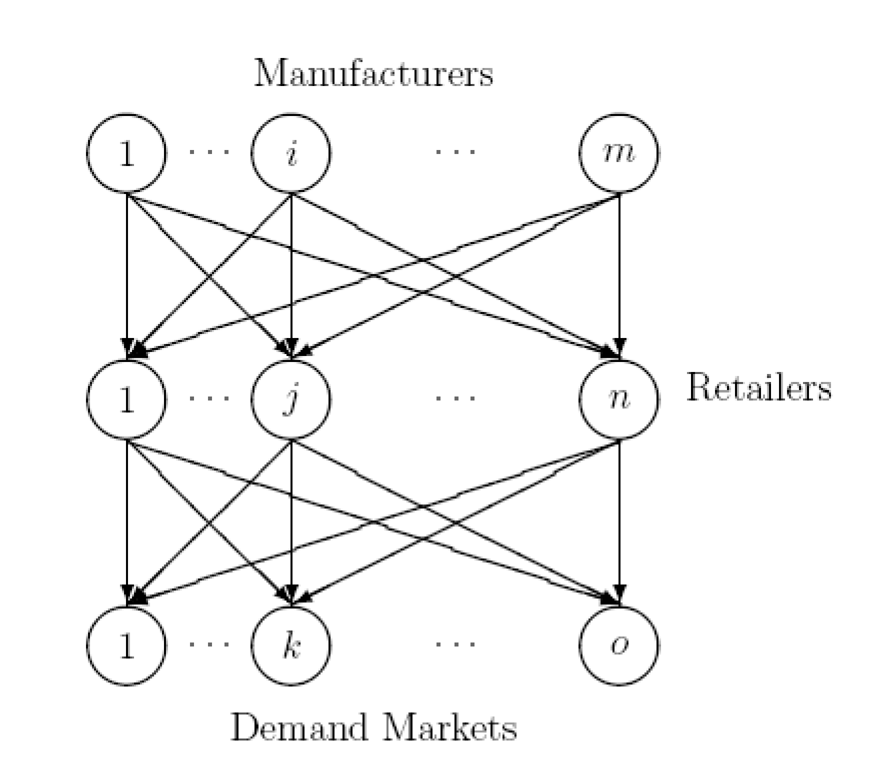
- Nagurney, A., Dong, J. and Zhang, D. (2002), A Supply Chain Network Equilibrium Model, *Transportation Research E* 38, 281-303.



The Network Structure of the Supply Chain at Equilibrium

The Development of the Supply Chain Network Equilibrium Model with Fixed Demands

- Commodities with price-insensitive demand
  - Electricity, gasoline, milk, etc.



The Behavior of Manufacturers and their Optimality Conditions

- Manufacturer's optimization problem

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho_{1ij}^* q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}),$$

- The Optimality conditions of the manufacturers

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} - \rho_{1ij}^* \right] \times \left[q_{ij} - q_{ij}^* \right] \geq 0, \quad \forall Q^1 \in R_+^{mn}.$$

The Behavior of Retailers and their Optimality Conditions

- Retailer's optimization problem

$$\text{Maximize} \quad \sum_{k=1}^{o} \rho_{2j}^* q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho_{1ij}^* q_{ij}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \le \sum_{i=1}^{m} q_{ij},$$

- The Optimality conditions of the retailers

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ij}} + \rho_{1ij}^* - \gamma_j^*\right] \times \left[q_{ij} - q_{ij}^*\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho_{2j}^* + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \ge 0, \quad \forall (Q^1, Q^2, \gamma) \in R_+^{mn+no+n}.$$

The Equilibrium Conditions at the Demand Markets

- Conservation of flow equations must hold

$$d_k = \sum_{j=1}^{n} q_{jk}, \quad k = 1, \ldots, o,$$

- We say that vector (Q^{2*}, ρ_3*) is an equilibrium vector if for each *s, k pair*:

$$\rho_{2j}^* + c_{jk}(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0. \end{cases}$$

Supply Chain Network Equilibrium (For Fixed Demands at the Markets)

Definition: The equilibrium state of the supply chain network is one where the product flows between the tiers of the network coincide and the product flows satisfy the conservation of flow equations, the sum of optimality conditions of the manufacturers and the retailers, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

- Determine $(Q^{1*}, Q^{2*}, \gamma^*) \in \mathcal{K}^2$ satisfying

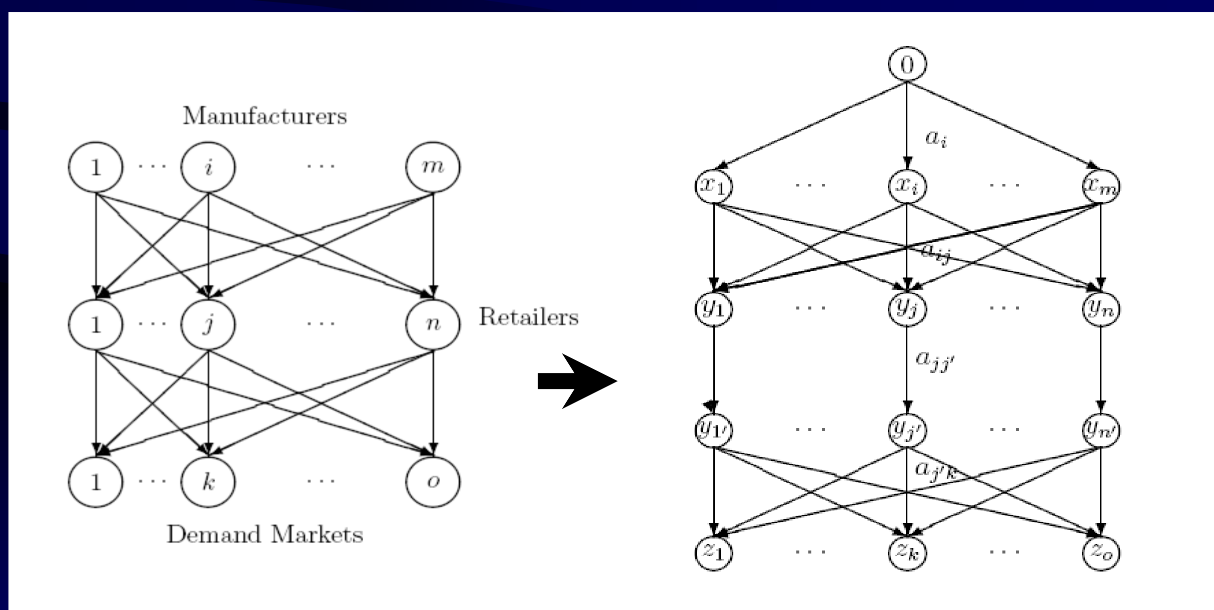
$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ij}} - \gamma_j^*\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*}) + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right] + \sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma) \in \mathcal{K}^2.$$

The Supernetwork Equivalence of Supply Chain Network Equilibrium and Transportation Network Equilibrium

- Nagurney, A. (2006), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, (*Transportation Research E (2006)* 42: (2006) pp 293-316)



Overview of the Transportation Network Equilibrium Model with Fixed Demand

- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.

- Dafermos, S. (1980), Traffic equilibrium and variational inequalities. *Transportation Science* 14, 42-54.

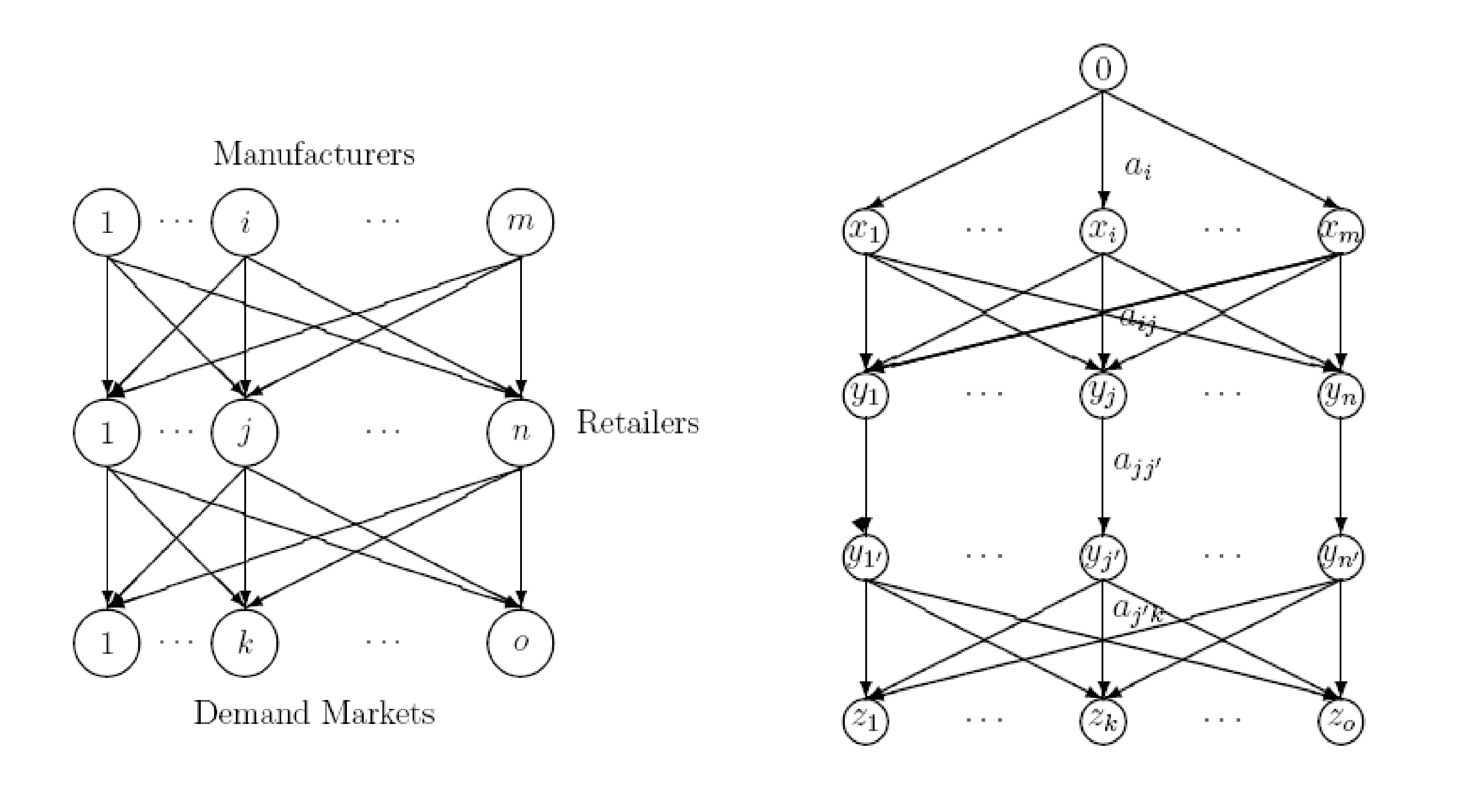
- In equilibrium, the following conditions must hold for each O/D pair and each path.

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases}$$

- A path flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality:

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^* \right] \geq 0, \quad \forall x \in \mathcal{K}^5.$$

Transportation Network Equilibrium Reformulation of the Supply Chain Network Model with Fixed Demands



The Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Cojocaru, M.-G., Daniele, P., Nagurney, A., (2005a). Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces with applications. *Journal of Optimization Theory and Applications* 27, no. 3, 1-15.

- Cojocaru, M.-G., Daniele, P., Nagurney, A., (2005b). Double-layered dynamics: A unified theory of projected dynamical systems and evolutionary variational inequalities. *European Journal of Operational Research*, in press.

The Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Cojocaru, M.-G., Daniele, P., Nagurney, A. (2005c). Projected dynamical systems, evolutionary variational inequalities, applications, and a computational procedure. *Pareto Optimality, Game Theory and Equilibria*. A. Migdalas, P. M. Pardalos, and L. Pitsoulis, editors, Springer Verlag, in press.

- Barbagallo, A., (2005). Regularity results for time-dependent variational and quasivariational inequalities and computational procedures. *To appear in Mathematical Models and Methods in Applied Sciences*.

The Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Daniele, P., Maugeri, A., Oettli, W., (1998). Variational inequalities and time-dependent traffic equilibria. *Comptes Rendue Academie des Science*, Paris 326, serie I, 10591062.

- Daniele, P., Maugeri, A., Oettli, W., (1999). Time-dependent traffic equilibria. *Journal of Optimization Theory and its Applications* 103, 543-555.

Finite-Dimentional Variational Inequalities and Projected Dynamical Systems Literature

- Dupuis, P., Nagurney, A., (1993). Dynamical systems and variational inequalities. *Annals of Operations Research* 44, 9-42.

- Nagurney, A., Zhang, D., (1996). Projected Dynamical Systems and Variational Inequalities with Applications. Kluwer Academic Publishers, Boston, Massachusetts.

- Nagurney, A., Zhang, D., (1997). Projected dynamical systems in the formulation, stability analysis, and computation of fixed demand traffic network equilibria. *Transportation Science* 31, 147-158.

Projected Dynamical Systems and Evolutionary Variational Inequalities

- Projected Dynamical Systems (PDSs) (Dupuis and Nagurney (1993))
  - PDS describes how the state of the network system approaches an equilibrium point on the curve of equilibria.
  - For almost every moment ‘t’ on the equilibria curve, there is a PDS_t associated with it.
  - A PDS_t is usually applied to study small scale time dynamics, i.e [t, t+τ]

Projected Dynamical Systems

PDSs: $$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))).$$

In this formulation, $\mathcal{K}$ is a convex polyhedral set in R^n, $F : \mathcal{K} \rightarrow R^n$ is a Lipschitz continuous function with linear growth and $\Pi_{\mathcal{K}} : R \times \mathcal{K} \rightarrow R^n$ is the Gateaux directional derivative

$$\Pi_{\mathcal{K}}(x, -F(x)) = \lim_{\delta \rightarrow 0^+} \frac{P_{\mathcal{K}}(x - \delta F(x)) - x}{\delta}$$

of the projection operator $P_{\mathcal{K}} : R^n \rightarrow \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$

Evolutionary Variational Inequalities

- Evolutionary Variational Inequalities (EVIs)
  - EVI provides a curve of equilibria of the network system over a finite time interval [0,T]
  - An EVI is usually used to model large scale time, i.e, [0, T]
  - EVIs have been applied to time-dependent equilibrium problems in transportation, and in economics and finance.

Evolutionary Variational Inequalities

Define $$\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt,$$

find $v \in \mathcal{K}$ such that $\ll F(v), u - v \gg \geq 0, \; \forall u \in \mathcal{K}.$

where

$$\mathcal{K} = \bigcup_{t \in [0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$$

Projected Dynamical Systems and Evolutionary Variational Inequalities

- Cojocaru, Daniele, and Nagurney (2005b) showed the following:

Theorem

Assume that $\hat{\mathcal{K}} \subseteq H$ is non-empty, closed, and convex. Assume also that $F : \hat{\mathcal{K}} \to H$ is a pseudo-monotone vector field, that is, for every pair of points $x, y \in \hat{\mathcal{K}}$, we have that

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq 0,$$

and that F is Lipschitz continuous, where H is a Hilbert space. Then the solutions of EVI (47) are the same as the critical points of the projected differential equation (48), that is, they are the functions $x^* \in \hat{\mathcal{K}}$ such that

$$\Pi_{\hat{\mathcal{K}}}(x^*(t), -F(x^*(t))) = 0,$$

and vice-versa.

Numerical Solution of Evolutionary Variational Inequalities

- The vector field *F* satisfies the requirement in the Theorem.

- We first discretize time horizon T.

- At each fixed time point, we solve the associated projected dynamical system PDS_t

- We use the Euler method to solve the projected dynamical system PDS_t.

The Euler Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$ and set $T = 0$. T is an iteration counter which may also be interpreted as a time period.

Step 1: Computation

Compute X^{T+1} by solving the variational inequality problem:

$$X^{T+1} = P_{\mathcal{K}}(X^T - a_T F(X^T)),$$

where $\{a_T\}$ is a sequence of positive scalars satisfying: $\sum_{T=0}^{\infty} a_T = \infty$, $a_T \to 0$, as $T \to \infty$

and $P_{\mathcal{K}}$ is the projection of X on the set $\mathcal{K}$ defined as:

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \|X - z\|.$$

Step 2: Convergence Verification

If $\|X^{T+1} - X^T\| \leq \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; else, set $T = T + 1$, and go to Step 1.

The Solution to the Transportation Network Model with Time-Varying Demands

Feasible set

$$\hat{\mathcal{K}} = \left\{ x \in L^2([0,T], R^Q) : 0 \leq x(t) \leq \mu \text{ a.e. in } [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0,T] \right\}.$$

Define

EVI: $\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$

determine $x^* \in \hat{\mathcal{K}}$ such that:

$$\langle\langle F(x^*), x - x^* \rangle\rangle \geq 0, \quad \forall x \in \hat{\mathcal{K}}.$$

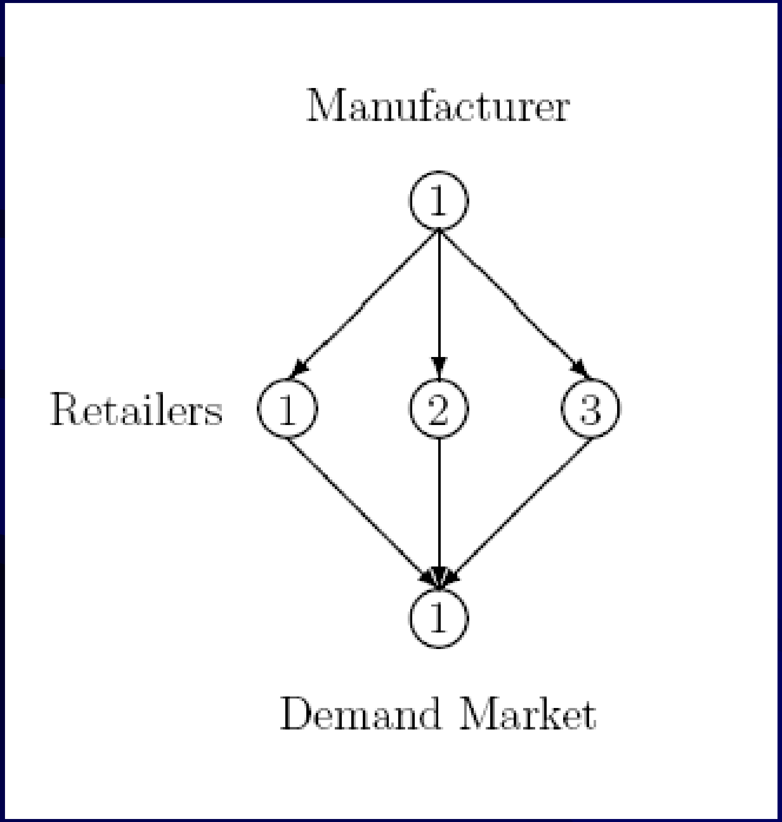
where *F(x)* denotes the vector of **path costs** as a function of path flows.

Solving Supply Chain Network Model with Time-Varying Demands

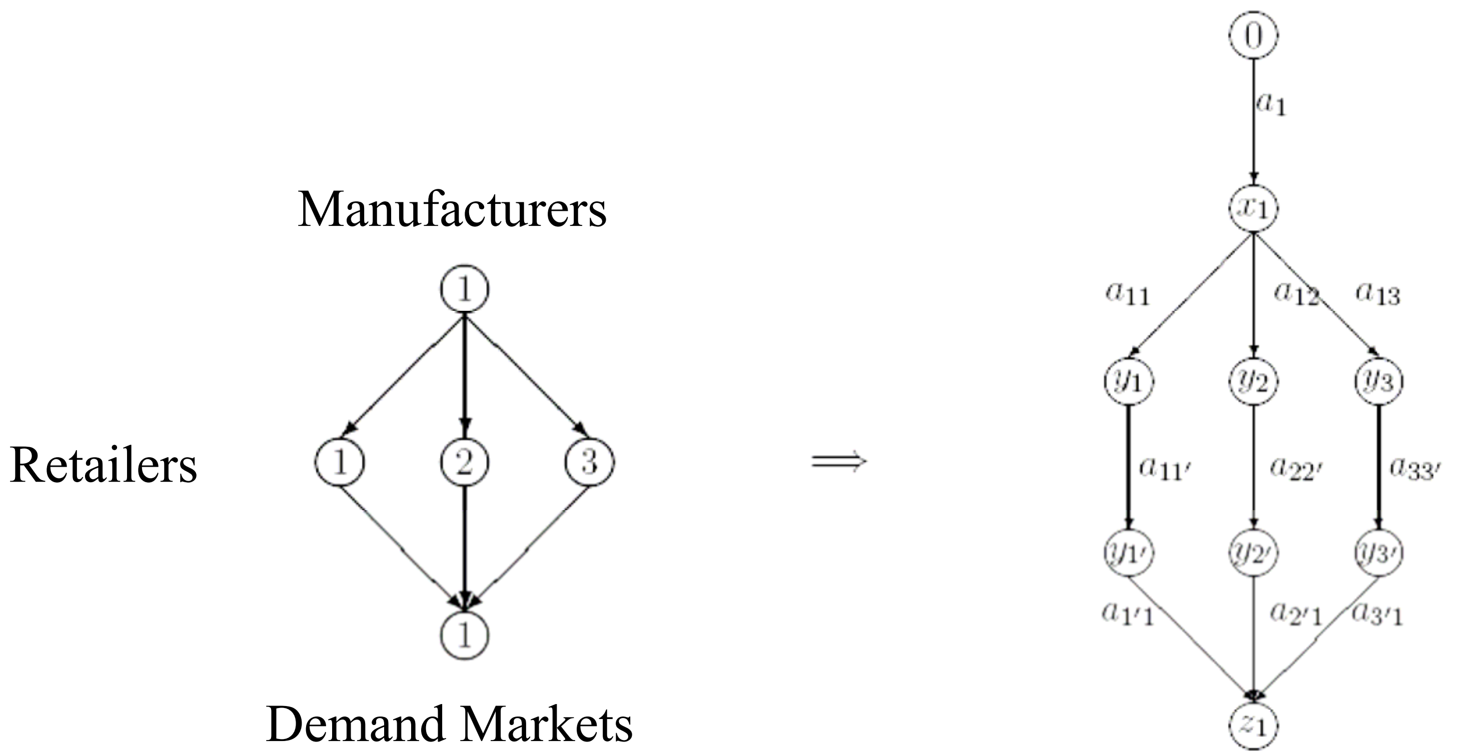
- First, construct the equivalent transportation network equilibrium model
- Solve the transportation network equilibrium model with time varying demands
- Convert the solution of the transportation network into the time-dependent supply chain network equilibrium model

Dynamic Supply Chain Network Examples with Computations

- Example 1



Numerical Example 1



The Equivalent Transportation Network

Numerical Example 1

- Production cost functions

$$f_1(q_1(t)) = 2.5(q_1(t))^2 + 2q_1(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5(q_{12}(t))^2 + 2.5q_{12}(t),$$

$$c_{13}(q_{13}(t)) = .5(q_{13}(t))^2 + 1.5q_{13}(t).$$

Numerical Example 1

- Handling cost functions of the retailers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

- Unit transaction cost between the retailers and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{21}(Q^2(t)) = q^1_{21}(t) + 5, \quad \hat{c}^1_{31}(Q^2(t)) = q^1_{31}(t) + 10.$$

Numerical Example 1

- Three paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

- The time-varying demand function

$$d_{w_1}(t) = d_1(t) = 41 + 10t.$$

The Solution of Numerical Example 1

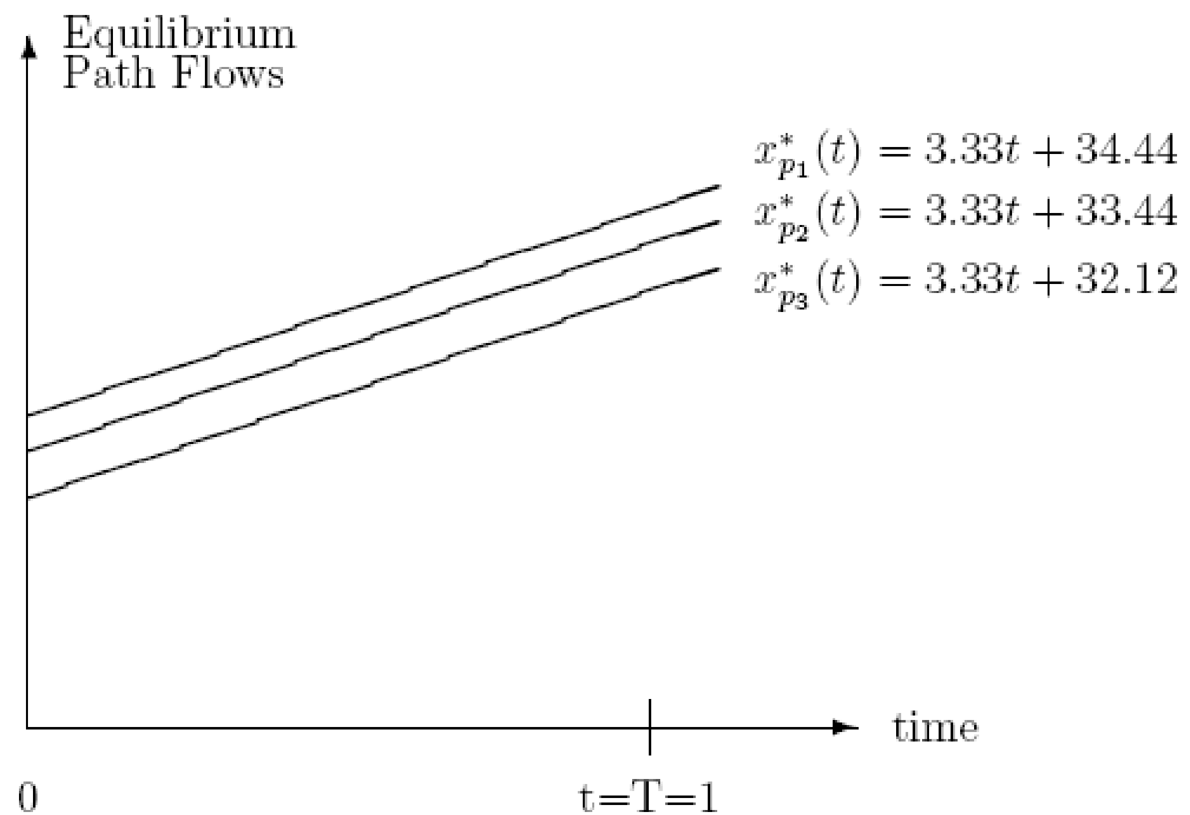
- Explicit Solution
  - Path flows

$$x^*_{p_1}(t) = 3.33t + 14.78,$$
$$x^*_{p_2}(t) = 3.33t + 13.78,$$
$$x^*_{p_3}(t) = 3.33t + 12.45,$$

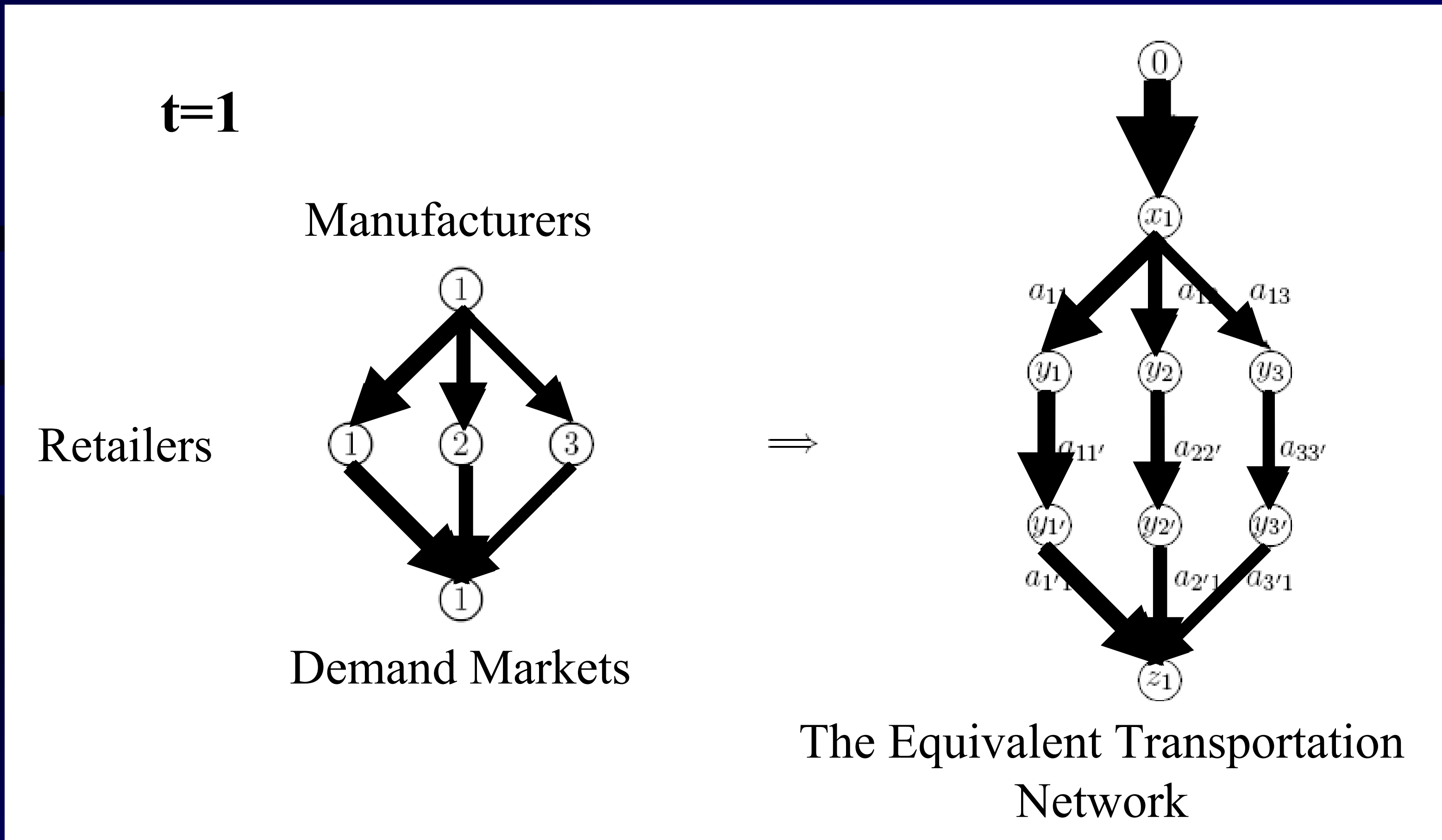
  - Travel disutility

$$\lambda^*_{w_1}(t) = 60t + 255.83, \quad \text{for} \quad t \in [0, T].$$

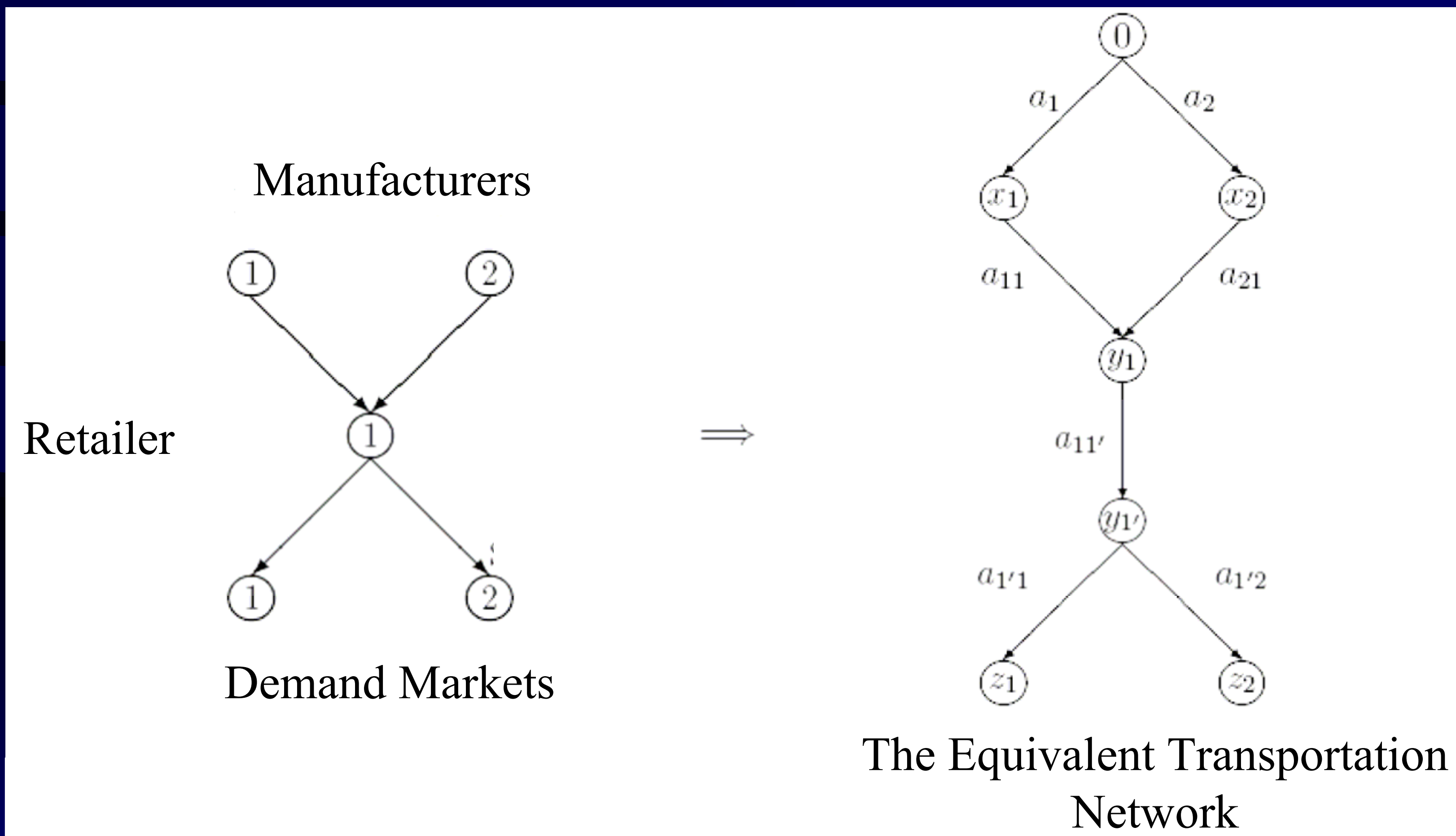
Time-Dependent Equilibrium Path Flows for Numerical Example 1



The Solution of Numerical Example 1



Numerical Example 2



Numerical Example 2

- Production cost functions

$$f_1(q(t)) = 2.5(q_1(t))^2 + q_1(t)q_2(t) + 2q_1(t), \quad f_2(q(t)) = 2.5(q_2(t))^2 + q_2(t)q_1(t) + 2q_2(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{21}(q_{21}(t)) = .5(q_{21}(t))^2 + 1.5q_{21}(t).$$

Numerical Example 2

- Handling cost function of the retailer

$$c_1(Q^1(t)) = .5(q_{11}(t))^2.$$

- Unit transaction costs between the retailer and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{12}(Q^2(t)) = q^1_{12}(t) + 1,$$

Numerical Example 2

- Four paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}),\ p_2 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$
$$p_3 = (a_1, a_{11}, a_{11'}, a_{1'2}),\ p_4 = (a_2, a_{21}, a_{11'}, a_{1'2}).$$

- The time-varying demand functions

$$d_{w_1}(t) = d_1(t) = 100 + 5t, \quad d_{w_2}(t) = d_2(t) = 80 + 4t.$$

The Solution of Numerical Example 2

- Numerical Solution
  - t=0

$$x^*_{p_1} = 49.90, \quad x^*_{p_2} = 50.10, \quad x^*_{p_3} = 39.90, \quad x^*_{p_4} = 40.10.$$

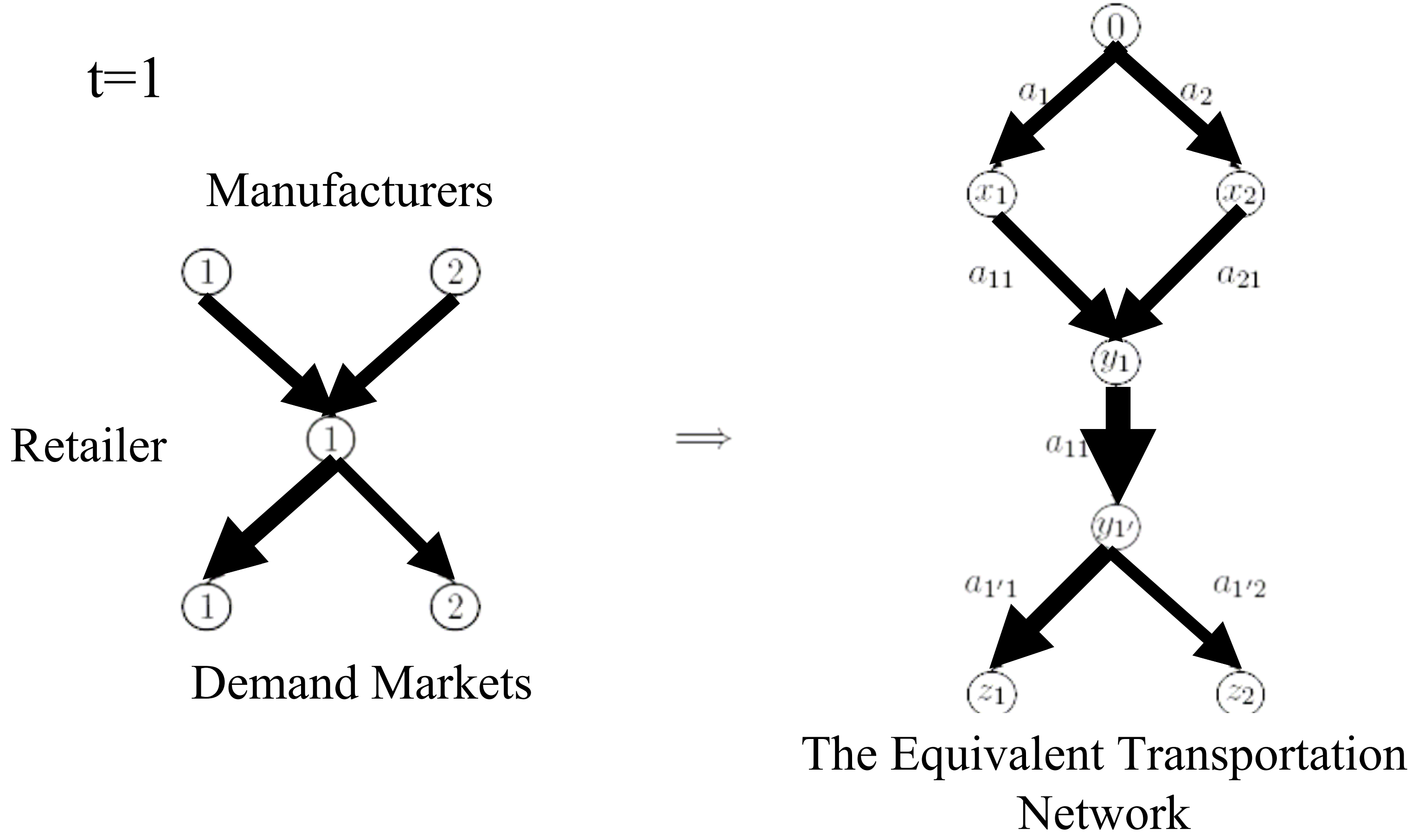
  - t=1/2

$$x^*_{p_1} = 51.15, \quad x^*_{p_2} = 51.35, \quad x^*_{p_3} = 40.90, \quad x^*_{p_4} = 41.10.$$

  - t=1

$$x^*_{p_1} = 52.40, \quad x^*_{p_2} = 52.60, \quad x^*_{p_3} = 41.90, \quad x^*_{p_4} = 42.10.$$

The Solution of Numerical Example 2



Conclusions

- We established the supernetwork equivalence of the supply chain networks with transportation networks.

- We utilized this isomorphism in the computation of the supply chain network equilibrium with time-varying demands.

Conclusions

- We are also investigating the applications to electric power networks.
  - Nagurney, A. and Matsypura, D., (2004), A Supply Chain Network Perspective for Electric Power Generation, Supply, Transmission, and Consumption, Proceedings of the International Conference on Computing, Communications and Control Technologies, Austin, Texas, Volume VI: (2004) pp 127-134.)
  - Nagurney, A., Liu, Z., Cojocaru, M-G., and Daniele, P., (2006), Dynamic Electric Power Supply Chains and Transportation Networks: An Evolutionary Variational Inequality Formulation (To appear in *Transportation Research E*.)

Thank You!

For more information, please see:

The Virtual Center for Supernetworks

http://supernet.som.umass.edu