General financial, economic and electric models: methodologies and suggestions for the recovery

Patrizia Daniele

Department of Mathematics and Computer Science
University of Catania - Italy

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Outline

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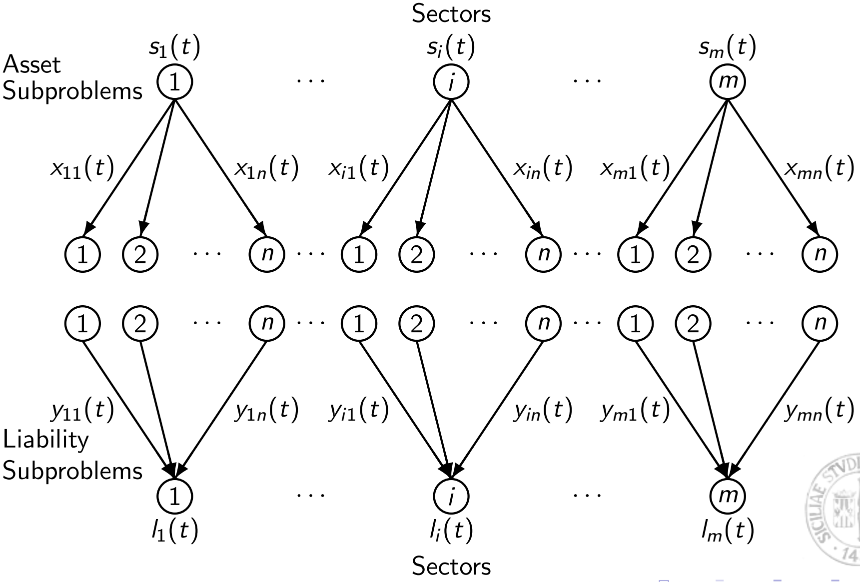
The Financial Model

- J. Dong, D. Zhang, A. Nagurney, Mathematical and Computer Modelling, 1996.
- A. Nagurney, Quantitative Finance, 2001.
- P. Daniele, Innovations in Financial and Economic Networks, A. Nagurney Ed., 2003.
- P. Daniele, S. Giuffré, S. Pia, Journal of Industrial and Management Optimization, 2005.
- A. Barbagallo, P. Daniele, A. Maugeri, Nonlinear Analysis: Theory, Methods & Applications, 2012.

Financial Network



Notation

- m sectors: households, domestic business, banks, financial institutions, state and local governments
- n instruments: mortgages, mutual funds, saving deposits, money market funds
- $s_i(t)$: total financial volume held by sector i at time t as assets
- $l_i(t)$: total financial volume held by sector i at time t as liabilities.
- $x_{ij}(t)$: amount of instrument j held as an asset in sector i's portfolio
- $y_{ij}(t)$: amount of instrument j held as a liability in sector i's portfolio
- $r_j(t)$: price of instrument j held as an asset
- $(1 + h_j(t))r_j(t)$: price of instrument j held as a liability, $h_j \geq 0$

Set of feasible assets and liabilities:

$$P_i = \Bigg\{ (x_i(t), y_i(t)) \in L^2([0, T], \mathbb{R}^{2n}) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t),$$

$$\sum_{j=1}^{n} y_{ij}(t) = l_i(t) \quad \text{a.e.} \quad \text{in } [0, T], \quad x_i(t) \geq 0, \; y_i(t) \geq 0, \quad \text{a.e.} \quad \text{in } [0, T] \Bigg\}$$

$$\forall i = 1, \dots, m.$$

Set of feasible assets and liabilities:

$$P_i = \left\{ (x_i(t), y_i(t)) \in L^2([0,T], \mathbb{R}^{2n}) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \right.$$

$$\left. \sum_{j=1}^{n} y_{ij}(t) = l_i(t) \quad \text{a.e. } \quad \text{in } [0,T], \quad x_i(t) \geq 0, \; y_i(t) \geq 0, \quad \text{a.e. } \quad \text{in } [0,T] \right\}$$

$$\forall i = 1, \ldots, m.$$

Set of feasible instrument prices:

$$\mathcal{R} = \{ r \in L^2([0,T], \mathbb{R}^n) : \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t),$$

$$j = 1, \ldots, n, \quad \text{a.e. } \quad \text{in } [0,T] \},$$

Utility Function:

$$
\begin{aligned}
& U_i(t, x_i(t), y_i(t), r(t)) \\
= & \underbrace{u_i(t, x_i(t), y_i(t))}_{\text{measure of the risk}} + \sum_{j=1}^{n} \underbrace{r_j(t)(1-\tau_{ij}(t))[x_{ij}(t)-(1+h_j(t))y_{ij}(t)]}_{\substack{\text{difference between the asset holdings} \\ \text{and the liability holdings}}},
\end{aligned}
$$

Utility Function:

$$U_i(t, x_i(t), y_i(t), r(t))$$

$$= \underbrace{u_i(t, x_i(t), y_i(t))}_{\text{measure of the risk}} + \sum_{j=1}^{n} \underbrace{r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 + h_j(t))y_{ij}(t)]}_{\substack{\text{difference between the asset holdings} \\ \text{and the liability holdings}}},$$

Equilibrium condition:

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x^*_{ij}(t) - (1 + h_j(t))y^*_{ij}(t)\right]+F_j(t)\begin{cases} \geq 0 & \textit{if } \; r^*_j(t) = \underline{r}_j(t) \\ = 0 & \textit{if } \; \underline{r}_j(t) < r^*_j(t) < \overline{r}_j(t) \\ \leq 0 & \textit{if } \; r^*_j(t) = \overline{r}_j(t) \end{cases}$$

Definition

$(x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^{m} P_i \times \mathcal{R}$ equilibrium of the dynamic financial model $\iff \forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, and a.e. in $[0, T]$,

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) - \mu_i^{(1)}(t) \geq 0,$$

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t))r_j^*(t) - \mu_i^{(2)}(t) \geq 0,$$

$$x_{ij}^*(t)\Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) - \mu_i^{(1)}(t)\Big] = 0,$$

$$y_{ij}^*(t)\Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t))r_j^*(t) - \mu_i^{(2)}(t)\Big] = 0,$$

Theorem (Variational Formulation)

$(x^*, y^*, r^*) \in \prod_{i=1}^{m} P_i \times \mathcal{R}$ *dynamic financial equilibrium* $\Longleftrightarrow$ *solves:*

Find $(x^*, y^*, r^*) \in \prod_{i=1}^{m} P_i \times \mathcal{R}$*:*

$$\sum_{i=1}^{m} \int_0^T \left\{ \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \right] \times [x_{ij}(t) - x_{ij}^*(t)] \right.$$

$$\left. + \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t)) r_j^*(t)(1+h_j(t)) \right] \times [y_{ij}(t) - y_{ij}^*(t)] \right\} dt$$

$$+ \sum_{j=1}^{n} \int_0^T \sum_{i=1}^{m} \left\{ (1-\tau_{ij}(t)) \left[x_{ij}^*(t) - (1+h_j(t)) y_{ij}^*(t) \right] + F_j(t) \right\} \times \left[r_j(t) - r_j^*(t) \right] dt \geq 0,$$

$$\forall (x, y, r) \in \prod_{i=1}^{m} P_i \times \mathcal{R}$$

Conditions

Deficit formula

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^{*}(t)-(1+h_j(t))y_{ij}^{*}(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$

$$\forall j=1,\ldots,n,$$

- $\rho_j^{(1)*}(t)$: deficit per unit
- $\rho_j^{(2)*}(t)$: positive surplus per unit

Balance law

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right]$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t) + \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t)$$

Balance law

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$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t) + \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t)$$

Liability formula

$$\sum_{i=1}^{m} l_i(t) = \frac{(1-\theta(t))\sum_{i=1}^{m} s_i(t) + \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))}$$

- $\tau_{ij}(t) = \theta(t),\ i = 1,\ldots,m\ j = 1,\ldots,n,$
- $h_j(t) = i(t),\ j = 1,\ldots,n$

Propositions

1. When the prices are minimal, namely they coincide with the floor prices, the economy collapses.

$$\left.\begin{array}{l}\rho_j^{(1)*}(t)(\underline{r}_j(t) - r_j^*(t)) = 0,\\ \rho_j^{(2)*}(t)(r_j^*(t) - \overline{r}_j(t)) = 0,\end{array}\right. \qquad \forall j = 1, \ldots, n.$$

Propositions

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2. Minimal prices imply the increase in the public debt.

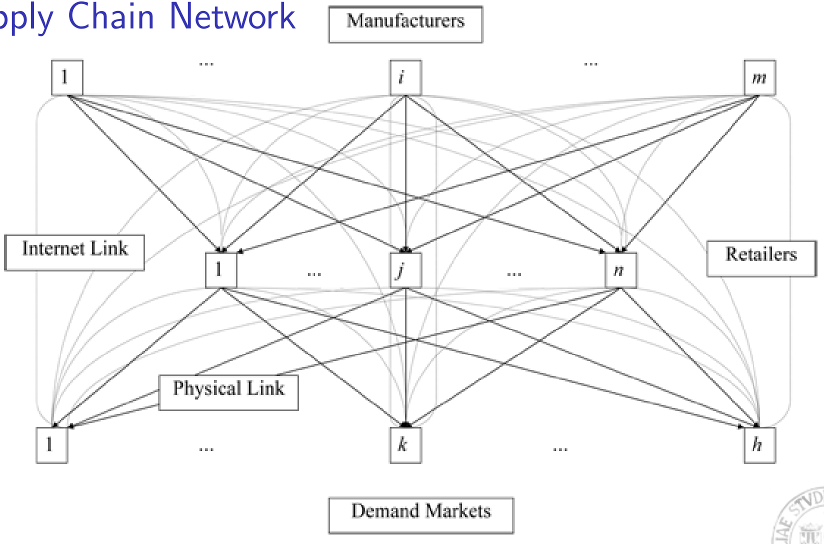
$$\left\{ \sum_{i=1}^{m} (1 - \tau_{ij}(t)) \left[x_{ij}^*(t) - (1 + h_j(t)) y_{ij}^*(t) \right] \right\} \underline{r}_j(t) + F_j(t) \underline{r}_j(t)$$

$$= \underbrace{\rho_j^{(1)*}(t) \underline{r}_j(t) > 0,}_{\text{deficit}} \quad j = 1, \dots, n, \text{ a.e. in } [0, T].$$

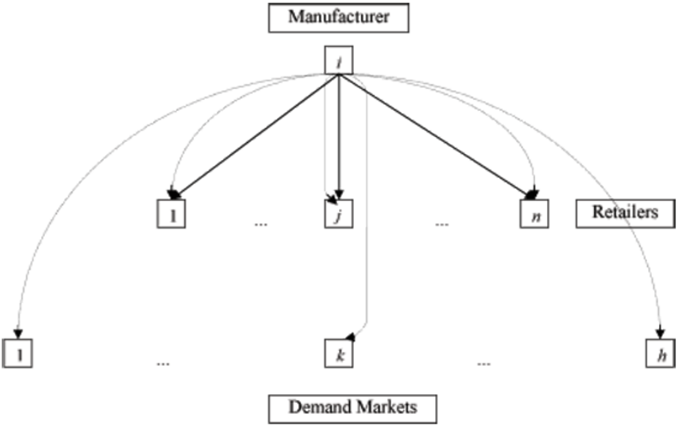
Supply Chain Networks

- A. Nagurney, J. Dong, Edward Elgar Publishers, Chelthenham, England, 2002.
- A. Nagurney, Z. Liu, Network Science, in Nonlinear Science and Dynamic Game Theory Applied to the Study of Infrastructure Systems, T.L. Friesz, Editor, Springer, Berlin, Germany, 2007.
- J.M. Cruz, European Journal of Operational Research, 2008.
- P. Daniele, Transportation Research Part E, 2010.

Supply Chain Network



Behavior of Manufacturer i



Notation

- $G_i(t)$: production of manufacturer i at time $t \in [0, T]$;
- $G(t) \in L^2([0, T], \mathbb{R}^m_+)$: production outputs of all manufacturers;
- $p_i(G((t)) : L^2([0, T], \mathbb{R}^m_+) \to \mathbb{R}^m_+$: production cost;
- $g_{ijl}(t)$: product shipment associated with manufacturer i, retailer j and mode l;
- $g(t) \in L^2([0, T], \mathbb{R}^{2mn}_+)$: shipment between manufacturers and retailers;
- $c_{ijl}(g_{ijl}(t))$: transaction cost between manufacturer i and retailer j via mode l;
- $b_{ik}(t)$: product shipment associated with manufacturer i and demand market k;
- $b(t) \in L^2([0, T], \mathbb{R}^{mh}_+)$: shipment between manufacturers and consumers;
- $c_{ik}(b_{ik}(t))$: transaction cost between manufacturer i and demand market k;

Notation

- $s_i(t)$: excess of production of manufacturer i at time $t \in [0, T]$;
- $s(t) \in L^2([0, T], \mathbb{R}^m_+)$: production excesses;
- $\overline{s}(t) \in L^2([0, T], \mathbb{R}^m_+)$: upper level;
- $c_i(s_i(t))$: storing cost;

Notation

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Conservation of flow equation:

$$G_i(t) = \sum_{j=1}^{n} \sum_{l=1}^{2} g_{ijl}(t) + \sum_{k=1}^{h} b_{ik}(t) + s_i(t), \quad \forall i.$$

Notation

- $\rho^1_{ijl}(t)$: price charged for the product by manufacturer i to retailer j in the transaction via mode l;
- $\rho^1_{ik}(t)$: price charged for the product by manufacturer i to consumers at the demand market k;
- $\rho^1(t) = \left(\rho^1_{ik}(t)\right)_{\substack{i=1,\dots,m \\ k=1,\dots,h}}$: price vector;

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Set of feasible quantities:

$$\mathbb{K} = \left\{(g(t), b(t), s(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh+m}_+)\right\}$$

Revenue of manufacturer i

$$
\begin{aligned}
R^i(g(t), b(t), s(t)) &= \sum_{j=1}^{n} \sum_{l=1}^{2} \rho_{ijl}^1(t)\, g_{ijl}(t) - p_i(g(t), b(t), s(t)) \\
&- \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(g_{ijl}(t)) \\
&+ \sum_{k=1}^{h} \rho_{ik}^1(t)\, b_{ik}(t) - \sum_{k=1}^{h} c_{ik}(b_{ik}(t)),
\end{aligned}
$$

Equilibrium Law

$$\forall i = 1, \ldots, m \text{ and a.e. in } [0, T]$$

$$\text{if } s_i^*(t) = 0 \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) \geq 0;$$

$$\text{if } 0 < s_i^*(t) < \bar{s}_i(t) \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) = 0;$$

$$\text{if } s_i^*(t) = \bar{s}_i(t) \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) \leq 0.$$

Equilibrium Law

$$\forall i = 1, \dots, m \text{ and a.e. in } [0, T]$$
$$\text{if } s_i^*(t) = 0 \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) \geq 0;$$
$$\text{if } 0 < s_i^*(t) < \overline{s}_i(t) \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) = 0;$$
$$\text{if } s_i^*(t) = \overline{s}_i(t) \Rightarrow R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t)) \leq 0.$$

Optimality conditions for manufacturer i :

$$\text{Find } (g^*(t), b^*(t), s^*(t)) \in \mathbb{K} :$$

$$\max_{(g(t),b(t)) \in L^2([0,T],\mathbb{R}_+^{2mn+mh})} \left(R^i(g(t), b(t), s(t)) - c_i(s_i(t))\right)$$
$$= R^i(g^*(t), b^*(t), s^*(t)) - c_i(s_i^*(t))$$

Theorem (Variational Formulation)

If $R^i(\cdot,\cdot,s(t))$ is concave and differentiable, $\forall i = 1,\dots,m$ and grad $R^i : L^2([0,T],\mathbb{R}_+^{2mn+mh}) \to L^2([0,T],\mathbb{R}_+^{2mn})$, then $(g^*(t),b^*(t),s^*(t)) \in \mathbb{K}$ is a solution to the maximum problem $\Longleftrightarrow$ it is a solution to the EVI:

$$\text{Find } (g^*(t),b^*(t),s^*(t)) \in \mathbb{K}:$$

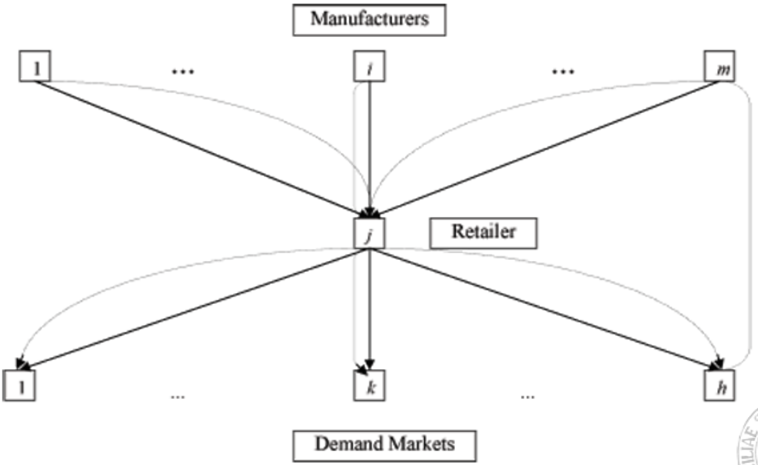
$$\int_0^T \langle -\sum_{i=1}^m \frac{\partial R^i(g^*(t),b^*(t),s^*(t))}{\partial g}, g(t)-g^*(t)\rangle\, dt$$

$$+\int_0^T \langle -\sum_{i=1}^m \frac{\partial R^i(g^*(t),b^*(t),s^*(t))}{\partial b}, b(t)-b^*(t)\rangle\, dt$$

$$+\int_0^T \langle R(g^*(t),b^*(t),s^*(t)) - c(s^*(t)), s(t)-s^*(t)\rangle\, dt \geq 0$$

$$\forall (g(t),b(t),s(t)) \in \mathbb{K}.$$

Behavior of Retailer j



Notation

- $d_{jkl}(t)$: quantity of product purchased by demand market k from retailer j via mode l at time $t \in [0, T]$;
- $d(t) \in L^2([0, T], \mathbb{R}^{2nh}_+)$: quantity vector;
- $c_{jkl}(d_{jkl}(t))$: transaction cost between retailer j and demand market k via mode l;
- $\overline{c}_{ijl}(g_{ijl}(t))$: transaction cost incurred by retailer j between retailer j and manufacturer i via mode l;
- $c_j\left(g_j(t), d_j(t)\right)$: handling cost associated to retailer j;
- $q_j(t)$: price imposed by retailer j to the product in his own outlet;

Optimality conditions for retailer j

$$\max_{(g_j,d_j)\in\mathbb{K}_{1j}\times\mathbb{K}_{3j}} \Big\{ q_j(t)\sum_{k=1}^{h}\sum_{l=1}^{2} d_{jkl}(t) - c_j\left(g_j(t),d_j(t)\right) - \sum_{i=1}^{m}\sum_{l=1}^{2}\overline{c}_{ijl}(g_{ijl}(t)) - \sum_{i=1}^{m}\sum_{l=1}^{2}\rho^1_{ijl}\, g_{ijl}(t) - \sum_{k=1}^{h}\sum_{l=1}^{2} c_{jkl}(d_{jkl}(t))\Big\},$$

where $\mathbb{K}_{1j} = \left\{ g_j(t) = (g_{ijl}(t))_{\substack{i=1,\dots,m\\ l=1,2}} \in L^2([0,T],\mathbb{R}^{2m}_+)\right\}$ and

$$\mathbb{K}_{3j} = \Big\{ d_j(t) = (d_{jkl}(t))_{\substack{k=1,\dots,h\\ l=1,2}} \in L^2([0,T],\mathbb{R}^{2h}_+), \text{ and } \sum_{k=1}^{h}\sum_{l=1}^{2} d_{jkl}(t) \le \sum_{i=1}^{m}\sum_{l=1}^{2} g_{ijl}(t) \text{ a.e. in } [0,T]\Big\}.$$

Notation

- $$R_{1j}(g_j(t), d_j(t)) = -c_j\,(g_j(t), d_j(t)) - \sum_{i=1}^{m}\sum_{l=1}^{2} \overline{c}_{ijl}(g_{ijl}(t)) - \sum_{i=1}^{m}\sum_{l=1}^{2} \rho^1_{ijl}\, g_{ijl}(t);$$
- $$R_{3j}(d_j(t)) = q_j(t)\sum_{k=1}^{h}\sum_{l=1}^{2} d_{jkl}(t) - \sum_{k=1}^{h}\sum_{l=1}^{2} c_{jkl}(d_{jkl}(t));$$
- $$\mathbb{K}_1 = \left\{ g(t) = (g_j(t))_{j=1,\ldots,n} : g_j(t) \in \mathbb{K}_{1j},\ j = 1,\ldots,n \right\}$$
- $$\mathbb{K}_3 = \left\{ d(t) = (d_j(t))_{j=1,\ldots,n} : d_j(t) \in \mathbb{K}_{3j},\ j = 1,\ldots,n \right\}$$

Theorem (Variational Formulation)

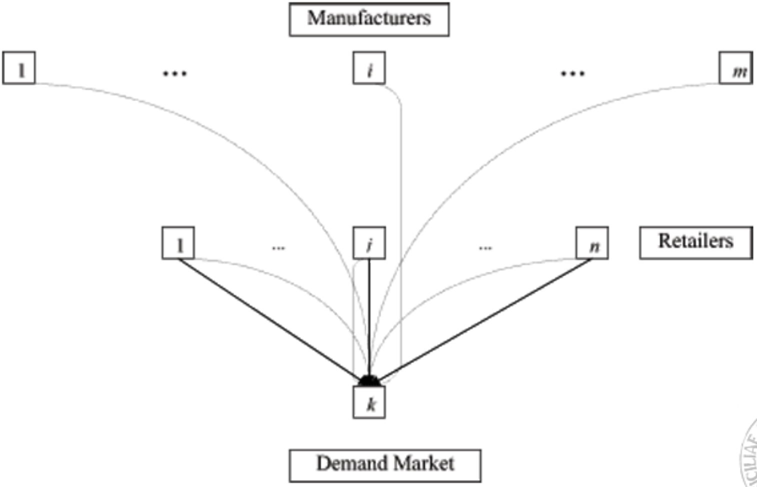
If $R_{1j}(\cdot,\cdot)$ and $R_{3j}(\cdot)$ are concave and differentiable $\forall j = 1,\ldots,n$, and

$$\begin{aligned} grad\ R_{1j} &: L^2([0,T],\mathbb{R}_+^{2mn+2nh}) \to L^2([0,T],\mathbb{R}_+^{2mn}), \\ grad\ R_{1j} &: L^2([0,T],\mathbb{R}_+^{2mn+2nh}) \to L^2([0,T],\mathbb{R}_+^{2nh}), \\ grad\ R_{3j} &: L^2([0,T],\mathbb{R}_+^{2nh}) \to L^2([0,T],\mathbb{R}_+^{2nh}), \end{aligned}$$

then $(g^*(t), d^*(t)) \in \mathbb{K}_1 \times \mathbb{K}_3$ is a solution to the maximum problem $\iff$ it is a solution to the EVI:

$$
\begin{gathered}
\text{Find } (g^*(t), d^*(t)) \in \mathbb{K}_1 \times \mathbb{K}_3 : \\
\int_0^T \langle -\operatorname{grad}_g \sum_{j=1}^n R_{1j}(g^*(t), d^*(t)), g(t) - g^*(t) \rangle \, dt \\
+ \int_0^T \langle -\operatorname{grad}_d \sum_{j=1}^n R_{1j}(g^*(t), d^*(t)), d(t) - d^*(t) \rangle \, dt \\
+ \int_0^T \langle -\operatorname{grad} \sum_{j=1}^n R_{3j}(d^*(t)), d(t) - d^*(t) \rangle \, dt \geq 0 \\
\forall (g(t), d(t)) \in \mathbb{K}_1 \times \mathbb{K}_3 .
\end{gathered}
$$

Behavior of Demand Market k



Notation

- $\tilde{c}_{jkl}(b(t), d(t))$: transaction cost between demand market k and retailer j via mode l;
- $\tilde{c}(b(t), d(t)) = (\tilde{c}_{jkl}(b(t), d(t)))_{\substack{j=1,\dots,n \\ k=1,\dots,h,\ l=1,2}}$: cost vector;
- $\hat{c}_{ik}(b(t), d(t))$: transaction cost between demand market k and manufacturer i;
- $\hat{c}(b(t), d(t)) = (\hat{c}_{ik}(b(t), d(t)))_{\substack{i=1,\dots,m \\ k=1,\dots,h}}$: cost vector;
- $\rho_k^3(t)$: demand price;
- $f_k(\rho^3(t))$: product at demand market k;
- $f(\rho_3(t))$: demand vector;
- $\overline{\rho_k^3}(t)$: maximum price;
- $\tau_k(t) \geq 0$: excess of demand;

Conservation of flow equation:

$$f_k(\rho^{3*}(t)) = \sum_{i=1}^{m} b^*_{ik}(t) + \sum_{j=1}^{n} \sum_{l=1}^{2} d^*_{jkl}(t) + \tau_k(t), \quad \forall k = 1, \ldots, h$$

Equilibrium conditions for demand market k

$$\rho^1_{ik}(t) + \hat{c}_{ik}(b^*(t), d^*(t)) \begin{cases} = \rho^{3*}_k(t), & \text{if} \quad b^*_{ik}(t) > 0 \\ \geq \rho^{3*}_k(t), & \text{if} \quad b^*_{ik}(t) = 0; \end{cases}$$

$$q_j(t) + \tilde{c}_{jkl}(b^*(t), d^*(t)) \begin{cases} = \rho^{3*}_k(t), & \text{if} \quad d^*_{jkl}(t) > 0 \\ \geq \rho^{3*}_k(t), & \text{if} \quad d^*_{jkl}(t) = 0; \end{cases}$$

$$\underbrace{f_k(\rho^{3*}(t)) - \left(\sum_{j=1}^{n} \sum_{l=1}^{2} d^*_{jkl}(t) + \sum_{i=1}^{m} b^*_{ik}(t) \right)}_{\tau_k(t)} \begin{cases} = 0, & 0 \leq \rho^{3*}_k(t) < \overline{\rho^{3*}_k}(t) \\ \geq 0, & \rho^{3*}_k(t) = \overline{\rho^3_k}(t). \end{cases}$$

Variational Formulation

$$\text{Find } (b^*(t), d^*(t), \rho^{3*}(t)) \in L^2([0,T], \mathbb{R}_+^{mh+2nh+h}) :$$

$$\int_0^T \Big\{ \sum_{j=1}^{n}\sum_{k=1}^{h}\sum_{l=1}^{2} \left[q_j(t) + \tilde{c}_{jkl}(b^*(t), d^*(t)) - \rho_k^{3*}(t)\right] \times \left[d_{jkl}(t) - d_{jkl}^*(t)\right]$$

$$+ \sum_{i=1}^{m}\sum_{k=1}^{h} \left[\rho_{ik}^1(t) + \hat{c}_{ik}(b^*(t), d^*(t)) - \rho_k^{3*}(t)\right] \times \left[b_{ik}(t) - b_{ik}^*(t)\right]$$

$$+ \sum_{k=1}^{h} \left[f_k(\rho^{3*}(t)) - \sum_{j=1}^{n}\sum_{l=1}^{2} d_{jkl}^*(t) - \sum_{i=1}^{m} b_{ik}^*(t) \right] \times \left[\rho_k^3(t) - \rho_k^{3*}(t)\right] \Big\} dt \geq 0,$$

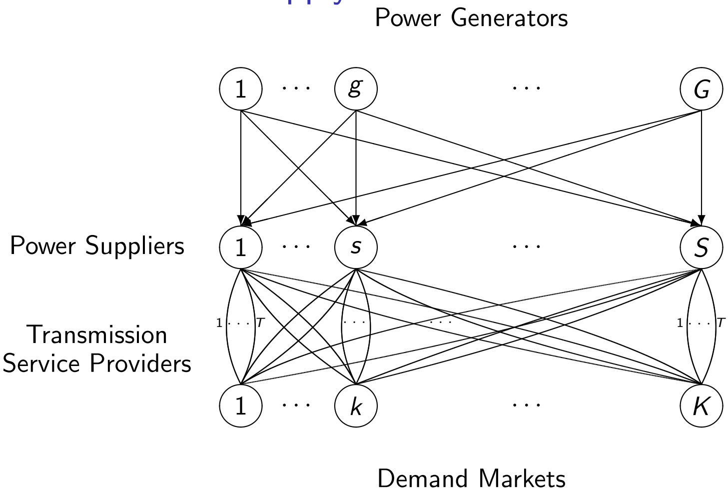
$$\forall (b(t), d(t), \rho^3(t)) \in L^2([0,T], \mathbb{R}_+^{mh+2nh+h}),\ \forall \rho^3(t) \leq \overline{\rho^3}(t) \text{ a.e. in } [0,T]$$

Equilibrium for the supply chain

Definition

The equilibrium state of the supernetwork consisting of the supply chain with electronic commerce and with data depending on time is one where the flows between the levels of the supernetwork coincide and the product shipments and the prices satisfy the sum of the evolutionary variational inequalities.

Electric Power Supply Chain



Behavior of Power Generator g

- $Q^1 = (q_{gs})_{\substack{g=1,\ldots,G \\ s=1,\ldots,S}} \in \mathbb{R}^{GS}$: vector of electric power flows between power generators and power suppliers
- ρ^*_{1gs} : unit price charged by power generator g for the transaction with power supplier s
- $f_g(Q^1)$: power generating cost function of power generator g
- $c_{gs}(q_{gs})$: transaction cost incurred by power generator g in transacting with power supplier s

Optimality conditions for Power Generator g

$$\begin{cases} \max \left\{ \displaystyle\sum_{s=1}^{S} \rho^*_{1gs} q_{gs} - f_g(Q^1) - \sum_{s=1}^{S} c_{gs}(q_{gs}) \right\} \\ q_{gs} \geq 0, \qquad s = 1, \dots, S \end{cases}$$

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Variational Formulation

$$\text{Find } Q^{1*} \in \mathbb{R}^{GS}_+ :$$

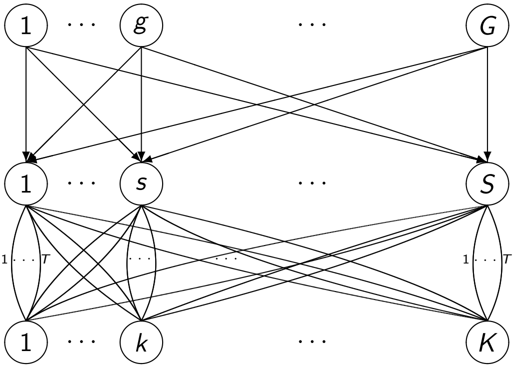
$$\sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(q^*_{gs})}{\partial q_{gs}} - \rho^*_{1gs} \right] \times [q_{gs} - q^*_{gs}] \geq 0, \quad \forall Q^1 \in \mathbb{R}^{GS}_+$$

Electric Power Supply Chain

Power Generators

Power Suppliers

Transmission Service Providers



Demand Markets

Behavior of Power Supplier s

- $Q^2 = (q^t_{sk})_{\substack{s=1,\ldots,S,\ k=1,\ldots,K,\\ t=1,\ldots,T}} \in \mathbb{R}^K$: vector of power flows between suppliers and demand markets
- ρ^{t*}_{2sk} : price charged by power supplier s to demand market k via transmission service provider t
- $c_s(Q^1)$: operating cost of power supplier s
- $c^t_{sk}(q^t_{sk})$: transaction cost incurred by power supplier s in transacting with demand market k via transmission provider t
- $\hat{c}_{gs}(q_{gs})$: transaction cost incurred by power supplier s in transacting with power generator g

Conservation of flow equations:

$$\sum_{k=1}^{K}\sum_{t=1}^{T} q_{sk}^{t} = \sum_{g=1}^{G} q_{gs}, \quad s = 1, \dots, S.$$

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Optimality conditions for Power Supplier s

$$\begin{cases} \max\Big\{ \sum_{k=1}^{K}\sum_{t=1}^{T} \rho_{2sk}^{t*} q_{sk}^{t} - c_s(Q^1) - \sum_{g=1}^{G} \rho_{1gs}^{*} q_{gs} \\ \qquad - \sum_{g=1}^{G} \hat{c}_{gs}(q_{gs}) - \sum_{k=1}^{K}\sum_{t=1}^{T} c_{sk}^{t}(q_{sk}^{t}) \Big\} \\ \sum_{k=1}^{K}\sum_{t=1}^{T} q_{sk}^{t} = \sum_{g=1}^{G} q_{gs} \\ q_{gs} \geq 0, \quad g = 1, \ldots, G \\ q_{sk}^{t} \geq 0, \quad k = 1, \ldots, K; t = 1, \ldots, T \end{cases}$$

Variational Formulation

$$\text{Find } (Q^{2*}, Q^{1*}) \in \mathcal{K}^2 = \Bigg\{ (Q^2, Q^1) \in \mathbb{R}_+^{STK+GS} :$$

$$\sum_{k=1}^{K} \sum_{t=1}^{T} q_{sk}^t = \sum_{g=1}^{G} q_{gs}, \quad s = 1, \ldots, S \Bigg\}$$

$$\sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{t=1}^{T} \left[\frac{\partial c_{sk}^t(q_{sk}^{t*})}{\partial q_{sk}^t} - \rho_{2sk}^{t*} \right] \times [q_{sk}^t - q_{sk}^{t*}]$$

$$+ \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial c_s(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} + \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \geq 0,$$

$$\forall (Q^2, Q^1) \in \mathcal{K}^2$$

Equilibrium Conditions for Demand Market k

- $\hat{c}^t_{sk}(Q^2)$: unit transaction cost incurred by consumers at demand market k in transacting with power supplier s via transmission provider t
- d_k : demand at demand market k
- ρ_{3k} : demand market price at demand market k

Equilibrium Conditions for Demand Market k

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Conservation equations:

$$d_k = \sum_{s=1}^{S} \sum_{t=1}^{T} q^t_{sk}, \quad k = 1, \ldots, K$$

Equilibrium conditions for consumers at demand market k

$$\rho_{2sk}^{t*} + \hat{c}_{sk}^{t}(Q^{2*}) \begin{cases} = \rho_{3k}^{*}, & \text{if} \quad q_{sk}^{t*} > 0, \\ \geq \rho_{3k}^{*}, & \text{if} \quad q_{sk}^{t*} = 0, \end{cases} \qquad \forall s = 1, \ldots, S, \ \forall t = 1, \ldots, T$$

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Variational Formulation

$$\text{Find } Q^{2*} \in \mathcal{K}^4 = \left\{ Q^2 \in R_+^{K(ST)} : d_k = \sum_{s=1}^{S} \sum_{t=1}^{T} q_{sk}^{t}, \quad k = 1, \dots, K \right\}$$

$$\sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{t=1}^{T} \left[\rho_{2sk}^{t*} + \hat{c}_{sk}^{t}(Q^{2*}) \right] \times [q_{sk}^{t} - q_{sk}^{t*}] \geq 0, \quad \forall Q^2 \in \mathcal{K}^4$$

Equilibrium Conditions for the Electric Power Supply Chain Network

Definition

The equilibrium state of the electric power supply chain network is one where the electric power flows between the tiers of the network coincide and the electric power flows satisfy the sum of the previous optimality conditions.

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The equilibrium state of the electric power supply chain network is one where the electric power flows between the tiers of the network coincide and the electric power flows satisfy the sum of the previous optimality conditions.

Theorem

The equilibrium conditions governing the electric power supply chain network coincide with the solution of the variational inequality obtained by the sum of the previous ones.

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Infinite Dimensional Separation Theory

X real linear topological space; $S \subseteq X$ convex; Y real normed space ordered by a convex cone C; Z real normed space; $f : S \to \mathbb{R}$, $g : S \to Y$, $h : S \to Z$ affine-linear mapping;

$$f(x_0) = \min_{x \in \mathbb{K}} f(x), \qquad \textit{Problem } 1$$

with

$$\mathbb{K} = \{x \in S : g(x) \in -C,\ h(x) = \theta_Z\}$$

and the dual problem:

$$\max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle], \qquad \textit{Problem } 2$$

where

$$C^* = \{u \in Y^* : \langle u, y \rangle \geq 0,\ \forall y \in C\}.$$

X real normed space; X^* topological dual space; $C \subseteq X$; $x \in X$;
Cone Generated by C:

$$\text{cone}\,(C) = \{\lambda x : x \in C,\ \lambda \in \mathbb{R},\ \lambda \geq 0\}$$

Tangent Cone to C at x :

$$T_C(x) = \Big\{h \in X : h = \lim_{n\to\infty} \lambda_n(x_n - x),\ \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0\ \forall n \in \mathbb{N},$$

$$x_n \in C,\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x\Big\}$$

Quasi-Relative Interior of a convex set $C \subseteq X$:

$$\mathrm{qri}C = \{x \in C : \; T_C(x) \text{ linear subspace of } X\}$$

Normal Cone to C at x:

$$N_C(x) = \{\xi \in X^* : \langle \xi, y - x \rangle \leq 0, \; \forall y \in C\}$$

Proposition

$x \in C \subseteq X$ convex, $x \in$ qri $C \Leftrightarrow N_C(x)$ linear subspace of X^*.

Theorem

$f : S \to \mathbb{R}$, $g : S \to Y$ convex functions, $h : S \to Z$ affine-linear mapping,

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) :$$

$$x \in S \setminus \mathbb{K},\ \alpha \geq 0,\ y \in C\},$$

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\} = \emptyset. \qquad \textit{Assumption S}$$

Then, Problem 2 is also solvable, $\langle \bar{u}, g(x_0) \rangle = 0$ and the extrema of the two problems are equal.

Theorem

$$M = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)),\ x \in S,\ \alpha \geq 0,\ y \in C\},$$

$f : S \to \mathbb{R}$, $g : S \to Y$, $h : S \to Z$ *such that:*

$$\exists\, \bar{x} \in \mathbb{K},\ \exists\, (\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z) \text{ such that}$$

$$\hat{\xi}(f(\bar{x}) - f(x_0)) + \langle \hat{y}^*, g(\bar{x}) \rangle + \langle \hat{z}^*, h(\bar{x}) \rangle < 0. \qquad \textit{Assumption N}$$

Then, Problem 2 is solvable and the extremal values of both problems are equal.

Applications to Equilibrium Models

Archetype problem:

$$\int_0^T \langle C(x_0(t)), x(t) - x_0(t) \rangle \, dt \geq 0 \quad \forall x \in \mathbb{K},$$

where

$$\mathbb{K} = \left\{ x \in L^2([0, T], \mathbb{R}^m) : x(t) \geq 0, \ \Phi x(t) = \rho(t) \text{ a.e. in } [0, T] \right\},$$

with $\rho(t) \in L^2([0, T], \mathbb{R}^l)$, $\rho(t) > 0$ a.e. in $[0, T]$, $\Phi = \{\Phi_{ij}\}_{\substack{i=1,\dots,l \\ j=1,\dots,m}}$, $\Phi_{ij} \in \{0, 1\}$, and in each column there is only one entry different from zero, and $C : \mathbb{K} \to L^2([0, T], \mathbb{R}^m)$ is the cost trajectory.

$$\int_0^T \langle C(x_0(t)), x(t) - x_0(t) \rangle \, dt \geq 0 \; \forall x \in \mathbb{K}$$

$$\Updownarrow$$

$$\min_{\mathbb{K}} f(x) = f(x_0) = 0$$

with

$$f(x) = \int_0^T \langle C(x_0(t)), x(t) - x_0(t) \rangle \, dt$$