A Unified Network Performance Measure With Importance Identification and the Ranking of Network Components

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Outline

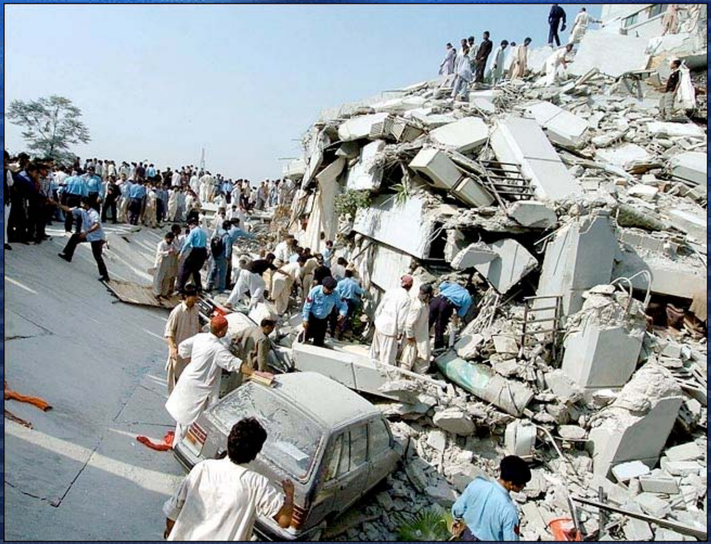
- Motivation
- Literature
- Network Equilibrium Model
- Network Efficiency Measure & Network Component Importance
- Numerical Examples
- Conclusion

Network Vulnerability

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.

- For example:
  - Hurricane Katrina, August 23, 2005
  - The biggest blackout in North America, August 14, 2003
  - 9/11 Terrorist Attacks, September 11, 2001

Earthquake Damage
prcs.org.pk



Tsunami
letthesunshinein.wordpress.com

Storm Damage
www.srh.noaa.gov

Infrastructure Collapse
www.10-7.com

An Urgent Need for a Network Efficiency/Performance Measure

In order to be able to assess the performance/efficiency of a network, it is imperative that appropriate measures be devised.

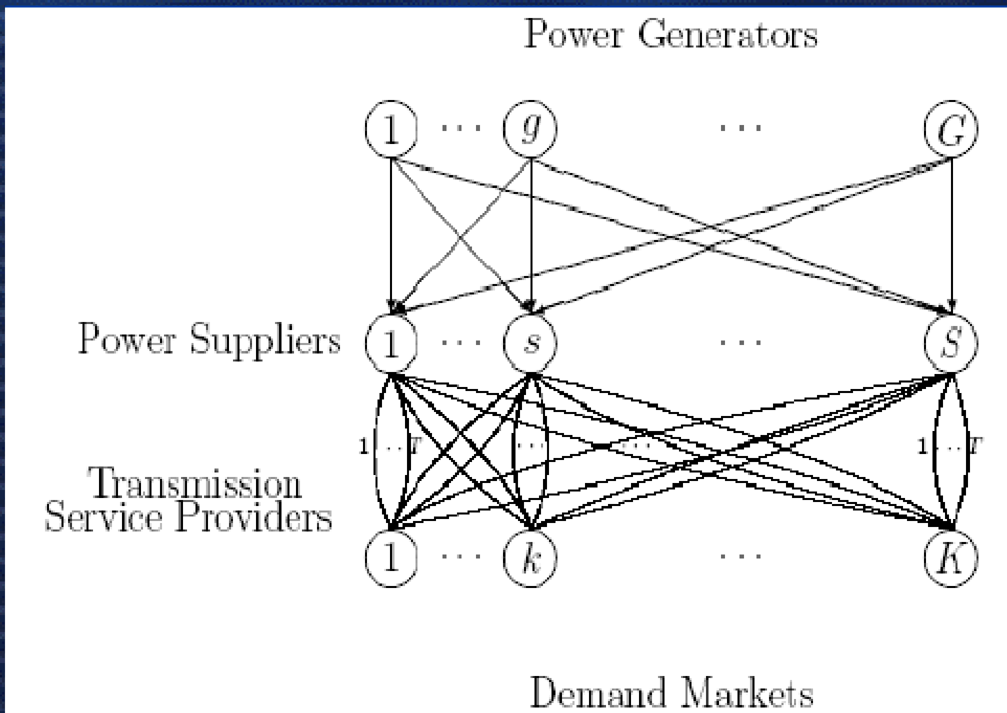
Appropriate network measures can assist in the identification of the importance of network components, that is, nodes and links, and the associated rankings. Such rankings can be very helpful in the case of the determination of network vulnerabilities as well as when to reinforce/enhance security.

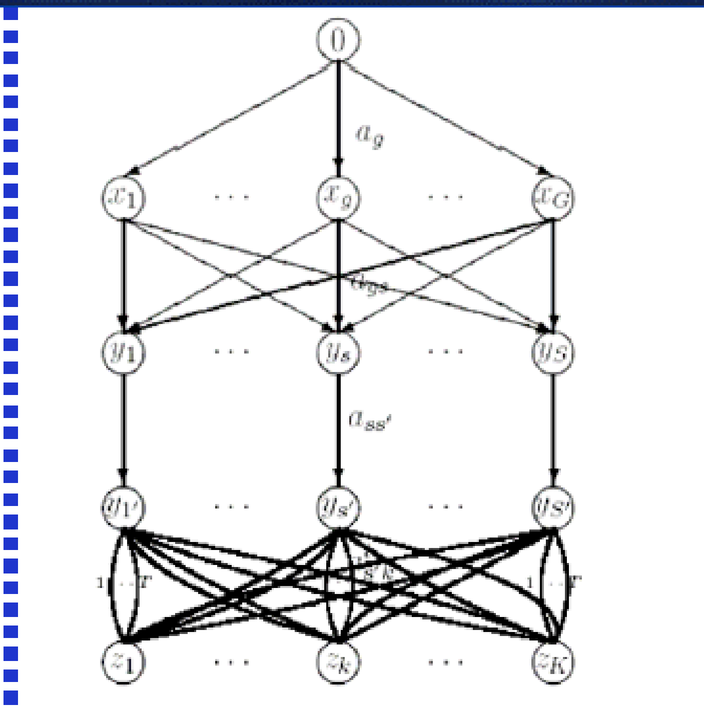
Transportation Network Equilibrium Paradigm

It has been recently shown that, as hypothesized over 50 years ago by Beckmann, McGuire, and Winsten (1956), that electric power generation and distribution networks can be reformulated and solved as transportation networks, Wu, Nagurney, Liu, and Stranlund, *Transportation Research D* (2006), Nagurney et al., *Transportation Research D*, in press.

It has been demonstrated that financial networks with intermediation can be reformulated and solved as transportation network problems; Liu and Nagurney, *Computational Management Science*, in press.

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks



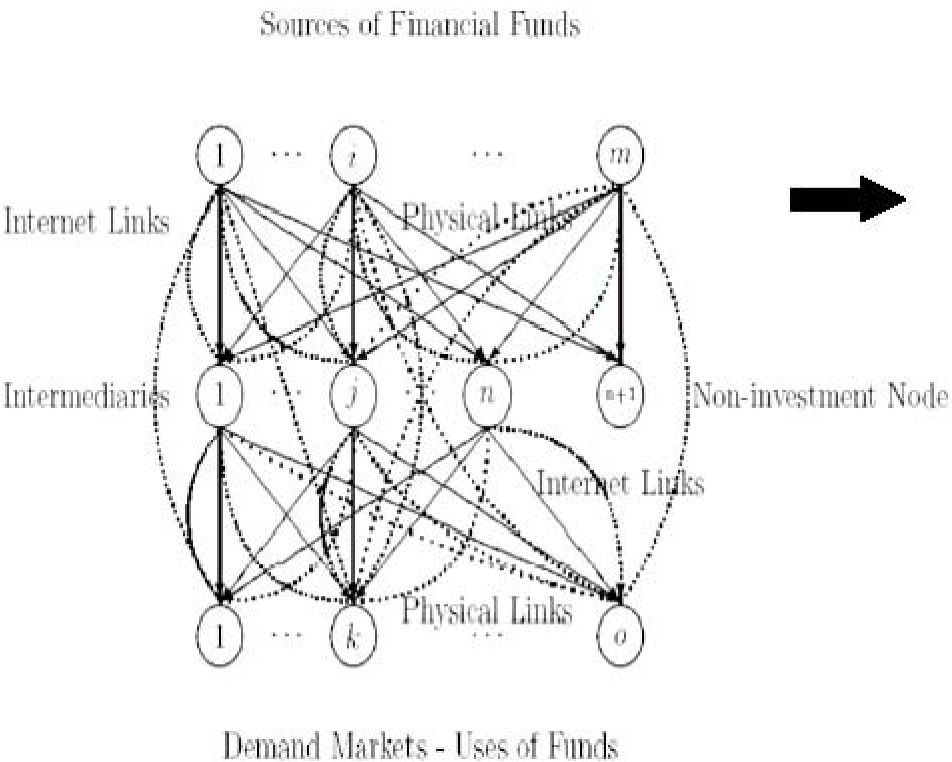


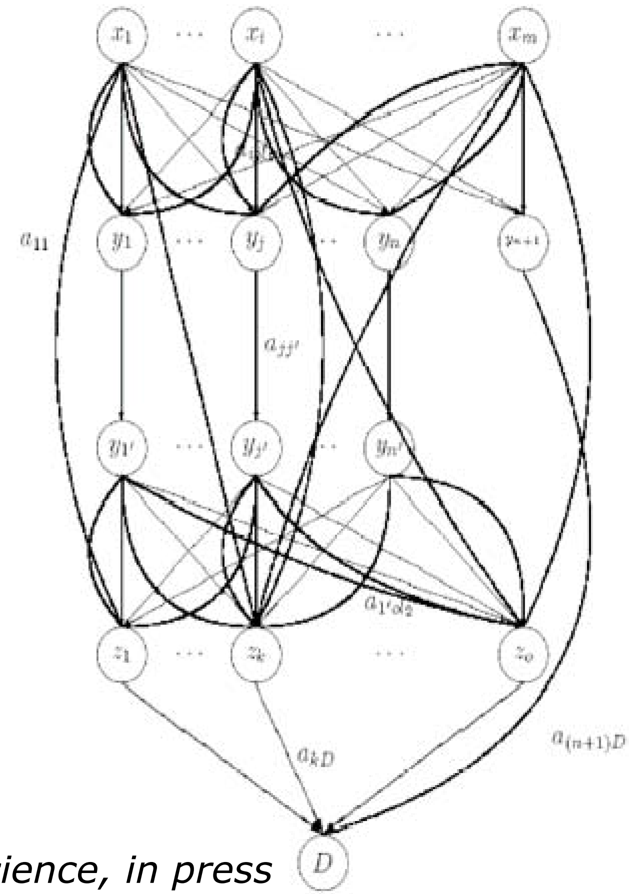
Electric Power Supply Chain Network

Transportation Network

Nagurney et al, to appear in *Transportation Research E*

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation





Liu and Nagurney, *Computational Management Science, in press*

Transportation Network Equilibrium Problem

We consider a network $\mathcal{G}$ with the set of directed links L with K elements, the set of origin/destination (O/D) pairs W with n_W elements, and the set of acyclic paths joining the O/D pairs by P with n_P elements.

We denote the set of paths joining O/D pair w by P_w. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We denote the nonnegative flow on path p by x_p and the flow on link a by f_a and we group the path flows into the vector $x \in R^{n_P}_+$ and the link flows into the vector $f \in R^K_+$. The link flows are related to the path flows through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L$$

where $\delta_{ap} = 1$ if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user cost on a path p is denoted by C_p and the user cost on a link a by c_a. We denote the demand associated with using O/D pair w by d_w and the disutility by λ_w.

The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path.

The following conservation of flow equations must also hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair.

Also, we assume, as given, the disutility (that is, the inverse demand) functions for the O/D pairs, which are assumed to be continuous, such that

$$\lambda_w = \lambda_w(d), \quad \forall w \in W$$

where d is the vector of demands.

Definition: Network Equilibrium – Elastic Demands

A path flow and demand pattern $(x^*, d^*) \in \mathcal{K}^1$*, where* $\mathcal{K}^1 \equiv \{(x,d)|(x,d) \in R_+^{n_P+n_W}$ *and* $\sum_{p \in P_w} x_p = d_w, \forall w \in W \}$*, is said to be a network equilibrium, in the case of elastic demands, if, once established, no user has any incentive to alter his "travel" decisions. The state is expressed by the following condition which must hold for each O/D pair* $w \in W$ *and every path* $p \in P_w$*:*

$$C_p(x^*) \begin{cases} = \lambda_w(d^*), & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w(d^*), & \text{if} \quad x_p^* = 0. \end{cases}$$

The above condition states that all utilized paths connecting an O/D pair have equal and minimal user costs and these costs are equal to the disutility associated with using that O/D pair. As established in Dafermos (1980), the above network equilibrium condition is equivalent to the following variational inequality problem.

Theorem

A path flow and demand pattern $(x^*, d^*) \in \mathcal{K}^1$ *is an equilibrium according to the above definition if and only if it satisfies the variational inequality: determine* $(x^*, d^*) \in \mathcal{K}^1$ *such that*

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] - \sum_{w \in W} \lambda_w(d^*) \times [d_w - d_w^*] \geq 0, \quad \forall (x,d) \in \mathcal{K}^1.$$

Definition: Network Equilibrium – Fixed Demands

A path flow pattern $x^* \in \mathcal{K}^2$, *where* $\mathcal{K}^2 \equiv \{x | x \in R_+^{n_P}$ *and* $\sum_{p \in P_w} x_p = d_w, \forall w \in W,$ *holds with* d_w *known and fixed for each* $w \in W\}$, *is said to be a network equilibrium, in the case of fixed demands, if the following condition holds for each O/D pair* $w \in W$ *and each path* $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0. \end{cases}$$

The interpretation of the above condition is that all used paths connecting an O/D pair have equal and minimal costs (see also Wardrop (1952) and Beckmann, McGuire, and Winsten (1956)). As proved in Smith (1979) and Dafermos (1980), the fixed demand network equilibrium condition is equivalent to the following variational inequality problem.

Theorem

A path flow pattern $x^* \in \mathcal{K}^2$ *is a network equilibrium according to the above definition if and only if it satisfies the variational inequality problem: determine* $x^* \in \mathcal{K}^2$ *such that*

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] \geq 0, \quad \forall x \in \mathcal{K}^2.$$

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Barrat, Barthélemy and Vespignani (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Chassin and Posse (2005)
- Holme, Kim, Yoon and Han (2002)
- Sheffi (2005)
- Taylor and D'este (2004)
- Jenelius, Petersen and Mattson (2006)
- Murray-Tuite and Mahmassani (2004)

Our Research on Network Efficiency and Network Vulnerability

- A Network Efficiency Measure with Application to Critical Infrastructure Networks, Nagurney and Qiang (2007a), to appear in *Journal of Global Optimization.*
- A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability, Nagurney and Qiang (2007b), *Proceedings of the POMS 18th Annual Conference*, May 4 to May 7, 2007.
- A Unified Network Performance Measure with Importance Identification and the Ranking of Network Components (2007), Qiang and Nagurney, *Optimization Letters,* in press.
- A Network Efficiency Measure for Congested Networks (2007), Nagurney and Qiang, *Europhysics Letters,* Accepted.

The Network Efficiency Measure of Latora and Marchiori (2001)

Latora and Marchiori (2001) proposed a network efficiency measure (the L-M measure) as follows:

Definition : The L-M Measure

The network performance/efficiency measure, $E(G)$, according to Latora and Marchiori (2001) for a given network topology G, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007a) (the N-Q Measure) proposed a network efficiency measure for networks with fixed demand, which captures demand and flow information under the network equilibrium.

Definition: The N-Q Measure

The N-Q network performance/efficiency measure, $\mathcal{E}(G,d)$, for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W}$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

The L-M Measure vs. the N-Q Measure

Theorem :

If positive demands exist for all pairs of nodes in the network G*, and each of these demands is equal to* 1 *and if* d_{ij} *is set equal to* λ_w*, where* $w = (i, j)$*, for all* $w \in W$ *then the proposed network efficiency measure and the L-M measure are one and the same.*

The Property of a Reasonable Network Efficiency Measure

Network Performance Property:

The performance/efficiency measure for a given network should be nonincreasing with respect to the equilibrium disutility for each O/D pair, holding the equilibrium disutilities for the other O/D pairs constant.

A Network Efficiency Measure by Zhu et al. (2006)

$$\hat{E}(G) = \frac{\sum_{w \in W} \lambda_w d_w}{\sum_{w \in W} d_w}.$$

- It is the average disutility of a network
- Undefined if the any O/D pair becomes disconnected
- Does not have the reasonable network efficiency property

Examining the Measure by Zhu et al. with the Property of Network Performance Measure

Let's take the partial derivative of the network efficiency measure by Zhu et al. (2006) with respect to λ_w for a network with elastic demands with the equilibrium disutilities for all the other O/D pairs being held constant, which yields the following:

$$\frac{\partial \hat{E}(G)}{\partial \lambda_w} = \frac{d_w \cdot (\sum_{w \in W} d_w) - (\sum_{w \in W} \lambda_w(d_w) d_w) \cdot (\lambda_w'(d_w))^{-1}}{(\sum_{w \in W} d_w)^2} + \frac{\lambda_w(d_w) \cdot (\lambda_w'(d_w))^{-1}}{\sum_{w \in W} d_w}.$$

It is reasonable to assume that $\lambda_w(d_w) \geq 0$, $d_w \geq 0$, and $\lambda_w'(d_w) < 0$, $\forall w \in W$. Obviously, the first term above is nonnegative and the second term is nonpositive. Therefore, the sign of $\frac{\partial \hat{E}(G)}{\partial \lambda_w}$ depends on the equilibrium demand and the disutility function for each w, which leads to the conclusion that the measure by Zhu et al. (2006) is not appropriate for elastic demand networks.

Examining the N-Q Measure with the Property of Network Performance Measure

Now let's check if the N-Q measure has the desired network performance property specified earlier. Let's assume that the disutility function for each $w \in W$ is assumed to depend, for the sake of generality, on the entire demand vector. With the assumption of the equilibrium disutilities for all the other O/D pairs being held constant, the partial derivative of the N-Q measure with regard to λ_w for the network with elastic demands is then given as follows:

$$\frac{\partial \mathcal{E}(G,d)}{\partial \lambda_w} = \frac{\frac{-d_w}{(\lambda_w(d))^2} + \sum_{v \in W} \frac{(\frac{\partial \lambda_w(d)}{\partial d_v})^{-1}}{\lambda_v(d)}}{n_W}.$$

Given the assumption that $d_w \geq 0$, $\lambda_w \geq 0$, and $\frac{\partial \lambda_w}{\partial d_v} < 0$, $\forall v \in W$, it is obvious that $\mathcal{E}(G,d)$ is a nonincreasing function of λ_w, $\forall w \in W$.

Importance of a Network Component

Definition : Importance of a Network Component According to the L-M Measure

The importance of a network component $g \in G$, $\bar{I}(g)$, is measured by the network efficiency drop, determined by the L-M measure, after g is removed from the network:

$$\bar{I}(g) = \frac{\triangle E}{E(G)} = \frac{E(G) - E(G-g)}{E(G)}.$$

where $G - g$ is the resulting network after component g is removed from network G.

Definition : Importance of a Network Component According to the N-Q Measure

The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop, determined by the N-Q measure, after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where $G - g$ is the resulting network after component g is removed from network G.

The Approach to Study the Importance of Network Components

The elimination of a link is represented in the N-Q measure by the removal of that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Hence, our measure is well-defined even in the case of disconnected networks.

Major Advantages of the N-Q Measure over the L-M Measure

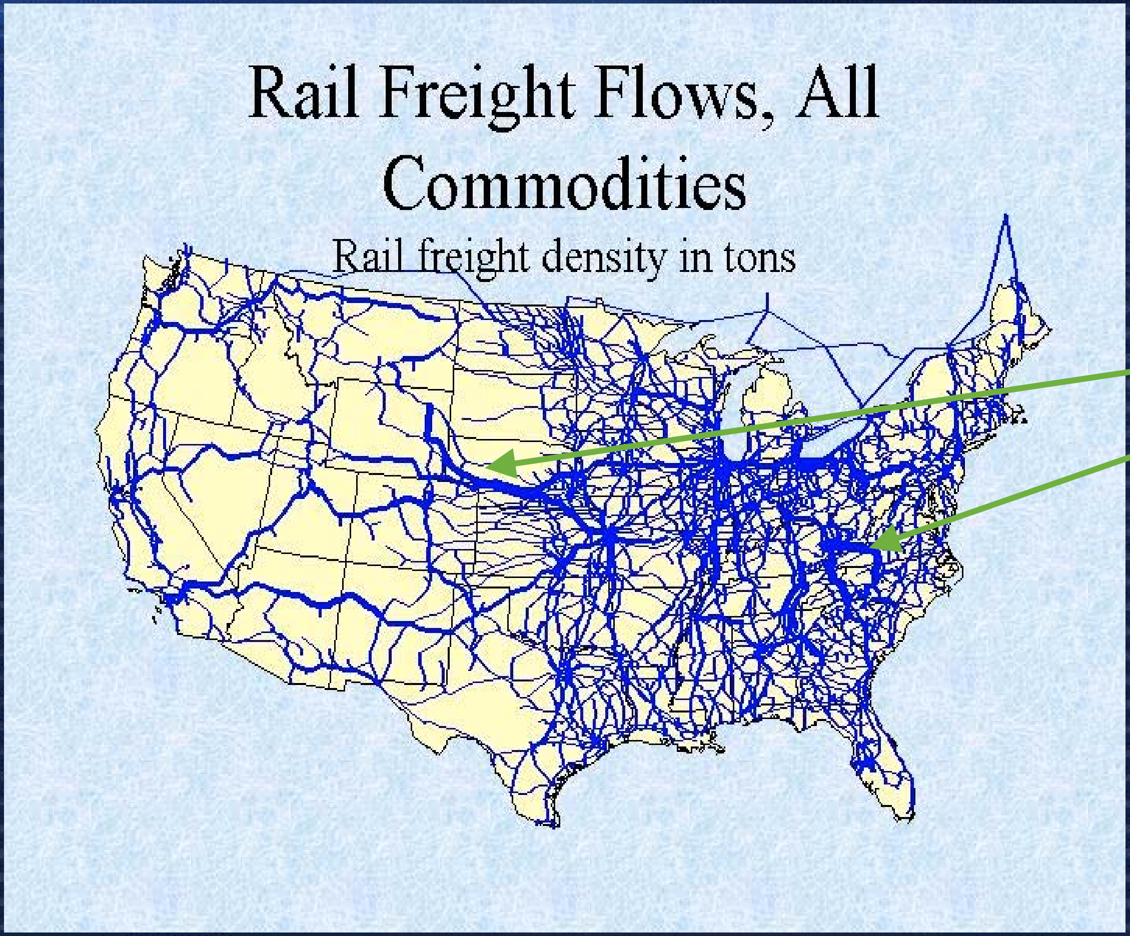
- The N-Q measure generalizes the L-M measure by capturing the flows, demand and user behavior information of the network besides the network topology structure.

- It has been shown that real-life networks displayed distinct disparities between topological properties and the flow patterns.

Major Advantages of the N-Q Measure over the Measure by Zhu et al.

- The N-Q measure is well defined even in the case where the network is disconnected.
- The N-Q measure is unified in the sense that it can gauge the network with either fixed or elastic demands.

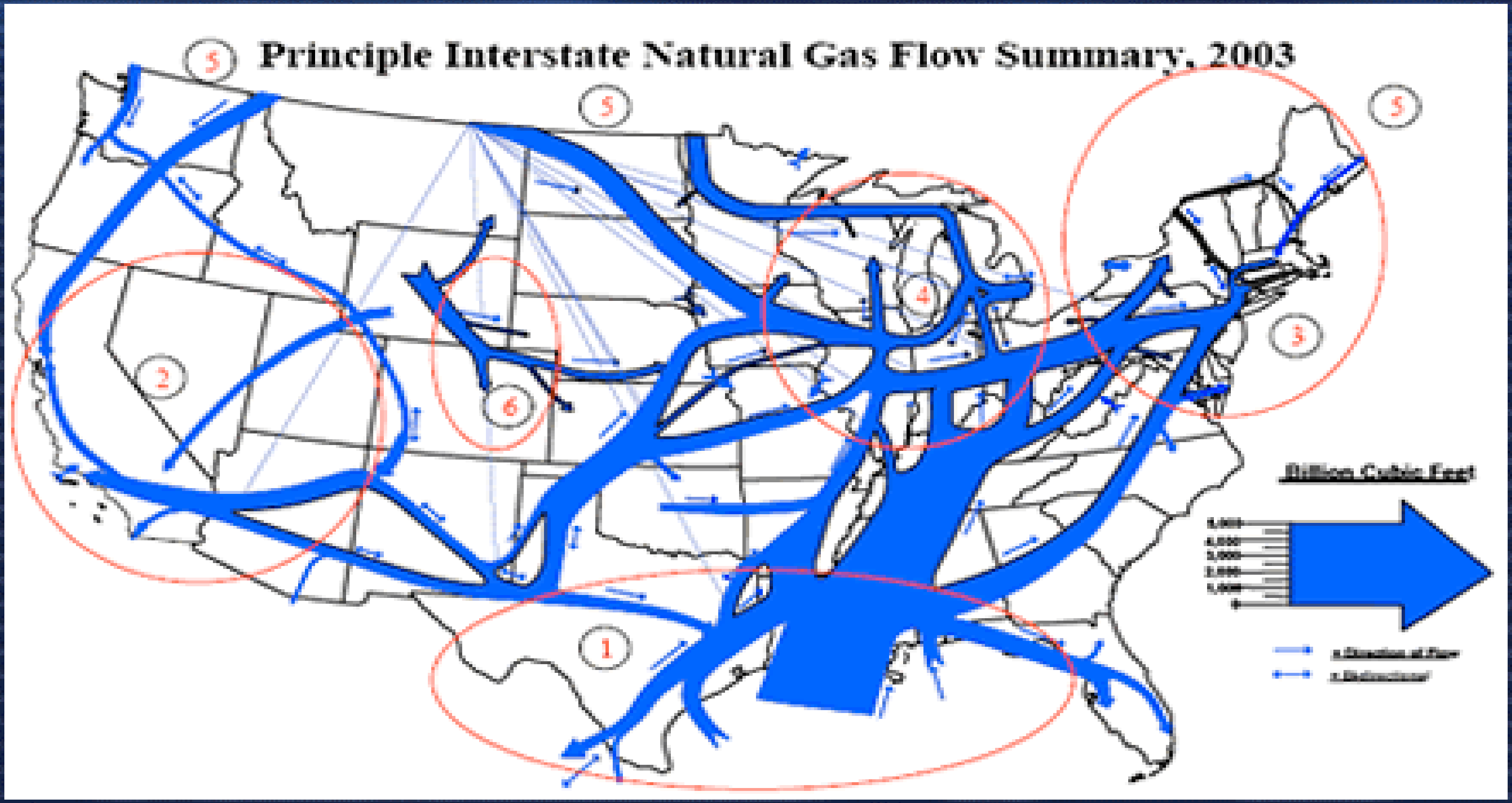
Railway Network in U.S.A



Some links are heavily used.

U.S. Department of Transportation
ops.fhwa.dot.gov

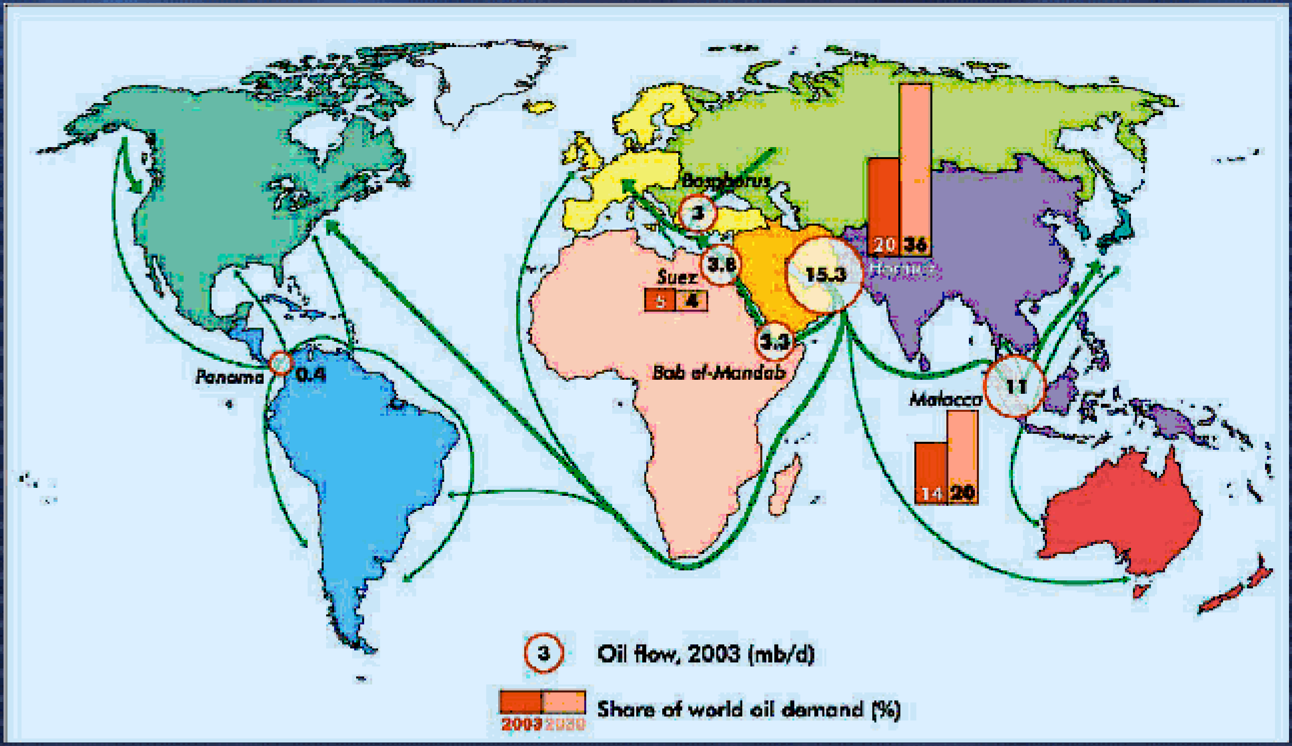
Natural Gas Pipeline Network in USA



Energy Information Administration, U.S. Department of Energy

www.eia.doe.gov

World Oil Trading Network



Treasurer of the Commonwealth
Australia
www.treasurer.gov.au

Link Importance Indicators

Definition : Link Importance Indicators According to Jenelius, Petersen, and Mattsson (2006)

In a network G, the global importance, I^1, the demand-weighted importance, I^2, and the relative unsatisfied demand, I^3, of link $k \in G$ are defined, respectively, as follows:

$$I^1(k) = \frac{1}{n_W} \sum_{w \in W} (\lambda_w(G-k) - \lambda_w(G)),$$

$$I^2(k) = \frac{\sum_{w \in W} d_w(\lambda_w(G-k) - \lambda_w(G))}{\sum_{w \in W} d_w},$$

$$I^3(k) = \frac{\sum_{w \in W} u_w(G-k)}{\sum_{w \in W} d_w},$$

where $\lambda_w(G)$ is the original equilibrium cost of O/D pair w while $\lambda_w(G-k)$ is the equilibrium cost of O/D pair w after link k is removed; $u_w(G-k)$ is the unsatisfied demand for O/D pair w after link k is removed.

Advantages of the N-Q Measure over Link Importance Indicators

- The N-Q measure is unified and can be applied to any network component, be it a node, or a link of a set of nodes and links;
- The N-Q measure is independent of whether the network is disconnected or not.

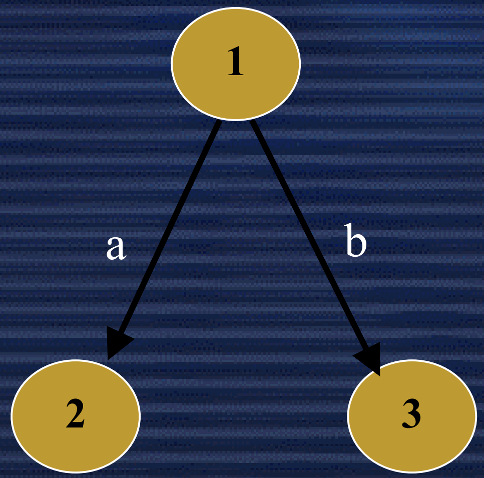
Example 1

Assume a network with two O/D pairs: w_1=(1,2) and w_2=(1,3) with demands given, respectively, by d_{w1}=100 and d_{w2}=20. The path for each O/D pair is: for w_1, $p_1=a$; for w_2, $p_2=b$.

The equilibrium path flows are $x_{p_1}^*$= 100, $x_{p_2}^*$=20.

The equilibrium path travel cost is

$C_{p_1}=C_{p_2}$=20.



$c_a(f_a)=0.01f_a+19$

$c_b(f_b)=0.05f_b+19$

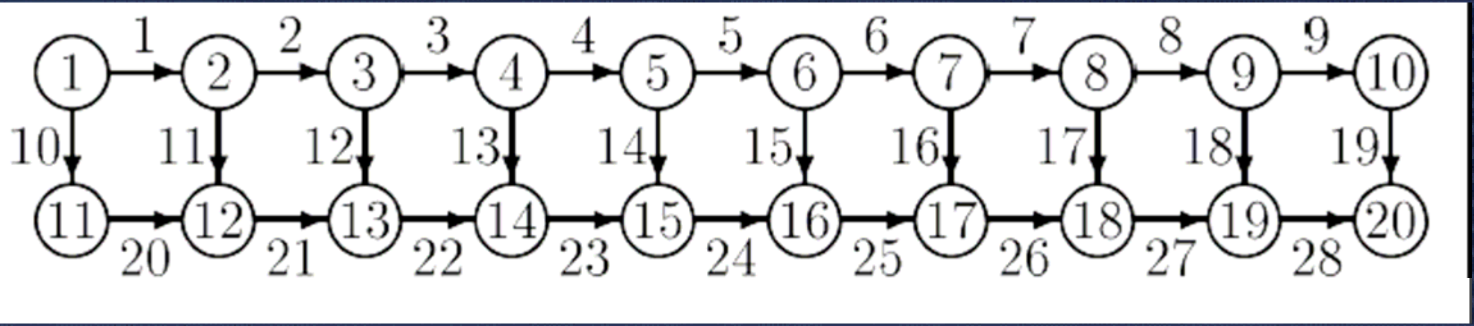
Importance and Ranking of Links and Nodes

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure	Importance Value from I^3	Importance Ranking from I^3
a	0.8333	1	0.5000	1	0.8333	1
b	0.1667	2	0.5000	1	0.1667	2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
1	1.0000	1	1.0000	1
2	0.8333	2	0.5000	2
3	0.1667	3	0.5000	2

Example 2

The network topology is the following:



$w_1 = (1,19)$, $w_2 = (1,20)$

$d_{w_1} = d_{w_2} = 100$

Link Cost Functions

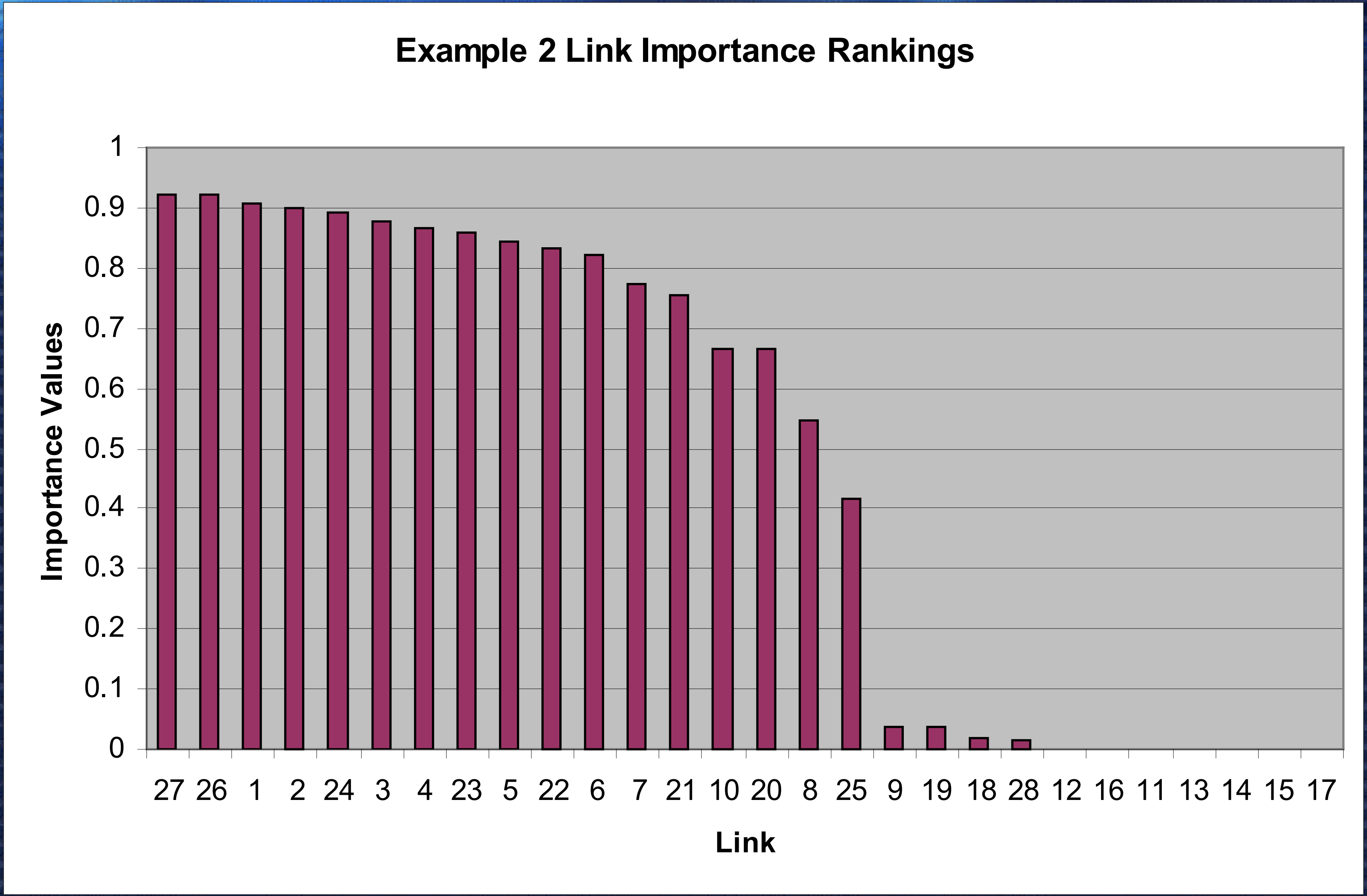
Link a	Link Cost Function $c_a(f_a)$
1	$.00005f_1^4 + 5f_1 + 500$
2	$.00003f_2^4 + 4f_2 + 200$
3	$.00005f_3^4 + 3f_3 + 350$
4	$.00003f_4^4 + 6f_4 + 400$
5	$.00006f_5^4 + 6f_5 + 600$
6	$7f_6 + 500$
7	$.00008f_7^4 + 8f_7 + 400$
8	$.00004f_8^4 + 5f_8 + 650$
9	$.00001f_9^4 + 6f_9 + 700$
10	$4f_{10} + 800$
11	$.00007f_{11}^4 + 7f_{11} + 650$
12	$8f_{12} + 700$
13	$.00001f_{13}^4 + 7f_{13} + 600$
14	$8f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003f_{15}^4 + 9f_{15} + 200$
16	$8f_{16} + 300$
17	$.00003f_{17}^4 + 7f_{17} + 450$
18	$5f_{18} + 300$
19	$8f_{19} + 600$
20	$.00003f_{20}^4 + 6f_{20} + 300$
21	$.00004f_{21}^4 + 4f_{21} + 400$
22	$.00002f_{22}^4 + 6f_{22} + 500$
23	$.00003f_{23}^4 + 9f_{23} + 350$
24	$.00002f_{24}^4 + 8f_{24} + 400$
25	$.00003f_{25}^4 + 9f_{25} + 450$
26	$.00006f_{26}^4 + 7f_{26} + 300$
27	$.00003f_{27}^4 + 8f_{27} + 500$
28	$.00003f_{28}^4 + 7f_{28} + 650$

Importance and Ranking of Links

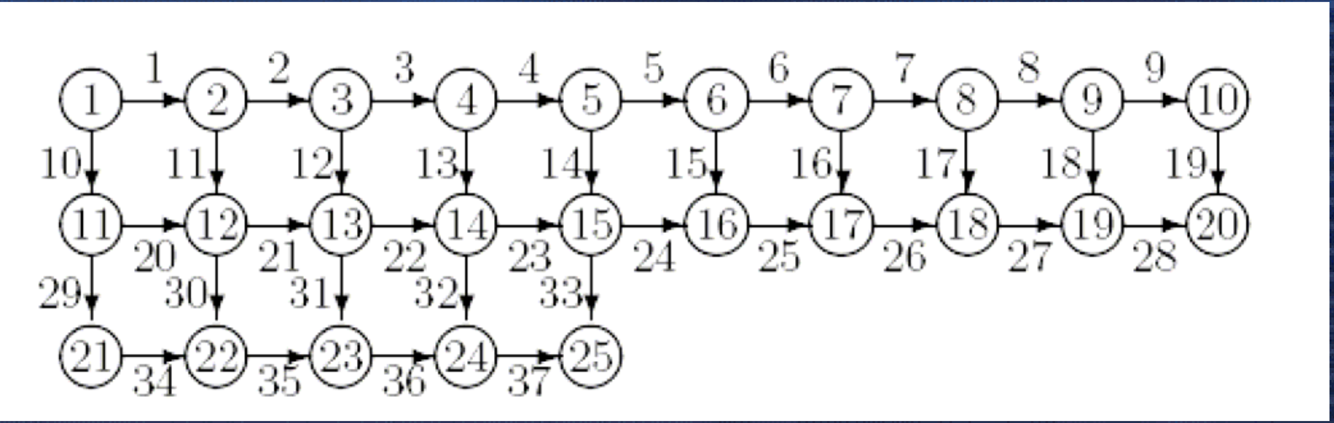
Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19



Example 2 Link Importance Rankings
Importance Values
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
27 26 1 2 24 3 4 23 5 22 6 7 21 10 20 8 25 9 19 18 28 12 16 11 13 14 15 17
Link

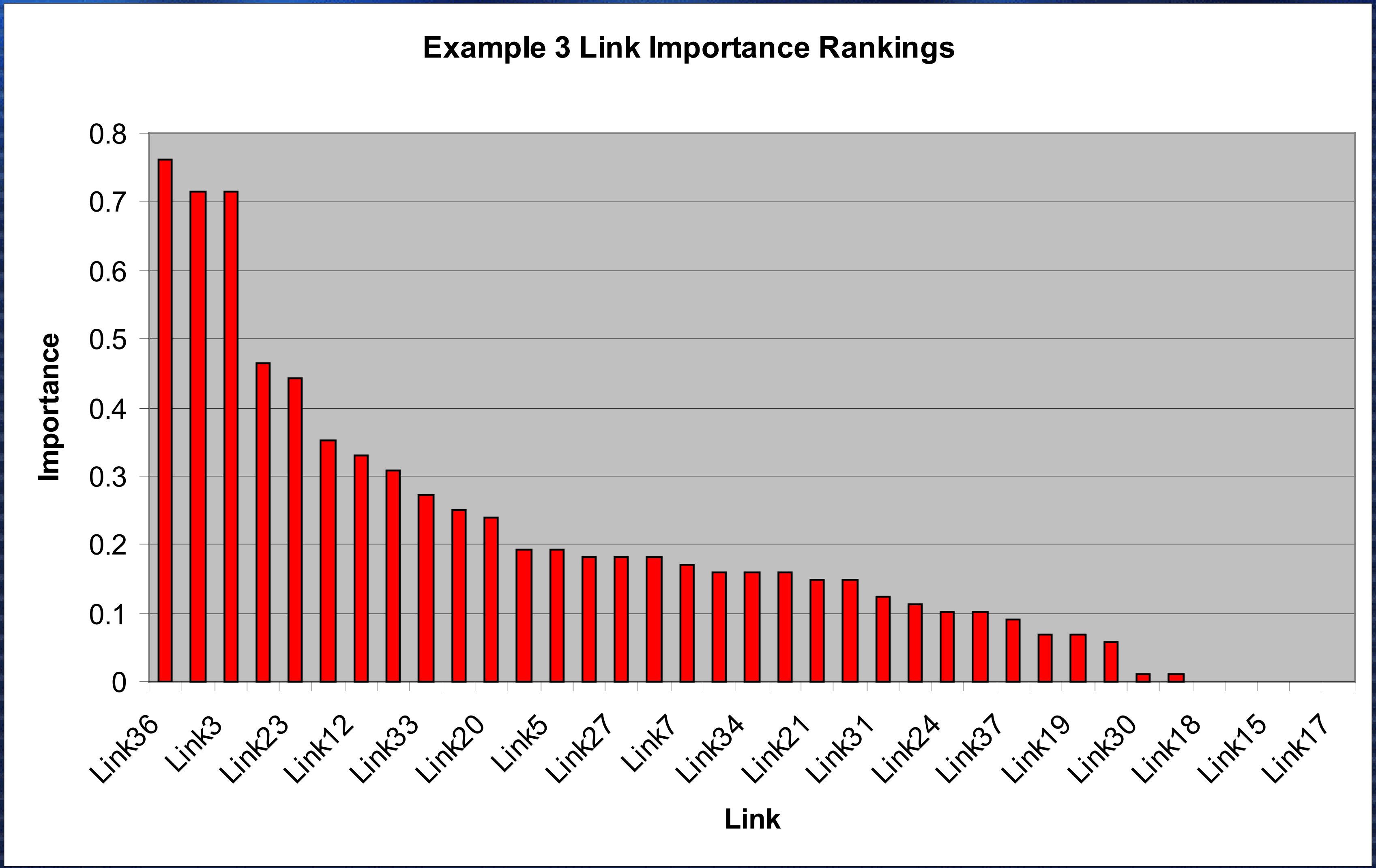
Example 3



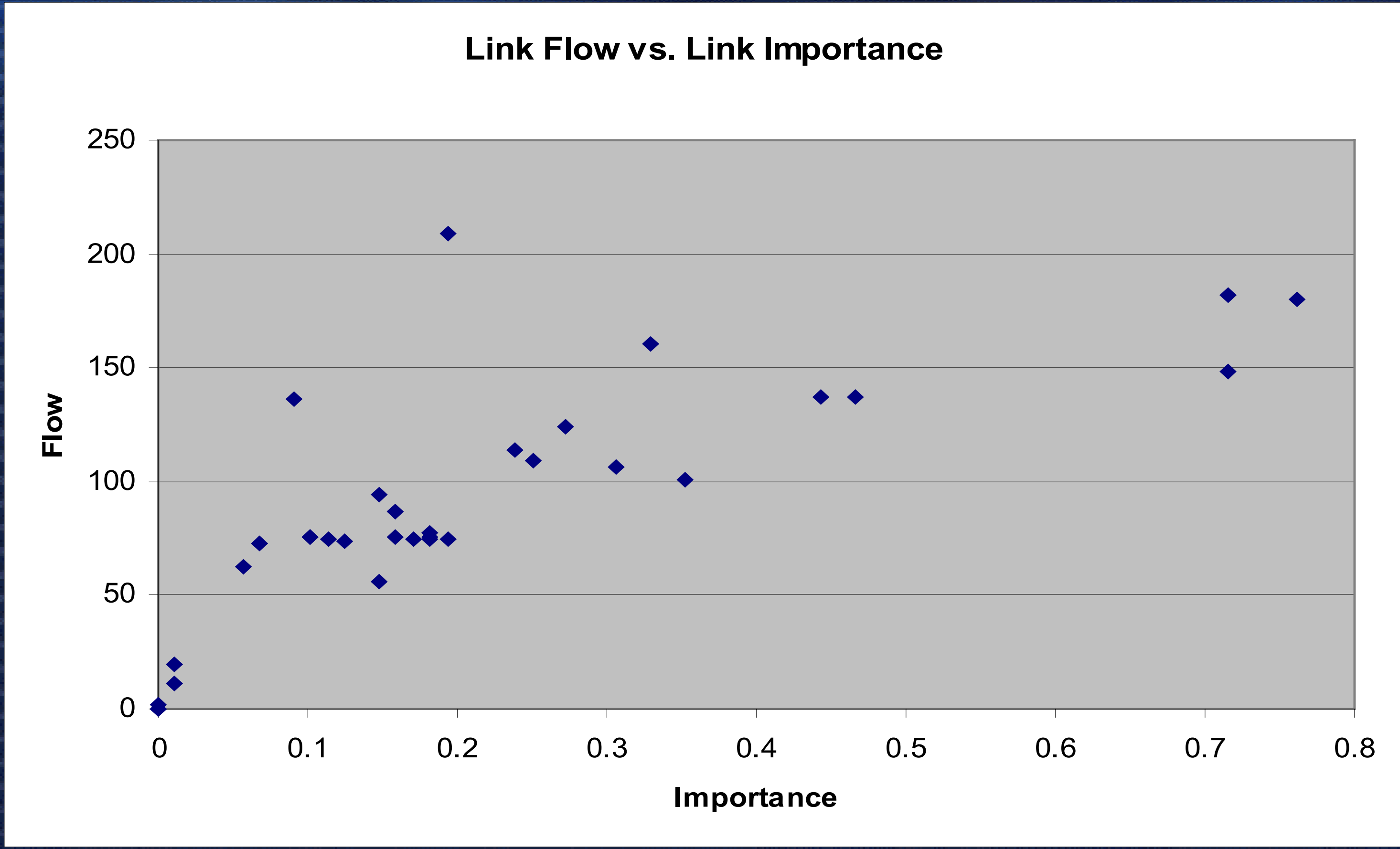
From Nagurney (1984)

w_1= (1,20), w_2= (1,25), w_3= (2,20), w_4= (3,25), w_5= (1,24), w_6= (11,25)

d_{w_1}= 50, d_{w_2}= 60, d_{w_3}= 100, d_{w_4}= 100, d_{w_5}= 100, d_{w_6}= 100

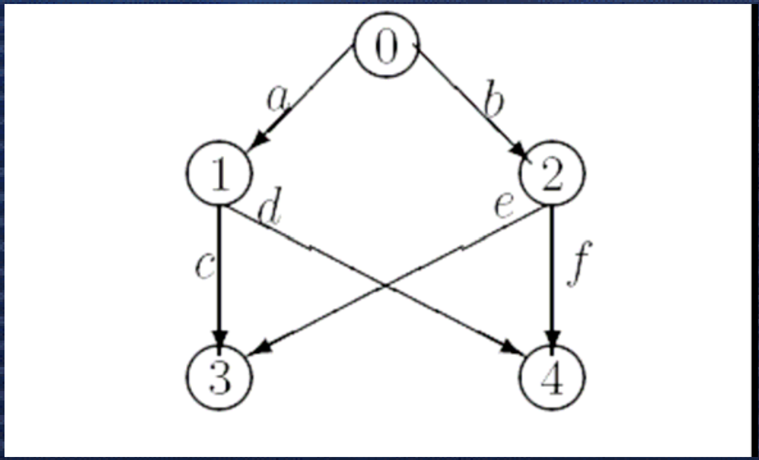

Example 3 Link Importance Rankings
Importance
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Link36
Link3
Link23
Link12
Link33
Link20
Link5
Link27
Link7
Link34
Link21
Link31
Link24
Link37
Link19
Link30
Link18
Link15
Link17
Link

Link Flow and Link Importance



Example 4 – Elastic Demand

The network topology is the following:



O/D pairs are: $w_1 = (0,3)$, $w_2 = (0,4)$

Link cost functions are: $c_a(f_a)=f_a$, $c_b(f_b)=f_b$, $c_c(f_c)=f_c$, $c_d(f_d)=f_d$, $c_e(f_e)=f_e$, $c_f(f_f)=f_f$

Inverse demand functions are: $\lambda_{w1}(d_{w_1}) = 100 - d_{w1}$

$\lambda_{w2}(d_{w_2}) = 40 - d_{w2}$

Importance and Rankings of Links

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
a	0.5327	1	N/A	N/A
b	0.5327	1	N/A	N/A
c	0.1475	2	N/A	N/A
d	0.0533	3	0.4516	1
e	0.1475	2	N/A	N/A
f	0.0533	3	0.4516	1

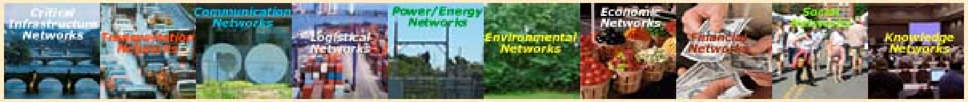
Importance and Rankings of Nodes

Node	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
0	1.0000	1	N/A	N/A
1	0.5327	2	0.2775	2
2	0.5327	2	0.2775	2
3	0.1475	3	0.3509	1
4	0.1475	3	0.3509	1

The Advantages of the Nagurney and Qiang Network Efficiency Measure

- It captures flows, costs, and behavior of travelers, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both;
- It can be applied to assess the efficiency/performance of a wide range of critical infrastructure networks;
- It is the unified measure that can be used to assess the network efficiency with either fixed or elastic demands.

The Virtual Center for Supernetworks



Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

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