On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations

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- The Solution of the Supply Chain Network Equilibrium Model via a Discrete-Time Algorithm developed for Elastic Demand Traffic Networks
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Introduction and Background

The topic of transportation network equilibrium and appears as early as in the work of Kohl (1841) and Pigou (1920), with the first rigorous mathematical treatment given by Beckmann, McGuire, and Winsten (1956) in their classical book and using Wardrop's principles.

Other seminal publications in terms of transportation network equilibrium modeling and methodological contributions include those of Dafermos and Sparrow (1969), Evans (1976), Florian (1977), Smith (1979), Dafermos (1980, 1982), and Boyce et al. (1983).

For additional research highlights in transportation network equilibrium, see the paper by Boyce, Mahmassani, and Nagurney (2004) as well as that of Florian and Hearn (1995) and the books by Patriksson (1994) and Nagurney (1999, 2000).

The richness of tools developed for transportation network equilibrium modeling, analysis, and computations have also been exploited in numerous other applications, including:

- spatial economic problems;

- telecommunication network problems;

- a variety of economic equilibrium problems including Walrasion price equilibrium and oligopolistic market equilibrum problems;

- and even recently in Internet marketing as well as in

- knowledge networks.

Here we ask the question:

Is there a relationship between **supply chain network equilibria** and **traffic network equilibria**?

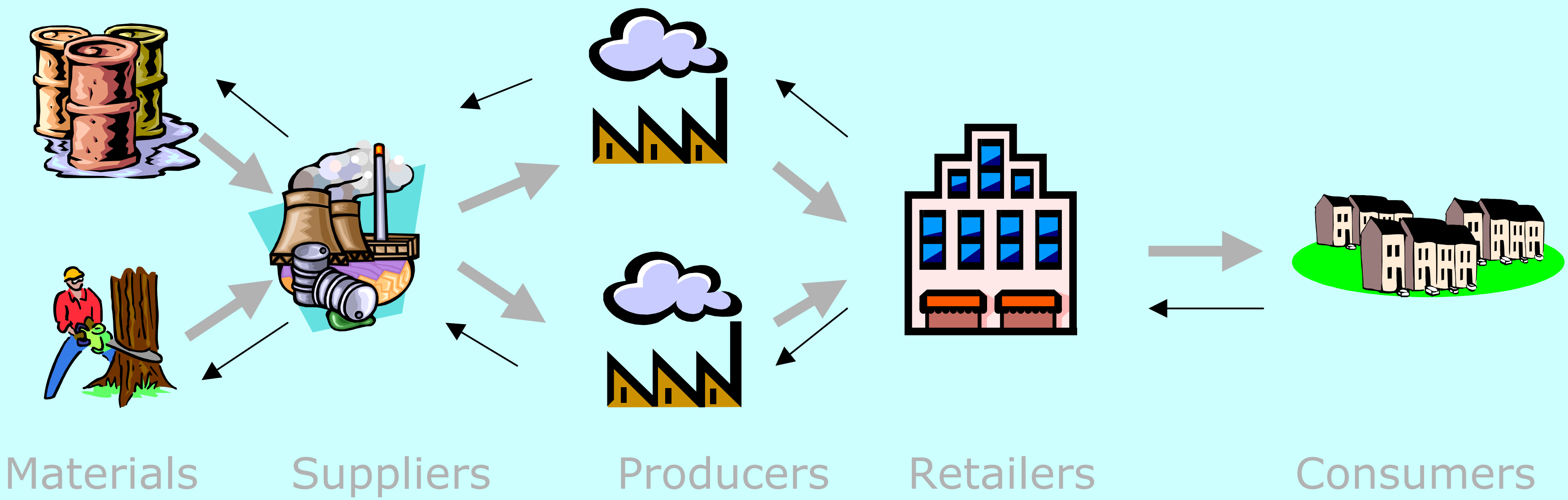
Some Background on Supply Chains

In supply chain modeling and analysis (cf. Federgruen and Zipkin (1986), Federgruen (1993), Lee and Billington (1993), Slats et al. (1995), Anupindi and Bassok (1996), Bramel and Simchi-Levi (1997), Lederer and Li (1997), Stadtler and Kilger (2000), Miller (2001), Mentzer (2001), Hensher, Button, and Brewer (2001) and the references therein), one, typically, associates the decision-makers with the nodes of the multitiered supply chain network.

In transportation networks, on the other hand, the nodes represent origins and destinations as well as intersections. Travelers or users of the transportation networks seek, in the case of user-optimization (cf. Wardrop (1952), Beckmann, McGuire, and Winsten (1956), and Dafermos and Sparrow (1969)), to determine their cost-minimizing routes of travel.

The "gaming" or competition on a transportation network takes place on paths associated with origin/destination pairs of nodes whereas that in the supply chain networks takes place on the nodes and links.

A Supply Chain Network



Examples of Supply Chains

IKEA

– 172 stores with products from 1,600 suppliers in 55 countries.

Carrefour

– the second largest retailer worldwide with 11,000 stores in 30 countries and over 250 warehouses. Products come from 15,000 suppliers and are distributed using 1.5 million trucks that carry more than 40 million pallets each year in the 4 major European countries.

Toyota

- Parts and materials from over 500 suppliers and has 6 parts facilities and 5 assembly facilities in North America. Vehicles are sold at more than 1700 North American dealers.

H&M

- 1,069 stores in 21 countries; Products come from 700 suppliers.

Ford Motor Company

- operates a global supply chain consisting of 2311 partners and suppliers, 5127 dealers in the U.S and 13000 dealers worldwide.

McDonald's

– 30,000 restaurants in 119 countries. In the U.S, it operates over 40 distribution centers and 12,000 restaurants.

Wal-Mart

– the world’s number one retailer; operates 4900 stores; Uses 68,000 U.S suppliers and 6000 international suppliers.

METRO GROUP

– 1,700 outlets in Germany. More than 8,000 suppliers supply 1 million separate items. Annually, more than 50 million orders to suppliers.

The topic of supply chain analysis is *interdisciplinary* by nature since it involves manufacturing, transportation and logistics, as well as retailing/marketing.

It has been the subject of a growing body of literature with the associated research being both conceptual in nature due to the complexity of the problem and the numerous agents, such as manufacturers, retailers, and consumers involved in the transactions, as well as analytical.

Lee and Billington (1993) expressed the need for decentralized models that allow for a generalized network structure and simplicity in the study of supply chains.

Anupindi and Bassok (1996), in turn, addressed the challenges of formulating systems consisting of decentralized retailers with information sharing.

Corbett and Karmarkar (2001) were concerned with the equilibrium number of firms in oligopolistic competition in a supply chain.

An Equilibrium Framework

Many researchers, in addition to, practitioners, have described the various networks that underly supply chain analysis and management with the goal being primarily that of *optimization*.

In 2002, Nagurney, Dong, and Zhang in *Transportation Research E* developed apparently the first supply chain network equilibrium model.

It *provides a benchmark* against which one can evaluate both price and product flows. The equilibrium model captures both the independent behavior of the various decision-makers as well as the effect of their interactions.

This model has been used to-date as the foundation for:

- the introduction of electronic commerce into this setting;

- the introduction of risk and uncertainty (cf. Nagurney, Loo, Dong, and Zhang (2002), Dong, Zhang, and Nagurney (2004), and the references therein) as well as

- the development of dynamic, multilevel supply chain network models (Nagurney, Ke, Cruz, Hancock, and Southworth (2002)).

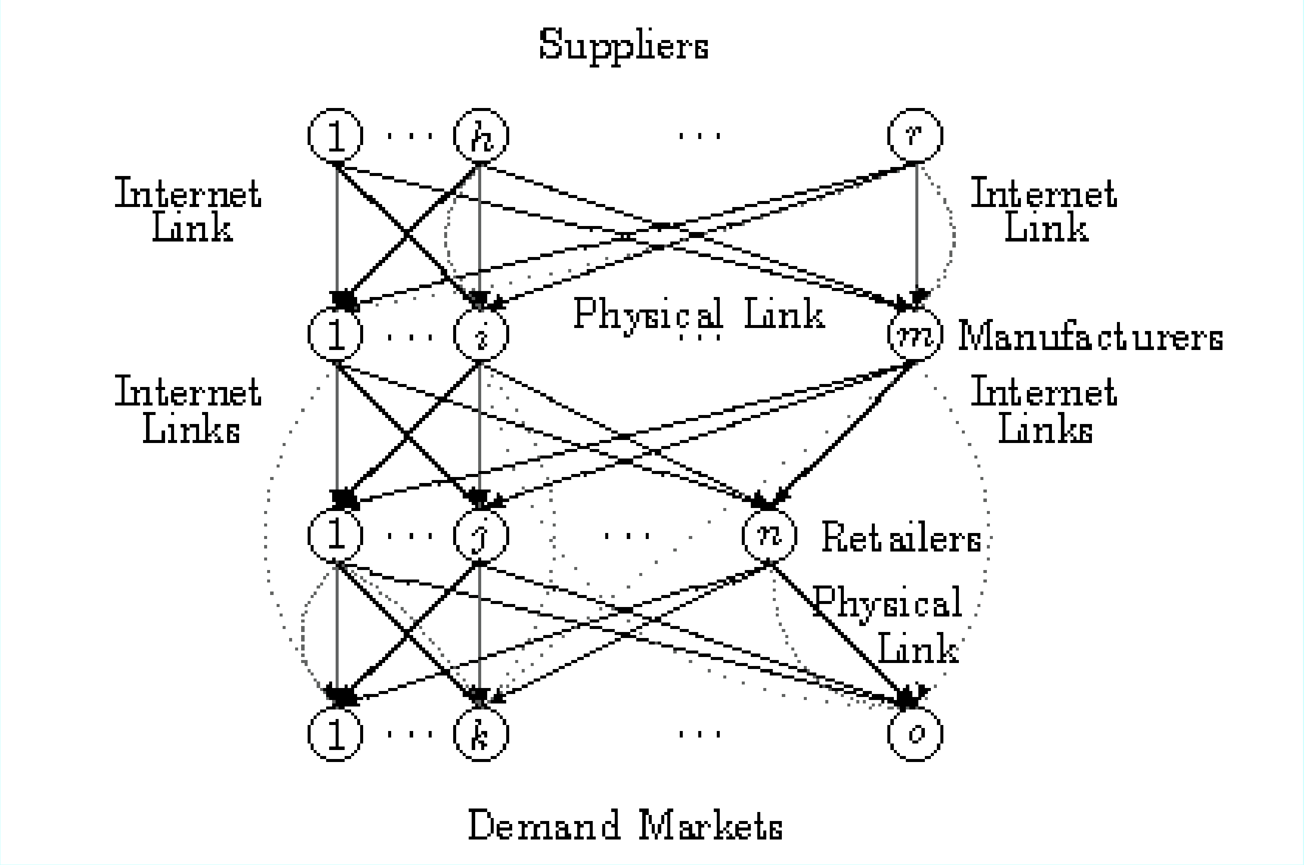
It has also been instrumental in yielding supply chain network perspectives for other application areas, including:

- recycling, notably, electronic recycling networks (see Nagurney and Toyasaki (2005));

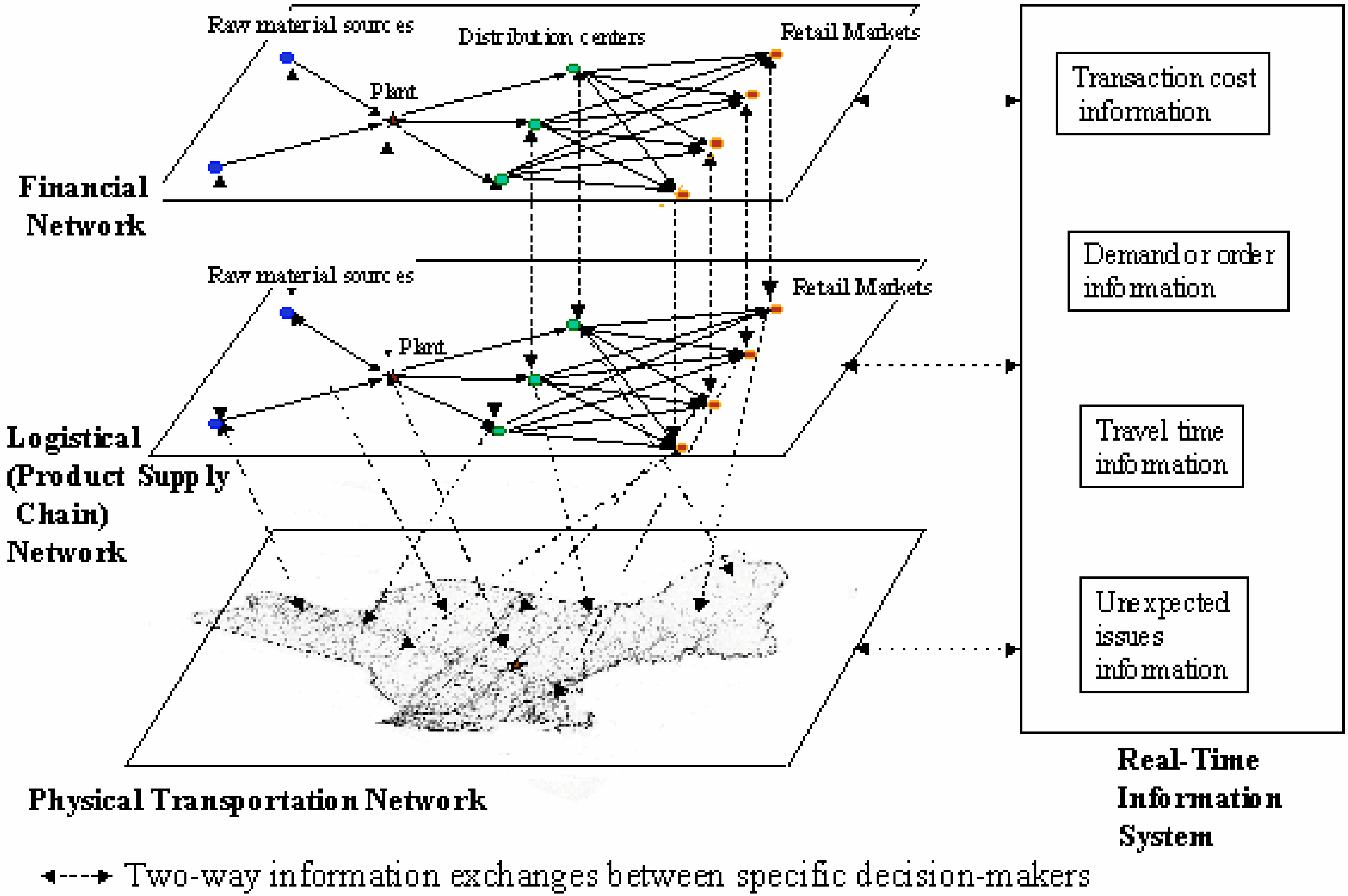
- power/electric grid networks consisting of suppliers, generators, distributors, transmitters, and consumers (Nagurney and Matsypura (2004)), and for

- the integration of social networks with supply chain networks (Wakolbinger and Nagurney (2004)).

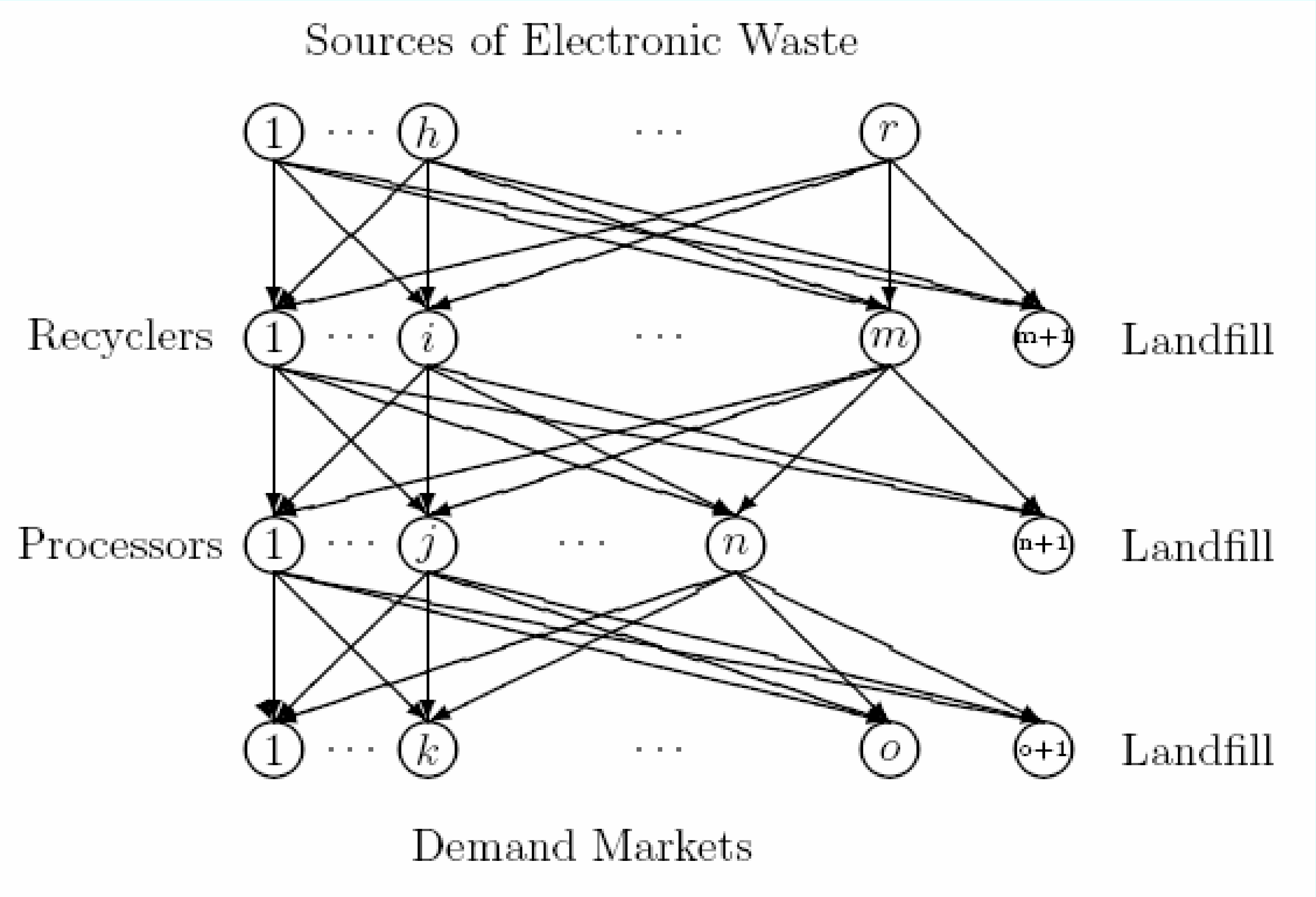
A Supply Chain Network with Electronic Commerce



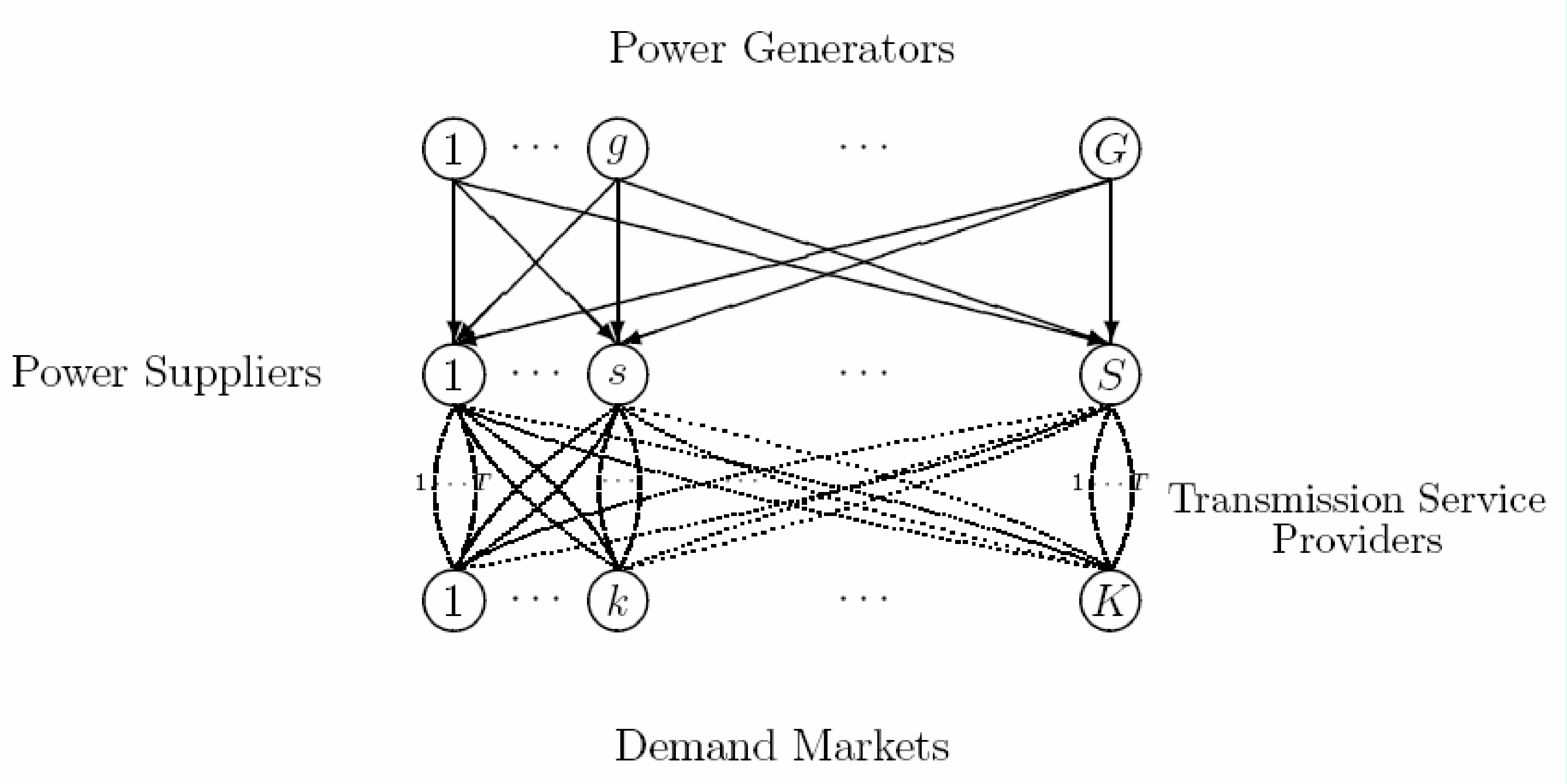
Supply Chain -Transportation Supernetwork Representation



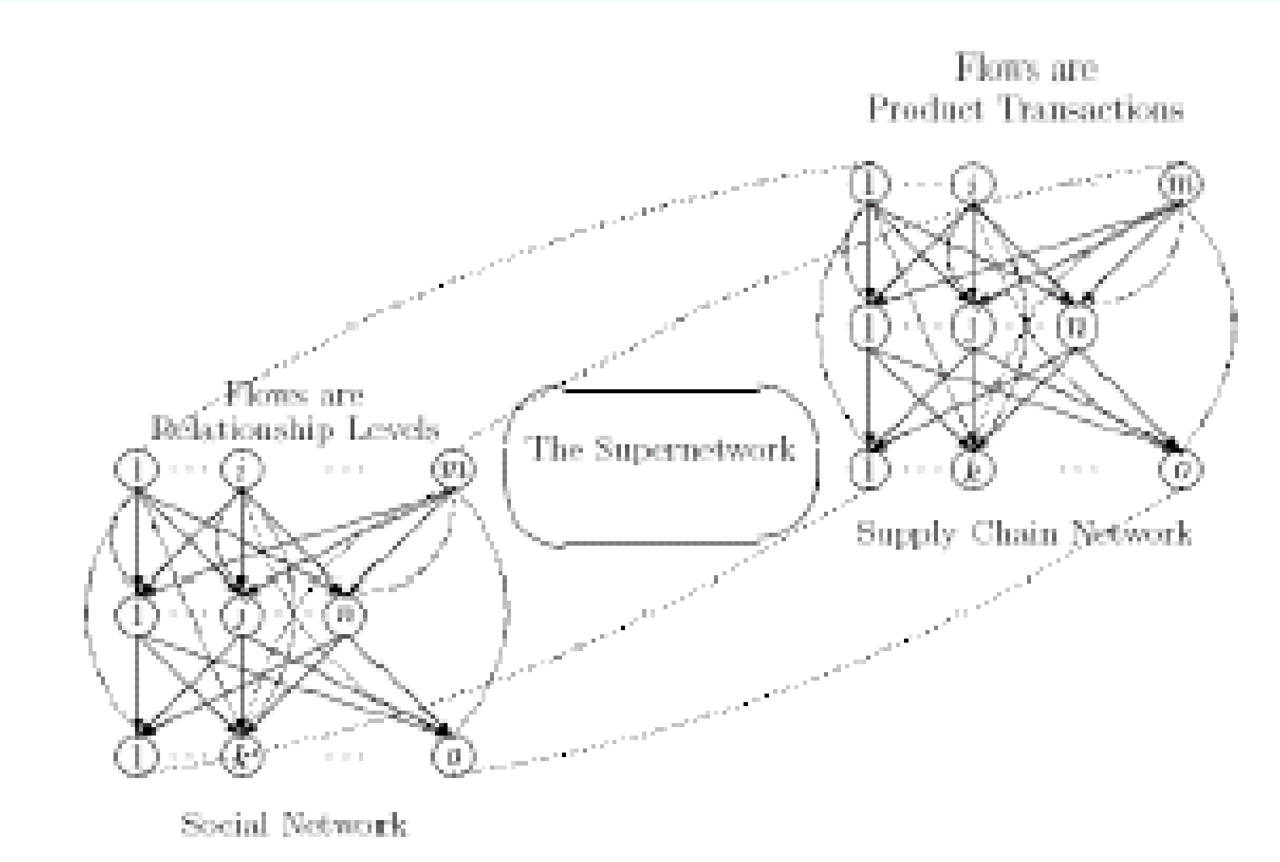
The 4-Tiered E-Cycling Network



The Electric Power Supply Chain Network



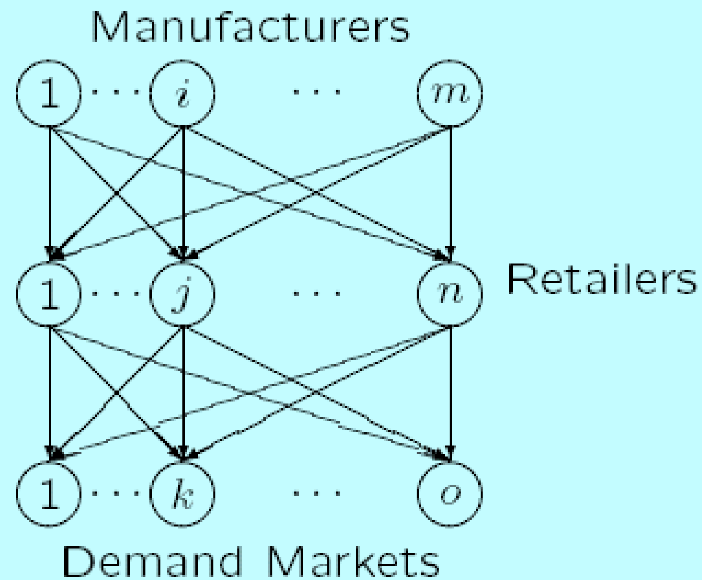
Integrated Supply Chain - Social Network System



The Supply Chain Network Equilibrium Model

We recall the supply chain network equilibrium model proposed in Nagurney, Dong, and Zhang in *Transportation Research E* (2002).

The manufacturers are involved in the production of a homogeneous product, which can then be purchased by the retailers, who, in turn, make the product available to consumers at the demand markets. The links in the supply chain network denote transportation/transaction links.



The Network Structure of the Supply Chain at Equilibrium

Supply Chain Network Equilibrium Model Notation

Notation	Definition
q	manufacturers' production output vector with components: $q_1, \ldots, q_m$
Q^1	mn-dimensional vector of product shipments between manufacturers and retailers with component ij denoted by q_{ij}
Q^2	no-dimensional vector of product shipments between retailers and demand markets with component jk denoted by q_{jk}
γ	n-dimensional vector of shadow prices associated with the retailers with component j denoted by γ_j
ρ_3	o-dimensional vector of demand market prices with component k denoted by ρ_{3k}
$f_i(q)$ $\equiv f_i(Q^1)$	production cost of manufacturer i with marginal production cost with respect to q_i denoted by $\frac{\partial f_i}{\partial q_i}$ and the marginal production cost with respect to q_{ij} denoted by $\frac{\partial f_i(Q^1)}{\partial q_{ij}}$
$c_{ij}(q_{ij})$	transaction cost between manufacturer i and retailer j with marginal transaction cost denoted by $\frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}}$
s	vector of the retailers' supplies of the product with components: $s_1, \ldots, s_n$
$c_j(s)$	handling cost of retailer j with marginal such cost with respect to $s_j \equiv c_j(Q^1)$ denoted by $\frac{\partial c_j}{\partial s_j}$ and the marginal handling cost with respect to q_{ij} denoted by $\frac{\partial c_j(Q^1)}{\partial q_{ij}}$
$c_{jk}(Q^2)$	unit transaction cost between retailer j and demand market k
$d_k(\rho_3)$	demand function at demand market k

The Behavior of the Manufacturers

Let ρ^*_{1ij} denote the price charged for the product by manufacturer i to retailer j (i.e., the supply price) and note the conservation of flow equations that express the relationship between the quantity of the product produced by manufacturer i and the associated shipments to the retailers:

$$q_i = \sum_{j=1}^{n} q_{ij}, \quad i = 1, \ldots, m. \tag{1}$$

Due to (1), and as noted in Table 1, we may express the production cost associated with manufacturer i, namely, f_i as follows: $f_i(Q^1) \equiv f_i(q)$ for all $i = 1, \ldots, m$. We can, thus, express the criterion of profit maximization for manufacturer i as:

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho^*_{1ij} q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}) \tag{2}$$

subject to $q_{ij} \geq 0$, for all j.

We assume that the production cost functions and the transaction cost functions for each manufacturer are continuously differentiable and convex. Hence, as discussed in Nagurney, Dong, and Zhang (2002), assuming that the manufacturers compete in a noncooperative fashion in the sense of Cournot (1838) and Nash (1950, 1951), **the optimality conditions for all manufacturers simultaneously** (see also Bazaraa, Sherali, and Shetty (1993) and Nagurney (1999)) may be expressed as the following variational inequality:

determine $Q^{1*} \in R^{mn}_{+}$ satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q^*_{ij})}{\partial q_{ij}} - \rho^*_{1ij}\right] \times \left[q_j - q^*_{ij}\right] \geq 0,$$

$$\forall Q^1 \in R^{mn}_{+}. \tag{3}$$

The Behavior of the Retailers

The retailers, in turn, are involved in transactions both with the manufacturers since they wish to obtain the product for their retail outlets, as well as with the consumers, who are the ultimate purchasers of the product.

The retailers associate a price with the product at their retail outlets, which is denoted by ρ^*_{2j}, for retailer j. This price is determined endogenously in the model along with the prices associated with the manufacturers, that is, the ρ^*_{1ij}, for all i and j. Assuming that the retailers are also profit-maximizers, **the optimization problem of retailer** j is given by:

$$\text{Maximize} \quad \rho^*_{2j} \sum_{k=1}^{o} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho^*_{1ij} q_{ij} \tag{4}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} q_{ij}, \tag{5}$$

and the nonnegativity constraints: $q_{ij} \geq 0$, and $q_{jk} \geq 0$, for all i and k.

The term γ_j is the Lagrange multiplier associated with constraint (5) for retailer j and, hence, has an interpretation as a shadow price as noted in Table 1.

We assume that the handling cost for each retailer is continuously differentiable and convex and that the retailers also compete with one another in a noncooperative manner, seeking to determine their optimal shipments from the manufacturers and to the demand markets. **The optimality conditions for all retailers simultaneously** coincide with the solution of the following variational inequality:

determine $(Q^{1*}, Q^{2*}, \gamma^*) \in R_+^{mn+no+o}$ satisfying:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1*})}{\partial q_{ij}} + \rho_{1ij}^* - \gamma_j^* \right] \times \left[q_{ij} - q_{ij}^* \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{o} \left[-\rho_{2j}^* + \gamma_j^* \right] \times \left[q_{jk} - q_{jk}^* \right]$$

$$+ \sum_{j=1}^{n} \left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^* \right] \times \left[\gamma_j - \gamma_j^* \right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma) \in R_+^{mn+no+o}. \qquad (6)$$

The Consumers at the Demand Markets and the Equilibrium Conditions

The consumers take the price charged by the retailers for the product, which, recall was denoted by ρ_{2j}^* for retailer j, plus the transaction cost associated with obtaining the product, in making their consumption decisions. The **equilibrium condition for consumers at demand market** k, (cf. Samuelson (1952) and Takayama and Judge (1971)) takes the form: For all retailers j; $j = 1, \ldots, n$:

$$\rho_{2j}^* + c_{jk}(Q^{2^*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0 \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0, \end{cases} \tag{7}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} q_{jk}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} q_{jk}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{8}$$

The Equilibrium Conditions of the Supply Chain

In equilibrium, we must have that the optimality conditions for all manufacturers, the optimality conditions for all retailers, and the equilibrium conditions for all the demand markets, must hold simultaneously. Hence, the shipments that the manufacturers ship to the retailers must, in turn, be the shipments that the retailers accept from the manufacturers. In addition, the amounts of the product purchased by the consumers at the demand markets must be equal to the amounts sold by the retailers.

We now state this more formally as done originally in Nagurney, Dong, and Zhang (2002):

Definition 1: Supply Chain Network Equilibrium

The equilibrium state of the supply chain network is one where: all manufacturers have achieved optimality (cf. (3)); all retailers have achieved optimality (cf. (6)), and, finally, for all pairs of retailers and demand markets, equilibrium conditions (7) and (8) hold.

We now recall the following theorem:

Theorem 1: Variational Inequality Formulation (Nagurney, Dong, and Zhang (2002))

The equilibrium conditions governing the supply chain network model are equivalent to the solution of the variational inequality problem given by:

determine $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*) \in R_+^{mn+m+n+o}$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1^*})}{\partial q_{ij}} - \gamma_j^*\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2^*}) + \gamma_j^* - \rho_{3k}^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* - d_k(\rho_3^*)\right] \times \left[\rho_{3k} - \rho_{3k}^*\right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma, \rho_3) \in R_+^{mn+no+n+o}. \qquad (9)$$

Corollary 1 (Nagurney, Dong, and Zhang (2002))

The market for the product clears for each retailer at the supply chain network equilibrium.

In addition, for notational convenience, and subsequent use, we let

$$d_k \equiv \sum_{j=1}^{n} q_{jk}, \quad k = 1, \ldots, n, \tag{11}$$

and

$$s_j \equiv \sum_{k=1}^{o} q_{jk}, \quad j = 1, \ldots, n. \tag{12}$$

The following result is then immediate:

Corollary 2

A solution $(Q^{1*}, Q^{2*}, \rho_3^*) \in \mathcal{K}$ to the variational inequality problem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ij}}\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*}) - \rho_{3k}^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (Q^1, Q^2, \rho_3) \in \mathcal{K}; \tag{13a}$$

equivalently, a solution $(q^*, Q^{1*}, s^*, Q^{2*}, d^*, \rho_3^*) \in \mathcal{K}^2$ to:

$$\sum_{i=1}^{m}\left[\frac{\partial f_i(q^*)}{\partial q_i}\right] \times [q_i - q_i^*] + \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\left[\frac{\partial c_j(s^*)}{\partial s_j}\right] \times \left[s_j - s_j^*\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*})\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$-\sum_{k=1}^{o} \rho_{3k}^* \times [d_k - d_k^*]$$

$$+\sum_{k=1}^{o}[d_k^* - d_k(\rho_3^*)]\times[\rho_{3k}-\rho_{3k}^*] \geq 0, \quad \forall (q, Q^1, s, Q^2, d, \rho_3) \in \mathcal{K}^2, \tag{13b}$$

where

$$\mathcal{K}^2$$

$$\equiv \{(q, Q^1, s, Q^2, d, \rho_3)|(q, Q^1, s, Q^2, s, \rho_3) \in R_+^{m+mn+n+no+2o}$$

$$\textit{and}\,(1),(10)-(12),\ \textit{hold}\}$$

also satisfies variational inequality (9).

The Traffic Network Equilibrium Model with Elastic Demand

We now review a traffic network equilibrium model with elastic demands in which it is assumed that the demand functions associated with the origin/destination (O/D) pairs are given. We present the single modal version of the model of Dafermos and Nagurney (1984).

Consider a network G with the set of links L consisting of K elements, the set of paths P consisting of Q elements, and W denoting the set of O/D pairs with Z elements. Let P_w denote the set of paths joining O/D pair w. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w, ω, etc.

The flow on a path p is denoted by x_p and the flow on a link a by f_a. The user travel cost on a path p is denoted by C_p and the user travel cost on a link a by c_a. The travel demand associated with traveling between O/D pair w is denoted by d_w and the travel disutility by λ_w.

We assume that the flows on links are related to the flows on the paths by the conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{15}$$

where $\delta_{ap} = 1$ if link a is contained in path p, and $\delta_{ap} = 0$, otherwise.

The user costs on paths are related to user costs on links as follows:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{16}$$

that is, the user cost on a path is equal to the sum of user costs on links that comprise the path.

Here we consider the general situation where the cost on a link may depend upon the entire vector of link flows, denoted by f, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{17}$$

Also, we assume, as given, travel demand functions, such that

$$d_w = d_w(\lambda), \quad \forall w \in W, \tag{18}$$

where λ is the vector of travel disutilities with the travel disutility associated with O/D pair being denoted by λ_w.

As given in Dafermos and Nagurney (1984); see also Aashtiani and Magnanti (1981), Fisk and Boyce (1982), Nagurney and Zhang (1996), and Nagurney (1999), a travel path flow and disutility pattern $(x^*, \lambda^*) \in R_+^{QZ}$ is said to be an equilibrium, if, once established, no user has any incentive to alter his travel choices. The state is characterized by the following **equilibrium conditions** which must hold for every O/D pair $w \in W$ and every path $p \in P_w$:

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0 \\ \geq 0, & \text{if} \quad x_p^* = 0 \end{cases} \tag{19}$$

and

$$\sum_{p \in P_w} x_p^* \begin{cases} = d_w(\lambda^*), & \text{if} \quad \lambda_w^* > 0 \\ \geq d_w(\lambda^*), & \text{if} \quad \lambda_w^* = 0. \end{cases} \tag{20}$$

Condition (19) states that all utilized paths connecting an O/D pair have equal and minimal travel costs and these costs are equal to the travel disutility associated with traveling between that O/D pair. Condition (20) states that the market clears for each O/D pair under a positive price or travel disutility.

The traffic network equilibrium conditions (18) and (19) can be formulated as the variational inequality: determine $(x^*, \lambda^*) \in R_+^{QZ}$ such that

$$\sum_{w \in W} \sum_{p \in P_w} [C_p(x^*) - \lambda_w^*] \times \left[x_p - x_p^*\right]$$

$$+ \sum_{w \in W} \sum_{p \in P_w} \left[x_p^* - d_w(\lambda^*)\right] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (x, \lambda) \in R_+^{QZ}. \tag{21}$$

Note that variational inequality (21) is in *path* flows.

Theorem 2 (Link Flow Variational Inequality Formulation – Dafermos and Nagurney (1984))

A travel link flow pattern and associated travel demand and disutility pattern is a traffic network equilibrium if and only if it satisfies the variational inequality problem: determine $(f^*, d^*, \lambda^*) \in \mathcal{K}^3$ satisfying

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) - \sum_{w} \lambda_w^* \times (d_w - d_w^*)$$

$$+ \sum_{w \in W} [d_w^* - d_w(\lambda^*)] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (f, d, \lambda) \in \mathcal{K}^3, \tag{22}$$

where $\mathcal{K}^3$

$\equiv \{(f, d, \lambda) \in R_+^{K+2Z} | \, \exists (x)$ *satisfying* (15); $d_w = \sum_{p \in P_w} x_p, \forall w\}$.

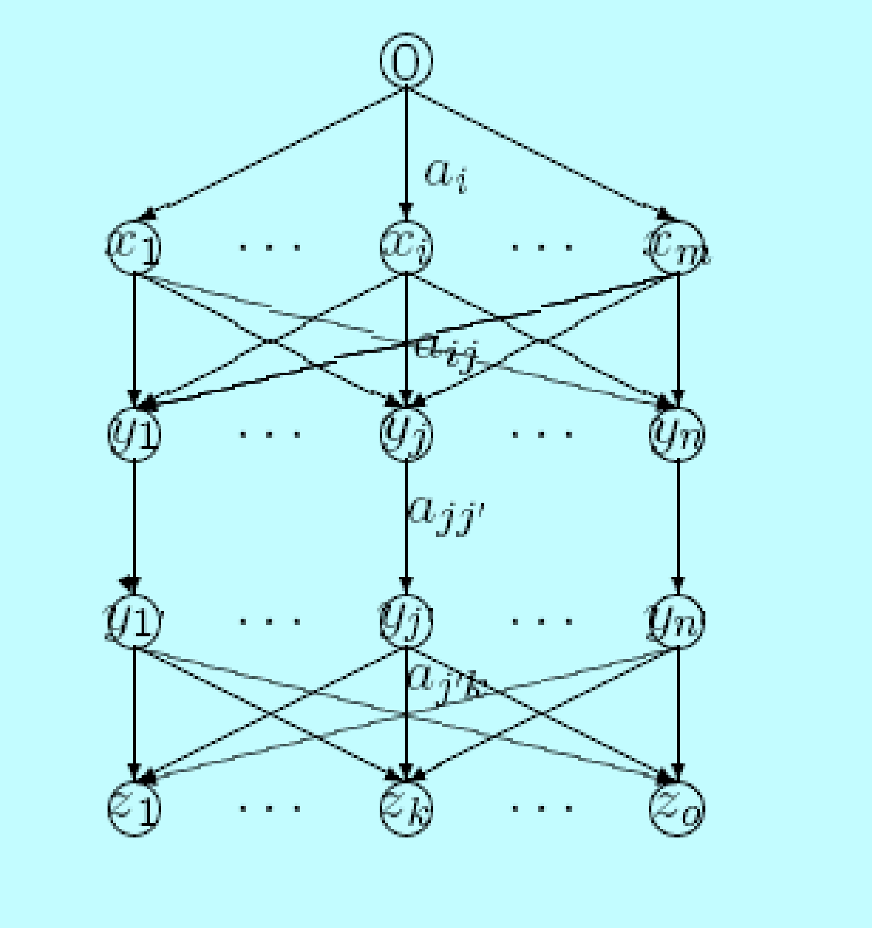
Supernetwork Equivalence of the Supply Chain Network Equilibrium Model

We now cast the supply chain network equilibrium problem into *supernetwork* form, which then reveals the configuration of the associated *isomorphic* traffic network equilibrium problem.

By constructing the associated supernetwork, which is an abstract network (see also Nagurney and Dong (2002)), the linkages to traffic network equilibrium become apparent through the identification of the origin/destination pairs, the paths connecting the origin/destination pairs, the costs on the links of the supernetwork, and the origin/destination pair demand functions and travel disutilities.

Note that isomorphic traffic networks have been identified in the case of spatial price equilibrium problems, single and multimodal ones, respectively, by Dafermos and Nagurney (1985) and Dafermos (1986); other applications can be found in the book by Nagurney (1999).

The G_S Supernetwork Representation of Supply Chain Network Equilibrium



In the supernetwork G_S of the isomorphic traffic network equilibrium model there is a single origin node 0, and o destination nodes at the bottom tier of nodes, denoted, respectively, by: $z_1, \ldots, z_o$.

There are o O/D pairs in G_S denoted, respectively, by $w_1 = (0, z_1)$, ..., $w_k = (0, z_k)$,..., $w_o = (0, z_o)$.

There are $1 + m + 2n + o$ nodes in the supernetwork; $K = m + mn + n + no$ links; $Z = o$ O/D pairs, and $Q = mo$ paths.

We now turn our attention to the definition of the links in the supernetwork in Figure 2 and the associated flows.

Let a_i denote the link from node 0 to node x_i with associated link flow f_{a_i}, for $i = 1, \ldots, m$. Let a_{ij} denote the link from node x_i to node y_j with associated link flow $f_{a_{ij}}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Also, let $a_{jj'}$ denote the link connecting node y_j with node $y_{j'}$ with associated link flow $f_{a_{jj'}}$ for j; $j' = 1, \ldots, n$.

Finally, let $a_{j'k}$ denote the link joining node $y_{j'}$ with node z_k for $j' = 1', \ldots, n'$ and $k = 1, \ldots, o$ and with associated link flow $f_{a_{j'k}}$.

We group the link flows into the vectors, respectively, as follows: we group the $\{f_{a_i}\}$ into the vector f^1; the $\{f_{a_{ij}}\}$ into the vector f^2; the $\{f_{a_{jj'}}\}$ into the vector f^3, and the $\{f_{a_{j'k}}\}$ into the vector f^4.

A typical path, hence, joining origin node 0 with destination node z_k, consists of four links. We, thus, note a typical path by $p_{ijj'k}$ which means that this path consists of links: $a_i, a_{ij}, a_{jj'}$, and $a_{j'k}$ with the associated flow on the path equal to $x_{p_{ijj'k}}$. Also, we let d_{w_k} denote the demand associated with O/D pair w_k and λ_{w_k} the travel disutility for w_k.

Note that the satisfaction of the conservation of flow equations (15) means that:

$$f_{a_i} = \sum_{jj'k} x_{p_{ijj'k}}, \quad i = 1, \ldots, m, \tag{23}$$

$$f_{a_{ij}} = \sum_{j'k} x_{p_{ijj'k}}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{24}$$

$$f_{a_{jj'}} = \sum_{ik} x_{p_{ijj'k}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n, \tag{25}$$

$$f_{a_{j'k}} = \sum_{ij} x_{p_{ijj'k}}, \quad j' = 1, \ldots, n; k = 1, \ldots, o. \tag{26}$$

Also, we have that

$$d_{w_k} = \sum_{ijj'} x_{ijj'k}, \quad k = 1, \ldots, o. \tag{27}$$

A path flow pattern induces a feasible link flow pattern if all path flows are nonnegative and (23)–(27) are satisfied.

Suppose now that we are given a feasible product shipment pattern for the supply chain model, $(q, Q^1, s, Q^2, d) \in \mathcal{K}^2$, that is, Q^1 and Q^2 consist of nonnegative product shipments and (1) and (10)–(12) are satisfied. We may construct a feasible link flow pattern on the network G_s as follows: the link flows and travel demands are defined as:

$$q_i \equiv f_{a_i}, \quad i = 1, \ldots, m, \tag{28}$$

$$q_{ij} \equiv f_{a_{ij}}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{29}$$

$$s_j \equiv f_{a_{jj'}}, \quad j = 1, \ldots, n; j' = 1, \ldots, n', \tag{30}$$

$$q_{jk} = f_{a_{j'k}}, \quad j' = 1', \ldots, n'; k = 1, \ldots, o, \tag{31}$$

$$d_k = \sum_{j=1}^{n} q_{jk}, \quad k = 1, \ldots, o. \tag{32}$$

Note that if (q, Q^1, s, Q^2, d) is feasible then the link flow pattern constructed according to (28) – (32) is also feasible and the corresponding path flow pattern that induces such a link flow (and demand) pattern is, hence, also feasible.

Remark

It is important to note that there is no explicit path flow concept in the supply chain network model. However, given the above relationships and identifications in the link flow and demand patterns, we will be able to obtain path flows and an associated new interpretation of the supply chain network equilibrium conditions.

We now assign travel costs on the links of the network G_S as follows: with each link a_i we assign a travel cost c_{a_i} defined by

$$c_{a_i} \equiv \frac{\partial f_i}{\partial q_i}, \quad i = 1, \ldots, m, \tag{33}$$

with each link a_{ij} we assign a travel cost $c_{a_{ij}}$ defined by:

$$c_{a_{ij}} \equiv \frac{\partial c_{ij}}{\partial q_{ij}}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{34}$$

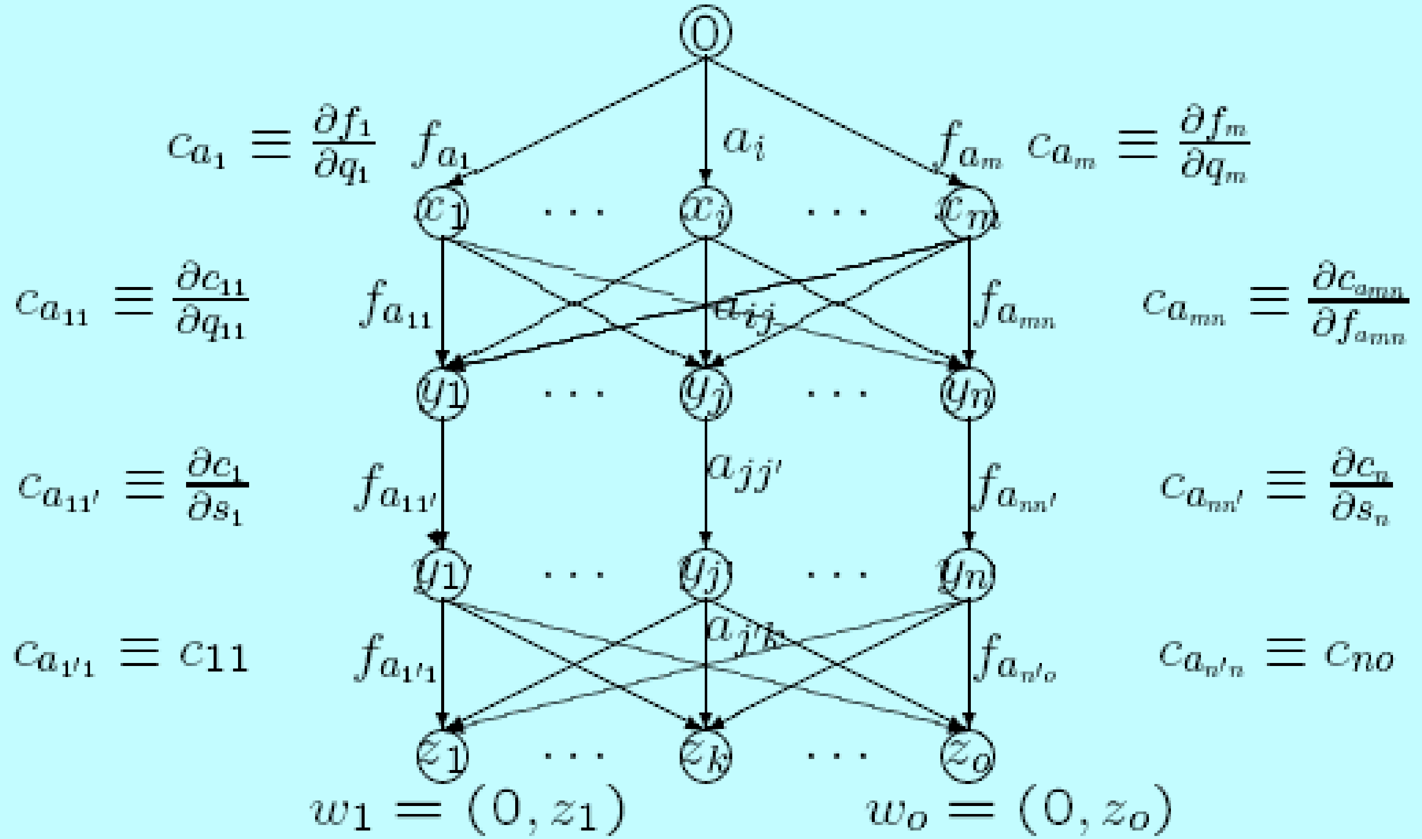
with each link jj' we assign a travel cost defined by

$$c_{a_{jj'}} \equiv \frac{\partial c_j}{\partial s_j}, \quad j = 1, \ldots, n; j' = 1, \ldots, n. \tag{35}$$

Finally, for each link $a_{j'k}$ we assign a travel cost defined by

$$c_{a_{j'k}} \equiv c_{jk}, \quad j' = 1, \ldots, n'; k = 1, \ldots, o. \tag{36}$$

The G_S Supernetwork Representation of Supply Chain Network Equilibrium with Link Flows and Travel Costs



Then a traveler traveling on path $p_{ijj'k}$, for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $j' = 1', \ldots, n'$; $k = 1, \ldots, o$, on network G_S in Figure 2 incurs a travel cost $C_{p_{ijj'k}}$ given by

$$C_{p_{ijj'k}} = c_{a_i} + c_{a_{ij}} + c_{a_{jj'}} + c_{a_{j'k}} = \frac{\partial f_i}{\partial q_i} + \frac{\partial c_{ij}}{\partial q_{ij}} + \frac{\partial c_j}{\partial s_j} + c_{jk}. \quad (37)$$

Also, we assign the travel demands associated with the O/D pairs as follows:

$$d_{w_k} \equiv d_k, \quad k = 1, \ldots, o \quad (38)$$

and the travel disutilities:

$$\lambda_{w_k} \equiv \rho_{3k}, \quad k = 1, \ldots, o. \quad (39)$$

Consequently, the equilibrium conditions (19) and (20) for the traffic network equilibrium model on the network G_S state that for every O/D pair w_k and every path connecting the O/D pair w_k:

$$C_{p_{ijj'k}} - \lambda^*_{w_k} = \frac{\partial f_i}{\partial q_i} + \frac{\partial c_{ij}}{\partial q_{ij}} + \frac{\partial c_j}{\partial q_j} + c_{jk} - \lambda^*_{w_k} \begin{cases} = 0, & \text{if} \quad x^*_{p_{ijj'k}} > 0 \\ \geq 0, & \text{if} \quad x^*_{p_{ijj'k}} = 0 \end{cases} \quad (40)$$

and

$$\sum_{p \in P_{w_k}} x^*_{p_{ijj'k}} \begin{cases} = d_{w_k}(\lambda^*), & \text{if} \quad \lambda^*_{w_k} > 0 \\ \geq d_{w_k}(\lambda^*), & \text{if} \quad \lambda^*_{w_k} = 0. \end{cases} \quad (41)$$

We now state the variational inequality formulation of the equilibrium conditions (40) and (41) in link form as in (22), which will make apparent the equivalence with variational inequalities (13a) and (13b) for the supply chain network equilibrium. According to Theorem 2, we have that a link flow, travel demand, and travel disutility pattern $(f^*, d^*, \lambda^*) \in \mathcal{K}^3$ is an equilibrium (according to (40) and (41)), if and only if it satisfies:

$$\sum_{i=1}^{m} c_{a_i}(f^{1*}) \times (f_{a_i} - f^*_{a_i}) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{a_{ij}}(f_{a_{ij}}) \times (f_{a_{ij}} - f^*_{a_{ij}})$$

$$+ \sum_{j=1}^{n} c_{a_{jj'}}(f^{3*}) \times (f_{a_{jj'}} - f^*_{a_{jj'}}) + \sum_{j'=1}^{n'}\sum_{k=1}^{n} c_{a_{j'k}}(f^{4*}) \times (f_{a_{j'k}} - f^*_{a_{j'k}})$$

$$- \sum_{k=1}^{o} \lambda^*_{w_k} \times (d_{w_k} - d^*_{w_k}) + \sum_{k=1}^{o} \left[d^*_{w_k} - d_{w_k}(\lambda^*)\right] \times \left[\lambda_{w_k} - \lambda^*_{w_k}\right] \geq 0,$$

$$\forall (f, d, \lambda) \in \mathcal{K}^3, \qquad (42)$$

which, through expressions (28) − (32), (33) − (36), and (38) − (39) yields:

$$\sum_{i=1}^{m}\left[\frac{\partial f_i(q^*)}{\partial q_i}\right]\times[q_i-q_i^*]+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}\right]\times\left[q_{ij}-q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\left[\frac{\partial c_j(s^*)}{\partial s_j}\right]\times\left[s_j-s_j^*\right]+\sum_{j=1}^{n}\sum_{k=1}^{o}c_{jk}(Q^{2*})]\times\left[q_{jk}-q_{jk}^*\right]$$

$$-\sum_{k=1}^{o}\rho_{3k}^*\times[d_k-d_k^*]+\sum_{k=1}^{o}[d_k^*-d_k(\rho_3^*)]\times[\rho_{3k}-\rho_{3k}^*]\geq 0,$$

$$\forall(q,Q^1,s,Q^2,d,\rho_3)\in\mathcal{K}^2. \tag{43}$$

But variational inequality (43) is precisely variational inequality (13b) governing the supply chain network equilibrium.

Hence, we have the following result:

Theorem 3

A solution $(q^*, Q^{1^*}, s^*, Q^{2^*}, d^*, \rho_3^*) \in \mathcal{K}^2$ of the variational inequality (13b) governing a supply chain network equilibrium coincides with the (via (28) – (32) and (38) – (39)) feasible link flow, travel demand, and travel disutility pattern for the supernetwork G_s constructed above and satisfies variational inequality (22). Hence, it is a traffic network equilibrium according to Theorem 2.

The equilibrium conditions (40) and (41) provide us with an entirely new interpretation of the supply chain network equilibrium conditions according to Definition 1.

Indeed, **(40) coincides with the well-known Wardrop (1952) conditions associated with traffic network equilibria and user-optimization** (see also Dafermos and Sparrow (1969)). Hence, we now have an *entirely new interpretation* of supply chain network equilibrium which states that all utilized paths in a supply chain supernetwork associated with a demand market will have equal and minimal costs.

This further suggests a type of *efficiency principle* regarding supply chain designs. Moreover, the flow on a path of the supernetwork representation of the supply chain network corresponds to the flow of product along a path which consists of such links as production links; transportation/transaction links between manufacturers and retailers; handling links associated with the retailers; and, finally, transportation/transaction links between retailers and the consumers at the demand markets.

Using an analogous supernetwork construction to the one detailed here we can now construct traffic network equilibrium representations (and formulations) of additional supply chain network problems as cited in the Introduction of this paper.

We note that Zhang, Dong, and Nagurney (2003) presented a supply chain network equilibrium model which recognized the importance of Wardrop's principles from traffic networks but considered a chain formulation, rather than a path formulation. Moreover, they reported no numerical results.

The Algorithm and Numerical Examples

We first recall an algorithm, the Euler method, which was proposed by Nagurney and Zhang (1996) for the solution of variational inequality problem (21) in path flows (or, equivalently, variational inequality (22) in link flows).

In particular, due to the simplicity of the feasible set (cf. (21)), which is simply the nonnegative orthant, the Euler method (see also Dupuis and Nagurney (1993)) here takes the form: at iteration τ compute the path flows $p \in P$ according to:

$$x_p^{\tau+1} = \max\{0, x_p^\tau + \alpha_\tau(\lambda_w^\tau - C_p(x^\tau))\}, \qquad (44)$$

and the travel disutilities for all O/D pairs $w \in W$ according to:

$$\lambda_w^{\tau+1} = \max\{0, \lambda_w^\tau + \alpha_\tau(d_w(\lambda^\tau) - \sum_{p \in P_w} x_p^\tau)\}, \qquad (45)$$

where $\{\alpha_\tau\}$ is a sequence of positive real numbers that satisfies: $\lim_{\tau \to \infty} \alpha_\tau = 0$ and $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$. Such a sequence is required for convergence (cf. Nagurney and Zhang (1996)).

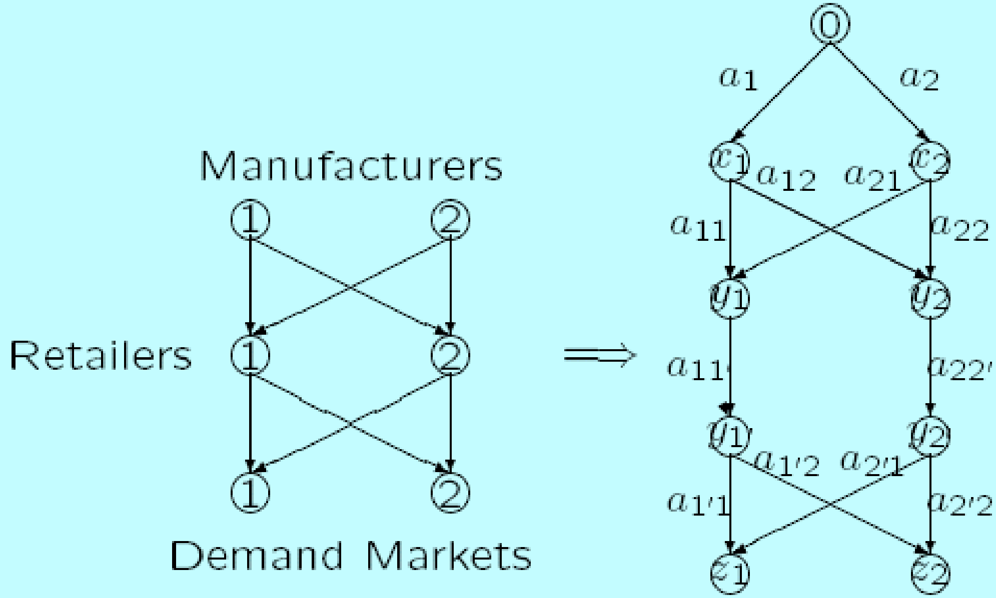
We now apply the Euler method described above to compute the equilibrium path, link flow, and travel disutility pattern in three numerical examples that were solved via the modified projection method (cf. Korpelevich (1977)) in Nagurney, Dong, and Zhang (2002).

The Euler method was implemented in FORTRAN and the computer system used was a Sun system located at the University of Massachusetts at Amherst.

The convergence criterion utilized was that the absolute value of the path flows and travel disutilities between two successive iterations differed by no more than 10^{-4}. The sequence $\{\alpha_\tau\}$ in the Euler method was set to: $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$.

Example 1

The first numerical example, depicted in Figure 3, consisted of two manufacturers, two retailers, and two demand markets. In Figure 3, we also provide the supernetwork representation and identify its nodes and links.



Supply Chain Network and Corresponding Supernetwork for Examples 1 and 2

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1 q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1 q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}) = .5q_{11}^2 + 3.5q_{11}, \; c_{12}(q_{12}) = .5q_{12}^2 + 3.5q_{12},$$

$$c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21}, \; c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2} q_{i2})^2.$$

The demand functions at the demand markets were:

$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000$, $d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000$,

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5,$$

$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5.$$

We utilized the supernetwork representation of this example depicted in Figure 3 with the links enumerated as in Figure 3 in order to solve the problem via the Euler method. There are 9 nodes and 12 links in the supernetwork in Figure 3.

We defined O/D pair $w_1 = (0, z_1)$ and O/D pair $w_2 = (0, z_2)$ and associated the demand price functions with the travel disutility functions as in (39) and the user link travel cost functions as given in (33)–(36) (analogous constructions were done for the subsequent examples).

There were four paths in P_{w_1} denoted by: p_1, p_2, p_3, and p_4, respectively, and also four paths in P_{w_2} denoted by: p_5, p_6, p_7, and p_8, respectively, and comprised of the links as follows:

for O/D pair w_1:

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}),$$

$$p_3 = (a_2, a_{21}, a_{11'}, a_{1'1}), \quad p_4 = (a_2, a_{22}, a_{22'}, a_{2'1}),$$

and for O/D pair w_2:

$$p_5 = (a_1, a_{11}, a_{11'}, a_{1'2}), \quad p_6 = (a_1, a_{12}, a_{22'}, a_{2'2}),$$

$$p_3 = (a_2, a_{21}, a_{11'}, a_{1'2}), \quad p_8 = (a_2, a_{22}, a_{22'}, a_{2'2}).$$

The Euler method converged in 194 iterations and yielded the following equilibrium pattern:

$$x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = x^*_{p_4} = 8.304,$$

$$x^*_{p_5} = x^*_{p_6} = x^*_{p_7} = x^*_{p_8} = 8.304,$$

and with the equilibrium travel disutilities given by:

$$\lambda^*_{w_1} = \lambda^*_{w_2} = 276.224.$$

The corresponding equilibrium link flows were:

$$f^*_{a_1} = f^*_{a_2} = 33.216, \quad f^*_{a_{11}} = f^*_{a_{12}} = f^*_{a_{21}} = f^*_{a_{22}} = 16.608,$$

$$f^*_{a_{11'}} = f^*_{a_{22'}} = 33.216, \quad f^*_{a_{1'1}} = f^*_{a_{1'2}} = f^*_{a_{2'1}} = f^*_{a_{2'2}} = 16.608.$$

We now provide the translations of the above equilibrium flows into the supply chain product shipment and price notation using (28)-(32).

The product shipments between the two manufacturers and the two retailers were:

$$Q^{1^*} = q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 16.608,$$

the product shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{2^*} = q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 16.608,$$

and the demand prices at the demand markets were:

$$\rho^*_{31} = \rho^*_{32} = 276.224.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

Moreover, these values are precisely the same values obtained for the equilibrium product shipments and demand market prices for Example 1 via the modified projection method in Nagurney, Dong, and Zhang (2002).

Example 2

We then modified Example 1 as follows: The production cost function for manufacturer 1 was now given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 12q_1,$$

whereas the transaction costs for manufacturer 1 were now given by:

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}.$$

The remainder of the data was as in Example 1. Hence, both the production costs and the transaction costs increased for manufacturer 1.

We next (as in the case of Example 1) constructed the supernetwork reformulation and applied the Euler method. Hence, since the number of manufacturers, retailers, and demand markets did not change the supernetwork representation for this example is as depicted in Figure 3.

The Euler method converged in 197 iterations and yielded the following equilibrium path flow and travel disutility pattern:

$$x^*_{p_1} = x^*_{p_2} = 7.234, \quad x^*_{p_3} = x^*_{p_4} = 8.635,$$

$$x^*_{p_5} = x^*_{p_6} = 7.274, \quad x^*_{p_7} = x^*_{p_8} = 8.595,$$

$$\lambda^*_{w_1} = \lambda^*_{w_2} = 276.397.$$

The equilibrium link flows were:

$$f^*_{a_1} = 29.014, \quad f^*_{a_2} = 34.460, \quad f^*_{a_{11}} = 14.507 = f^*_{a_{12}},$$

$$f^*_{a_{21}} = f^*_{a_{22}} = 17.230,$$

$$f^*_{a_{11'}} = f^*_{a_{22'}} = 31.737,$$

$$f^*_{a_{1'1}} = f^*_{a_{1'2}} = f^*_{a_{2'1}} = f^*_{a_{2'2}} = 15.870.$$

For easy reference, and comparison with the results for this example, but solved via the modified projection method in Nagurney, Dong, and Zhang (2002), provide the translations of the above results into the supply chain notation for equilibrium product shipments and demand market prices.

The product shipments between the two manufacturers and the two retailers were now:

$$Q^{1^*} = q^*_{11} = q^*_{12} = 14.507, \quad q^*_{21} = q^*_{22} = 17.230,$$

the product shipments (consumption amounts) between the two retailers and the two demand markets were now:

$$Q^{2^*} = q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 15.870,$$

and the demand prices at the demand markets were:

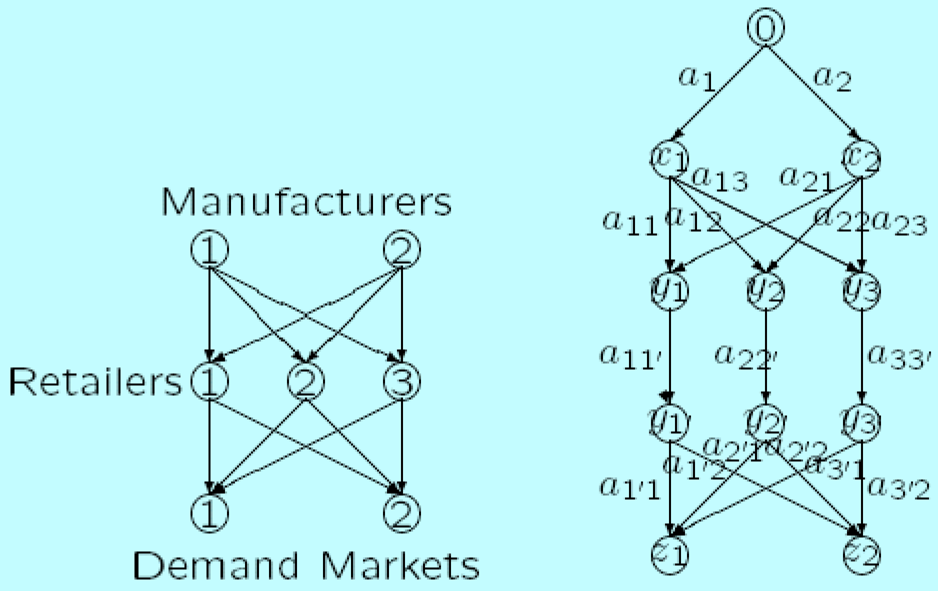
$$\rho^*_{31} = \rho^*_{32} = 276.646.$$

Hence, manufacturer 1 now produced less than it did in Example 1, whereas manufacturer 2 increased its production output. The demand price at the demand markets increased, with a decrease in the incurred demand.

The equilibrium product shipments and demand market prices computed via the Euler method were precisely equal to the correpsonding values obtained via the modified projection method for Example 2 by Nagurney, Dong, and Zhang (2002).

Example 3

The third supply chain network problem consisted of two manufacturers, three retailers, and two demand markets, as depicted in Figure 4. Its supernetwork representation is also given in Figure 4.



Supply Chain Network and Corresponding Supernetwork for Example 3

The data were constructed from Example 2, but we added data for the manufacturers' transaction costs associated with the third retailer; handling cost data for the third retailer, as well as the transaction cost data between the new retailer and the demand markets. Hence, the complete data for this example were given by:

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 12q_2.$$

The transaction cost functions faced by the two manufacturers and associated with transacting with the three retailers were given by:

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \; c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12},$$

$$c_{13}(q_{13}) = .5q_{13}^2 + 5q_{13},$$

$$c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21}, \; c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22},$$

$$c_{23}(q_{23}) = .5q_{23}^2 + 5q_{23}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2} q_{i2})^2,$$

$$c_3(Q^1) = .5(\sum_{i=1}^{2} q_{i3})^2.$$

The demand functions at the demand markets, again, were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000,$$

$$d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5,$$

$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5,$$

$$c_{31}(Q^2) = q_{31} + 5, \quad c_{32}(Q^2) = q_{32} + 5.$$

Note that in the supernetwork representation of this supply chain network problem, given in Figure 4, there are 11 nodes in the supernetwork and 17 links. There are two O/D pairs given by $w_1 = (0, z_1)$ and $w_2 = (1, z_2)$ with the nodes z_1 and z_2 corresponding to the bottom tiered nodes in the supernetwork in Figure 4.

There are now six paths connecting each O/D pair and given as follows:

for O/D pair w_1:

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}),$$

$$p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}),$$

$$p_4 = (a_2, a_{21}, a_{11'}, a_{1'1}), \quad p_5 = (a_2, a_{22}, a_{22'}, a_{2'1}),$$

$$p_6 = (a_2, a_{23}, a_{33'}, a_{3'1}),$$

and for O/D pair w_2:

$$p_7 = (a_1, a_{11}, a_{11'}, a_{1'2}), \quad p_8 = (a_1, a_{12}, a_{22'}, a_{2'2}),$$

$$p_9 = (a_1, a_{13}, a_{33'}, a_{3'2}),$$

$$p_{10} = (a_2, a_{21}, a_{11'}, a_{1'2}), \quad p_{11} = (a_2, a_{22}, a_{22'}, a_{2'2}),$$

$$p_{12} = (a_2, a_{23}, a_{33'}, a_{3'2}).$$

The Euler method converged in 331 iterations and yielded the following equilibrium path flow and travel disutility pattern:

$$x^*_{p_1} = x^*_{p_2} = 4.922, \quad x^*_{p_3} = 7.832, \quad x^*_{p_4} = x^*_{p_5} = 6.448,$$

$$x^*_{p_6} = 4.396,$$

$$x^*_{p_7} = x^*_{p_8} = 4.937, \quad x^*_{p_9} = 7.813, \quad x^*_{p_{10}} = x^*_{p_{11}} = 6.433,$$

$$x^*_{p_{12}} = 4.415,$$

and

$$\lambda^*_{w_1} = \lambda^*_{w_2} = 275.723.$$

The corresponding equilibrium link flows were:

$$f^*_{a_1} = 35.364 \quad f^*_{a_2} = 34.573, \quad f^*_{a_{11}} = f^*_{a_{12}} = 9.860,$$

$$f^*_{a_{13}} = 15.645, \quad f^*_{a_{21}} = f^*_{a_{22}} = 12.881,$$

$$f^*_{a_{23}} = 8.811, \quad f^*_{a_{11'}} = f^*_{a_{22'}} = 22.741, \quad f^*_{a_{33'}} = 24.456,$$

$$f^*_{a_{1'1}} = f^*_{a_{1'2}} = f^*_{a_{2'1}} = f^*_{a_{2'2}} = 11.370,$$

$$f^*_{a_{3'1}} = f^*_{a_{3'2}} = 12.228.$$

The above results translate into the following equilibrium product shipments and demand market prices on the supply chain network: The product shipments between the two manufacturers and the three retailers were:

$$Q^{1^*} = q_{11}^* = q_{12}^* = 9.860, \quad q_{13}^* = 15.645,$$

$$q_{21}^* = q_{22}^* = 12.881, \quad q_{23}^* = 8.811,$$

the product shipments between the three retailers and the two demand markets were:

$$Q^{2^*} = q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 11.370, \quad q_{31}^* = q_{32}^* = 12.228,$$

and the demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 275.723.$$

Note that the demand prices at the demand markets were now lower than in Example 2, since there is now an additional retailer and, hence, increased competition. The incurred demand also increased at both demand markets, as did the production outputs of both manufacturers. Since the retailers now handled a greater volume of product shipments, the prices charged for the product at the retailers, nevertheless, increased due to increased handling cost. The above computed values are very close to the analogous ones obtained for Example 3 in Nagurney, Dong, and Zhang (2002).

Summary, Conclusions, and Suggestions for Future Research

This paper demonstrated that a supply chain network equilibrium model from the literature could be reformulated as a transportation network equilibrium model with elastic demands in the case of known demand functions.

This identification was made through a supernetwork construction of the former to reveal the transportation network configuration for the supply chain and by showing the variational inequality equivalences of the respective governing equilibrium conditions.

In addition, a new interpretation of the supply chain network equilibrium conditions was provided which coincided with the well-known user-optimizing conditions in transportation network equilibrium modeling and analysis.

This connection allows us to transfer the plethora of algorithmic tools developed for transportation networks to the formulation, analysis, and solution of supply chain networks. Moreover, it allows us to transform a spectrum of supply chain network equilibrium models reported in the literature to their corresponding transportation network equilibrium counterparts.

In order to demonstrate the practical usefulness of the theoretical results in this paper, we also applied an algorithm proposed for the solution of transportation network equilibrium problems with elastic demands to compute the equilibrium product shipments and demand market prices for several supply chain network examples taken from the literature, using the supernetwork transformation.

Possible Future Research may also Include:

• the computation of large-scale supply chain network equilibrium problems with different cost and demand functional forms, with and without the supernetwork equivalence, in order to determine the "most" efficient computational algorithms for such problems;

• additional supply chain modeling efforts that may include the incorporation of raw material suppliers, distinct production processes, etc., that we expect will be made more transparent using a supernetwork equivalence, as well as

• the construction of system-optimization analogues for supply chain networks (analogous to those for transportation networks) and the identification of the costs / benefits of competition versus cooperation.

This paper is in press in *Transportation Research E* (2005).

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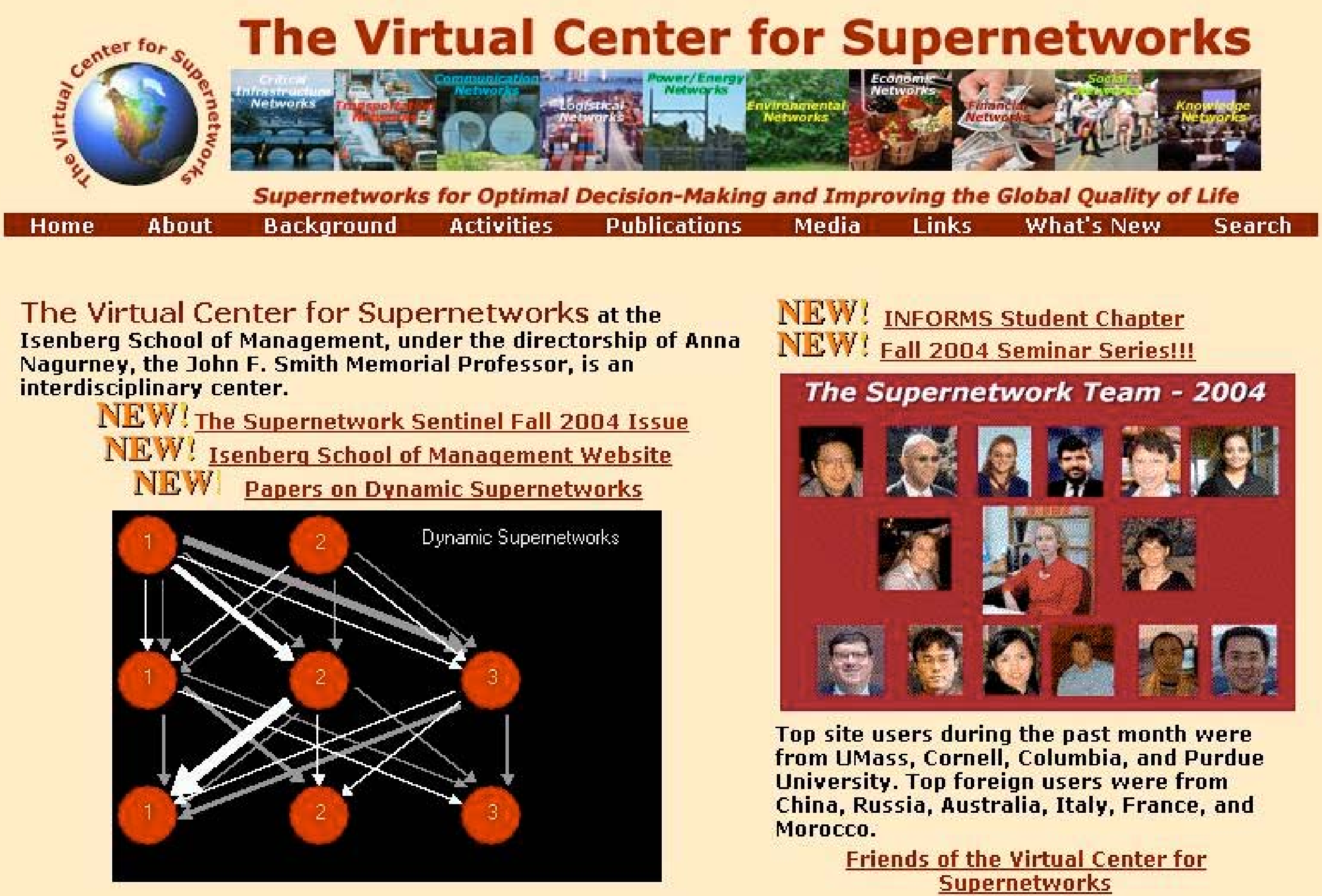

Thank you!

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Virtual Center for Supernetworks