Equilibria, Supernetworks, and Evolutionary Variational Inequalities

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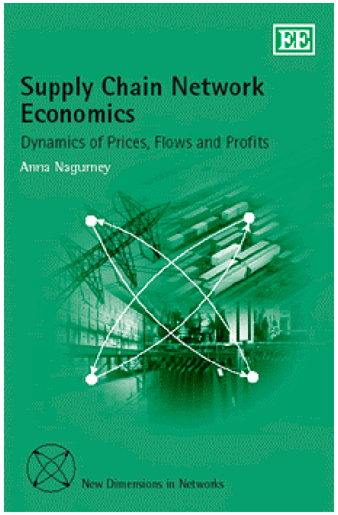
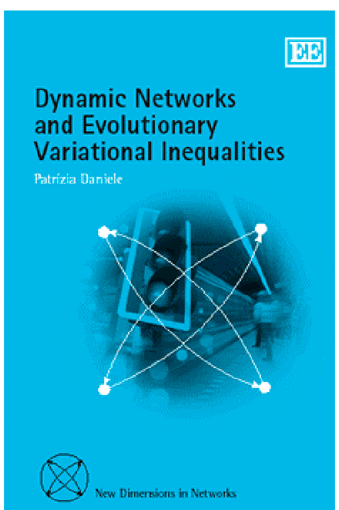
The New Books



- ***Supply Chain Network Economics***
  (New Dimensions in Networks)

  Anna Nagurney

  Available July 2006!



- ***Dynamic Networks And Evolutionary Variational Inequalities***
  (New Dimensions in Networks)

  Patrizia Daniele

Bellagio Research Team Residency March 2004

The Research Papers of this Presentation

- Nagurney, A. and Liu, Z. (2006), Dynamic Supply Chains, Transportation Network Equilibria, and Evolutionary Variational Inequalities
- Nagurney, A., Liu, Z., Cojocaru, M.-G., and Daniele, P. (2005), Dynamic Electric Power Supply Chains and Transportation Networks: An Evolutionary Variational Inequality Formulation (To appear in *Transportation Research E*.)
- Liu, Z. and Nagurney, A. (2005), Financial Networks with Intermediation and Transportation Network Equilibria: A Supernetwork Equivalence and Reinterpretation of the Equilibrium Conditions with Computations (To appear in *Computational Management Science*.)

Outline

- The static multitiered network equilibrium models
  - Supply chain networks with fixed demand
  - Electric power networks with fixed demand
  - Financial networks with intermediation
- The supernetwork equivalence of the supply chain networks, electric power networks and the financial networks with the transportation networks
- The dynamic network equilibrium models with time-varying demands
  - Evolutionary variational inequalities and projected dynamical systems.
  - The computation of the dynamic multi-tiered network equilibrium models with time-varying demands.

Some of the Related Literature

- Beckmann, M. J., McGuire, C. B., and Winsten, C. B. (1956), Studies in the Economics of Transportation. Yale University Press, New Haven, Connecticut.

- Nagurney, A (1999), Network Economics: A Variational Inequality Approach, Second and Revised Edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

- Nagurney, A., Dong, J., and Zhang, D. (2002), A Supply Chain Network Equilibrium Model, *Transportation Research E* 38, 281-303.

- Nagurney, A (2005), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, *Transportation Research E* 42: (2006) pp 293-316.

Some of the Related Literature (Cont'd)

- Nagurney, A. and Matsypura, D. (2004), A Supply Chain Network Perspective for Electric Power Generation, Supply, Transmission, and Consumption, *Proceedings of the International Conference on Computing, Communications and Control Technologies*, Austin, Texas, Volume VI: (2004) pp 127-134.

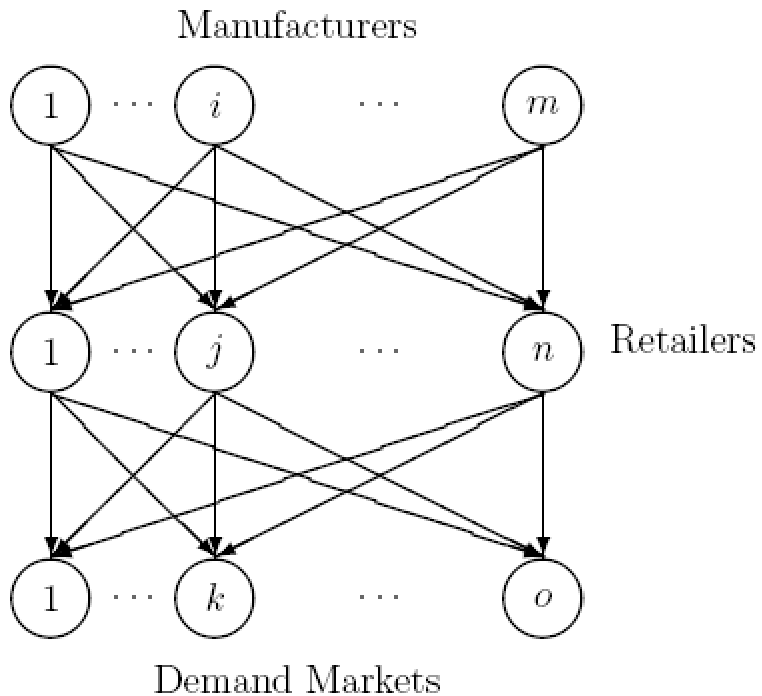
- Wu, K., Nagurney, A., Liu, Z. and Stranlund, J. (2006), Modeling Generator Power Plant Portfolios and Pollution Taxes in Electric Power Supply Chain Networks: A Transportation Network Equilibrium Transformation, *Transportation Research D* 11: (2006) pp 171-190.)

Some of the Related Literature (Cont'd)

- Nagurney A, Ke K. (2001), Financial networks with intermediation. *Quantitative Finance*, 1:309-317.
- Nagurney A, Ke K. (2003), Financial networks with electronic transactions: Modelling, analysis, and computations. *Quantitative Finance* 3:71-87.
- Nagurney A, Siokos S. (1997), Financial networks: Statics and Dynamics, Springer-Verlag, Heidelberg, Germany.

The Supply Chain Network Equilibrium Model with Fixed Demands

- Commodities with price-insensitive demand
  - gasoline, milk, etc.



The Behavior of Manufacturers and their Optimality Conditions

- Manufacturer's optimization problem

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho_{1ij}^{*} q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}),$$

- The Optimality conditions of the manufacturers

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^{*})}{\partial q_{ij}} - \rho_{1ij}^{*}\right] \times \left[q_{ij} - q_{ij}^{*}\right] \geq 0, \quad \forall Q^1 \in R_{+}^{mn}.$$

The Behavior of Retailers and their Optimality Conditions

- Retailer's optimization problem

$$\text{Maximize} \quad \sum_{k=1}^{o} \rho_{2j}^{*} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho_{1ij}^{*} q_{ij}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} q_{ij},$$

- The optimality conditions of the retailers

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1*})}{\partial q_{ij}} + \rho_{1ij}^{*} - \gamma_j^{*} \right] \times \left[q_{ij} - q_{ij}^{*} \right] + \sum_{j=1}^{n} \sum_{k=1}^{o} \left[-\rho_{2j}^{*} + \gamma_j^{*} \right] \times \left[q_{jk} - q_{jk}^{*} \right]$$

$$+ \sum_{j=1}^{n} \left[\sum_{i=1}^{m} q_{ij}^{*} - \sum_{k=1}^{o} q_{jk}^{*} \right] \times \left[\gamma_j - \gamma_j^{*} \right] \geq 0, \quad \forall (Q^1, Q^2, \gamma) \in R_{+}^{mn+no+n}.$$

The Equilibrium Conditions at the Demand Markets

- Conservation of flow equations must hold

$$d_k = \sum_{j=1}^{n} q_{jk}, \quad k = 1, \ldots, o,$$

- The vector (Q^{2*}, ρ_3^*) is an equilibrium vector if for each *j, k* pair:

$$\rho_{2j}^* + c_{jk}(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0. \end{cases}$$

Supply Chain Network Equilibrium (For Fixed Demands at the Markets)

- **Definition:** The equilibrium state of the supply chain network is one where the product flows between the tiers of the network coincide and the product flows satisfy the conservation of flow equations, the sum of the optimality conditions of the manufacturers and the retailers, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

- Determine $(Q^{1*}, Q^{2*}, \gamma^*) \in \mathcal{K}^1$ satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ij}}-\gamma_j^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*})+\gamma_j^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\gamma_j-\gamma_j^*\right]\geq 0,$$

$$\forall (Q^1, Q^2, \gamma) \in \mathcal{K}^1.$$

$\mathcal{K}^1$ is the feasible set where the non-negativity constraints and the conservation of flow equations hold.

Harvard University
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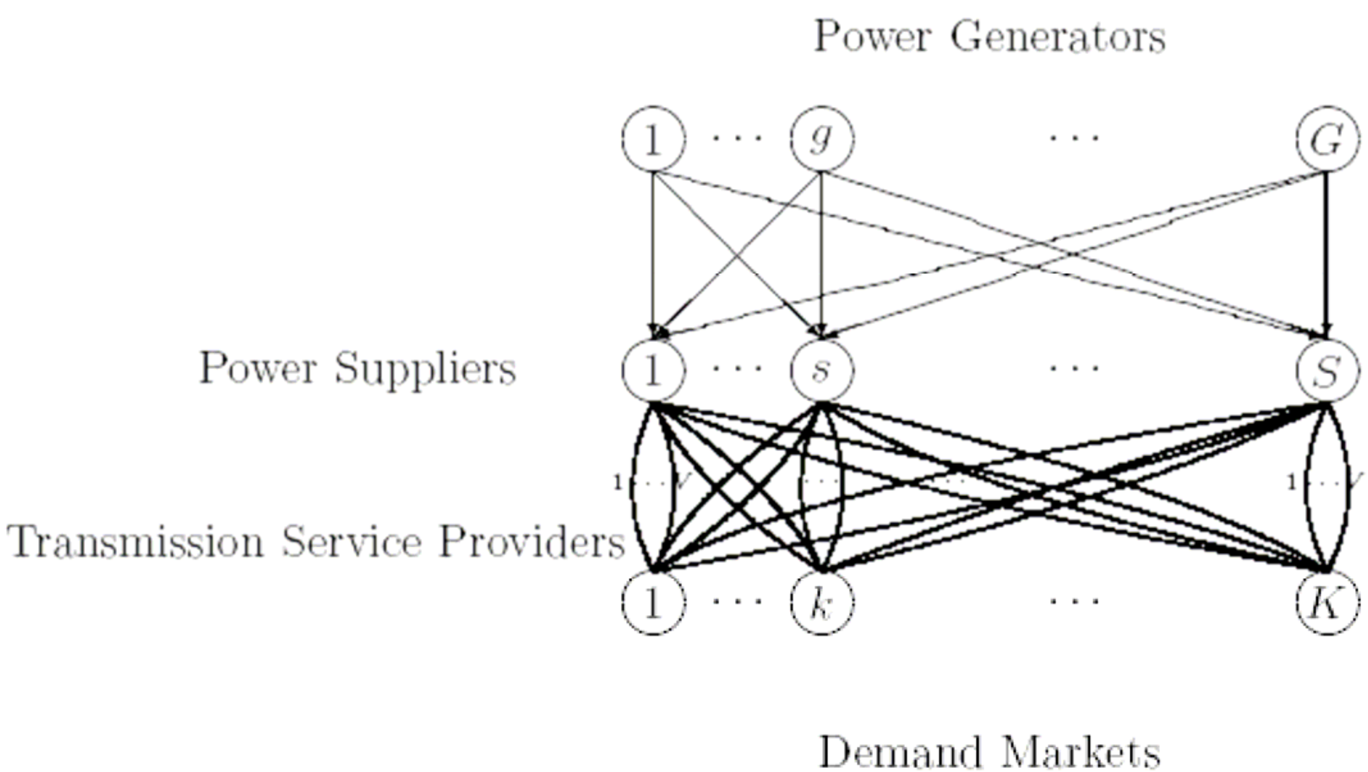
The Electric Power Supply Chain Network Equilibrium Model with Fixed Demands



The Behavior of Power Generator and Their Optimality Conditions

- Conservation of flow equations must hold for each power generator

$$\sum_{s=1}^{S} q_{gs} = q_g, \quad g = 1, \ldots, G. \qquad (1)$$

- Generator's optimization problem

$$\text{Maximize} \quad \sum_{s=1}^{S} \rho^*_{1gs} q_{gs} - f_g(Q^1) - \sum_{s=1}^{S} c_{gs}(q_{gs})$$

subject to:

$$q_{gs} \geq 0, \qquad s = 1, \ldots, S.$$

- The Optimality conditions of the generators

$$\sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(q^*_{gs})}{\partial q_{gs}} - \rho^*_{1gs} \right] \times [q_{gs} - q^*_{gs}] \geq 0, \quad \forall Q^1 \in R^{GS}_+.$$

The Behavior of Power Suppliers

- Supplier's optimization problem

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{v=1}^{V} \rho_{2sk}^{v*} q_{sk}^{v} - c_s(Q^1) - \sum_{g=1}^{G} \rho_{1gs}^{*} q_{gs} - \sum_{g=1}^{G} \hat{c}_{gs}(q_{gs}) - \sum_{k=1}^{K}\sum_{v=1}^{V} c_{sk}^{v}(q_{sk}^{v})$$

subject to:

$$\sum_{k=1}^{K}\sum_{v=1}^{V} q_{sk}^{v} = \sum_{g=1}^{G} q_{gs} \qquad (8)$$

$$q_{gs} \geq 0, \quad g = 1, \ldots, G,$$

$$q_{sk}^{v} \geq 0, \quad k = 1, \ldots, K; v = 1, \ldots, V.$$

- For notational convenience, we let

$$h_s \equiv \sum_{g=1}^{G} q_{gs}, \quad s = 1, \ldots, S. \qquad (13)$$

The Optimality Conditions of the Power Suppliers

- The optimality conditions of the suppliers

$$\sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{t=1}^{T} \left[\frac{\partial c_{sk}^t(q_{sk}^{t*})}{\partial q_{sk}^t} - \rho_{2sk}^{t*}\right] \times [q_{sk}^t - q_{sk}^{t*}]$$

$$+ \sum_{g=1}^{G}\sum_{s=1}^{S} \left[\frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} + \rho_{1gs}^*\right] \times [q_{gs} - q_{gs}^*] \geq 0, \qquad \forall (Q^1, Q^2, h) \in \mathcal{K}^3, \qquad (16)$$

where $\mathcal{K}^3 \equiv \{(h, Q^2, Q^1) | (h, Q^2, Q^1) \in R_+^{S(1+TK+G)} \text{and (8) and (13) hold}\}$.

The Equilibrium Conditions at the Demand Markets

- Conservation of flow equations must hold

$$d_k = \sum_{s=1}^{S} \sum_{t=1}^{T} q_{sk}^t, \quad k = 1, \ldots, K. \tag{17}$$

- The vector (Q^{2*}, ρ_3*) is an equilibrium vector if for each *s, k, v* combination:

$$\rho_{2sk}^{t*} + \hat{c}_{sk}^t(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{sk}^{t*} > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{sk}^{t*} = 0, \end{cases} \tag{18}$$

Electric Power Supply Chain Network Equilibrium
(For Fixed Demands at the Markets)

- **Definition**: The equilibrium state of the electric power supply chain network is one where the electric power flows between the tiers of the network coincide and the electric power flows satisfy the sum of the optimality conditions of the power generators and the suppliers, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

- Determine $(q^*, h^*, Q^{1*}, Q^{2*}) \in \mathcal{K}^5$ satisfying

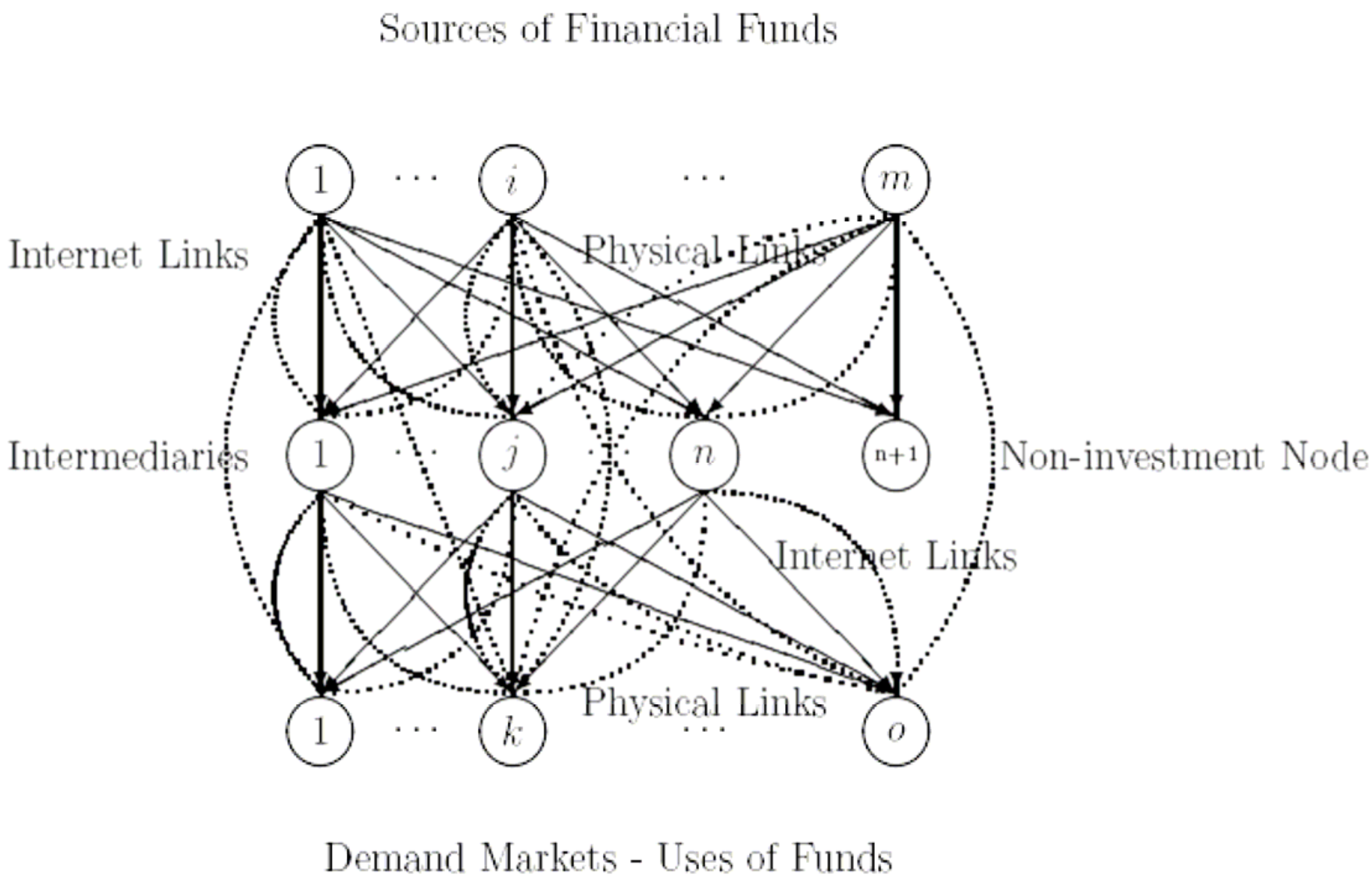
$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial c_{gs}(q_{gs}^*)}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} \right] \times [q_{gs} - q_{gs}^*]$$

$$+ \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[\frac{\partial c_{sk}^v(q_{sk}^{v*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}] \geq 0, \quad \forall (q, h, Q^1, Q^2) \in \mathcal{K}^5, \qquad (20)$$

$$\textit{where } \mathcal{K}^5 \equiv \{(q, h, Q^1, Q^2) | (q, h, Q^1, Q^2) \in R_+^{G+S+GS+VSK}$$

$$\textit{and } (1), (8), (13), \textit{ and } (17) \textit{ hold}\}.$$

The Financial Network Equilibrium Model with Intermediation



The Behavior of the Source Agents

- Source agent's optimization problem

$$\text{Maximize} \quad U^i(q_i) = \sum_{j=1}^{n}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n}\sum_{l=1}^{2} c_{ijl}(q_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}) - q_i^T V^i q_i$$

subject to:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \leq S^i$$

$$q_{ijl} \geq 0, \quad \forall j, l,$$

$$q_{ik} \geq 0, \quad \forall k,$$

$$q_{i(n+1)} \geq 0.$$

The Optimality Conditions of the Source Agents

- The optimality conditions of the source agents

determine $(Q^{1*}, Q^{2*}) \in \mathcal{K}^0$ such that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q_i^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}] \geq 0, \quad \forall (Q^1, Q^2) \in \mathcal{K}^0,$$

where $\mathcal{K}^0$ is the feasible set where the non-negativity constraints and the conservation of flow equations hold.

The Behavior of the Financial Intermediaries

- Financial intermediary's optimization problem

$$\text{Maximize} \quad U^j(q_j) = \sum_{k=1}^{o}\sum_{l=1}^{2}\rho^*_{2jkl}q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(q_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2}c_{jkl}(q_{jkl})$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl} - q_j^T V^j q_j$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl} \leq \sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl},$$

$$q_{ijl} \geq 0, \quad \forall i, l,$$

$$q_{jkl} \geq 0, \quad \forall k, l.$$

The Optimality Conditions of the Financial Intermediaries

- The optimality conditions of the financial intermediaries

determine $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{2mn+2no+n}$ satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{il}}^{j} \cdot q_j^* + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho_{1ijl}^* + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^*\right] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j} \cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} - \rho_{2jkl}^* + \gamma_j^*\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* - \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0, \quad \forall (Q^1, Q^3, \gamma) \in R_+^{2mn+2no+o}$$

The Equilibrium Conditions at the Demand Markets

- The conservation of flow equations must hold

$$d_k = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} + \sum_{i=1}^{m} q_{ik}, \quad k = 1, \dots, o.$$

- The equilibrium condition for the consumers at demand market k are as follows: for each intermediary j; j = 1, . . . ,n and mode of transaction l; l = 1, 2:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} = 0. \end{cases}$$

- In addition, for each source of funds i; i = 1, . . . ,m:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} = 0. \end{cases}$$

Financial Network Equilibrium

- **Definition:** The equilibrium state of the financial network with intermediation is one where the financial flows between tiers coincide and the financial flows and prices satisfy the sum of the optimality conditions of the source agents and the intermediaries, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

determine $(Q^{1*}, Q^{2*}, Q^{3*}, \gamma^*, d^*) \in \mathcal{K}^6$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}}\cdot q_i^* + 2V^j_{z_{il}}\cdot q_j^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma_j^*\right]\times\left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}}\cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right]\times\left[q_{ik} - q^*_{ik}\right]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) + \gamma_j^*\right]\times\left[q_{jkl} - q^*_{jkl}\right]$$

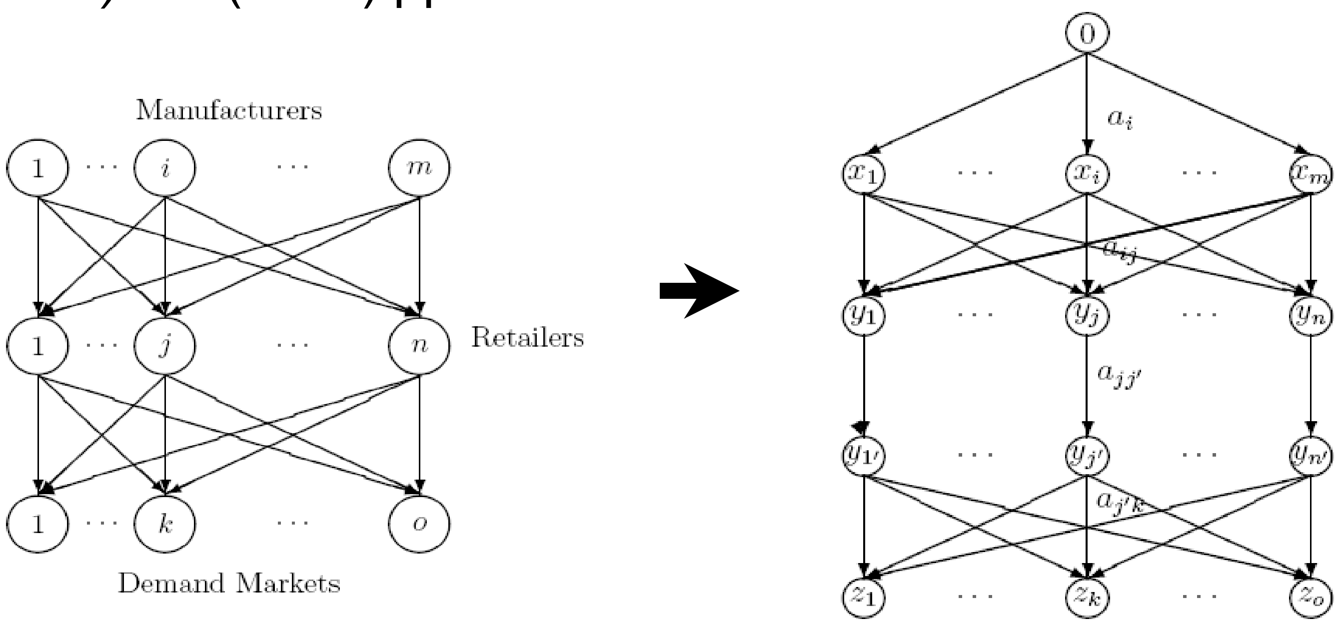
$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{n}\sum_{l=1}^{2} q^*_{jkl}\right]\times\left[\gamma_j - \gamma_j^*\right] - \sum_{k=1}^{o}\rho_{3k}(d^*)\times\left[d_k - d_k^*\right] \geq 0,$$

$$\forall (Q^1, Q^2, Q^3, \gamma, d) \in \mathcal{K}^6$$

$\mathcal{K}^6$ is the feasible set where the non-negativity constraints and the conservation of flow equations hold.

The Supernetwork Equivalence of Supply Chain Network Equilibrium and Transportation Network Equilibrium

- Nagurney, A. (2006), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, *Transportation Research E (2006)* 42: (2006) pp 293-316



Overview of the Transportation Network Equilibrium Model with Fixed Demands

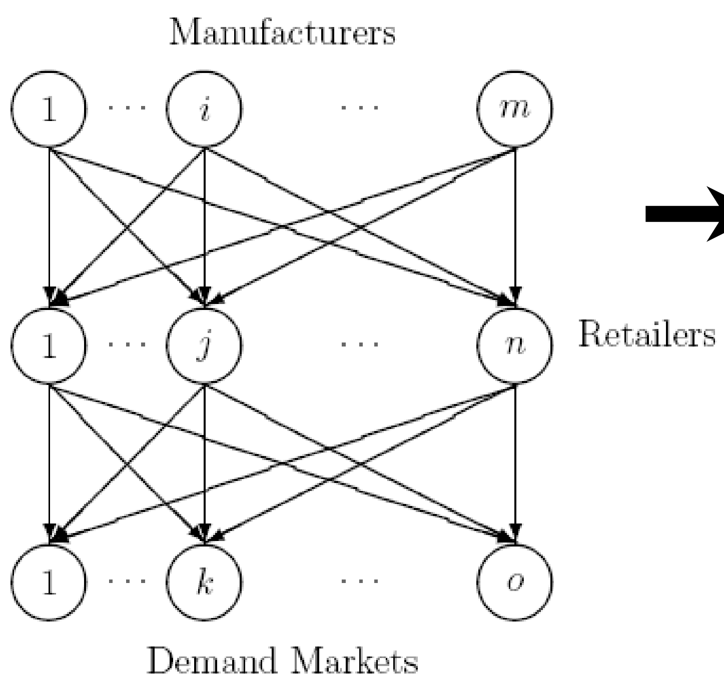
- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.
- Dafermos, S. (1980), Traffic equilibrium and variational inequalities. *Transportation Science* 14, 42-54.
- In equilibrium, the following conditions must hold for each O/D pair and each path.

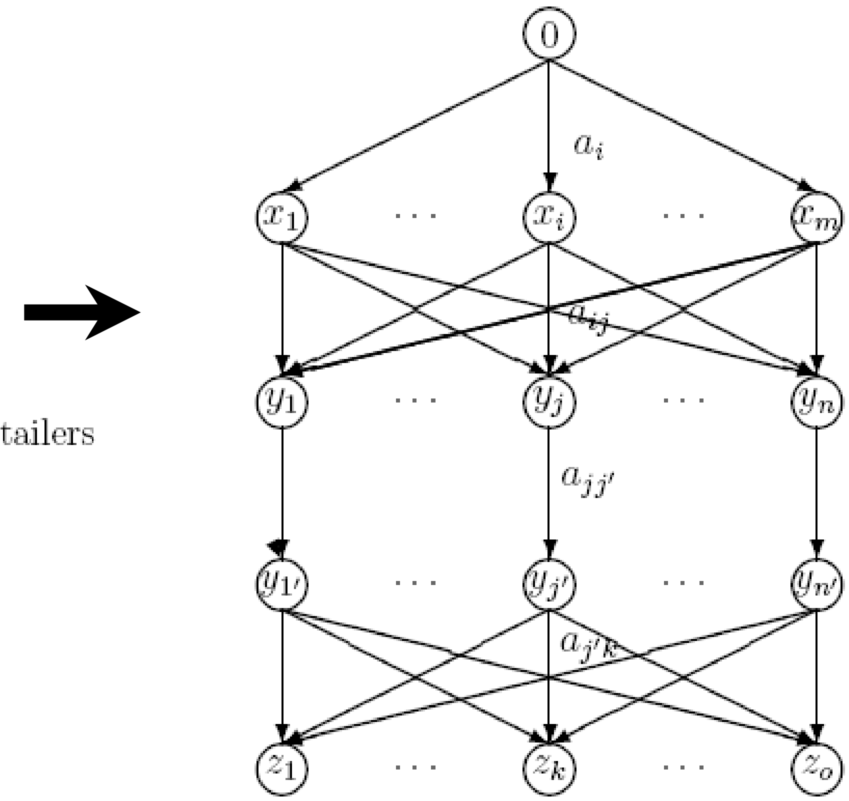
$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases}$$

- A path flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality:

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^* \right] \geq 0, \quad \forall x \in \mathcal{K}^7.$$

Transportation Network Equilibrium Reformulation of the Supply Chain Network Model with Fixed Demands

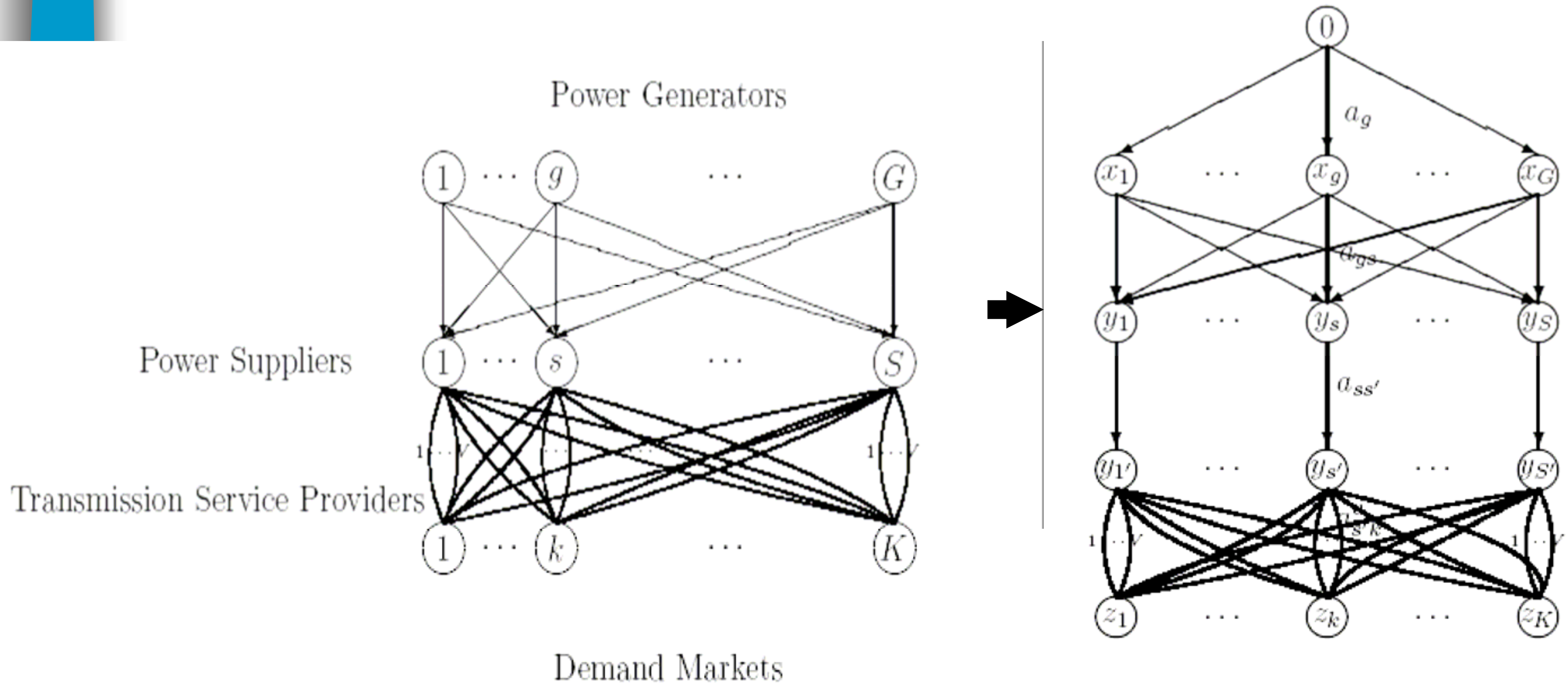




The Supernetwork Equivalence of the Electric Power Networks and the Transportation Networks

- The fifth chapter of the Beckmann, McGuire, and Winsten's classic book, "Studies in the Economics of Transportation" (1956), described some "unsolved problems" including a single commodity network equilibrium problem that the authors intuited could be generalized to capture electric power networks.
- We took up this challenge of establishing the relationship and application of transportation network equilibrium models to electric power networks.
- Nagurney, A and Liu, Z (2005), Transportation Network Equilibrium Reformulations of Electric Power Networks with Computations

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

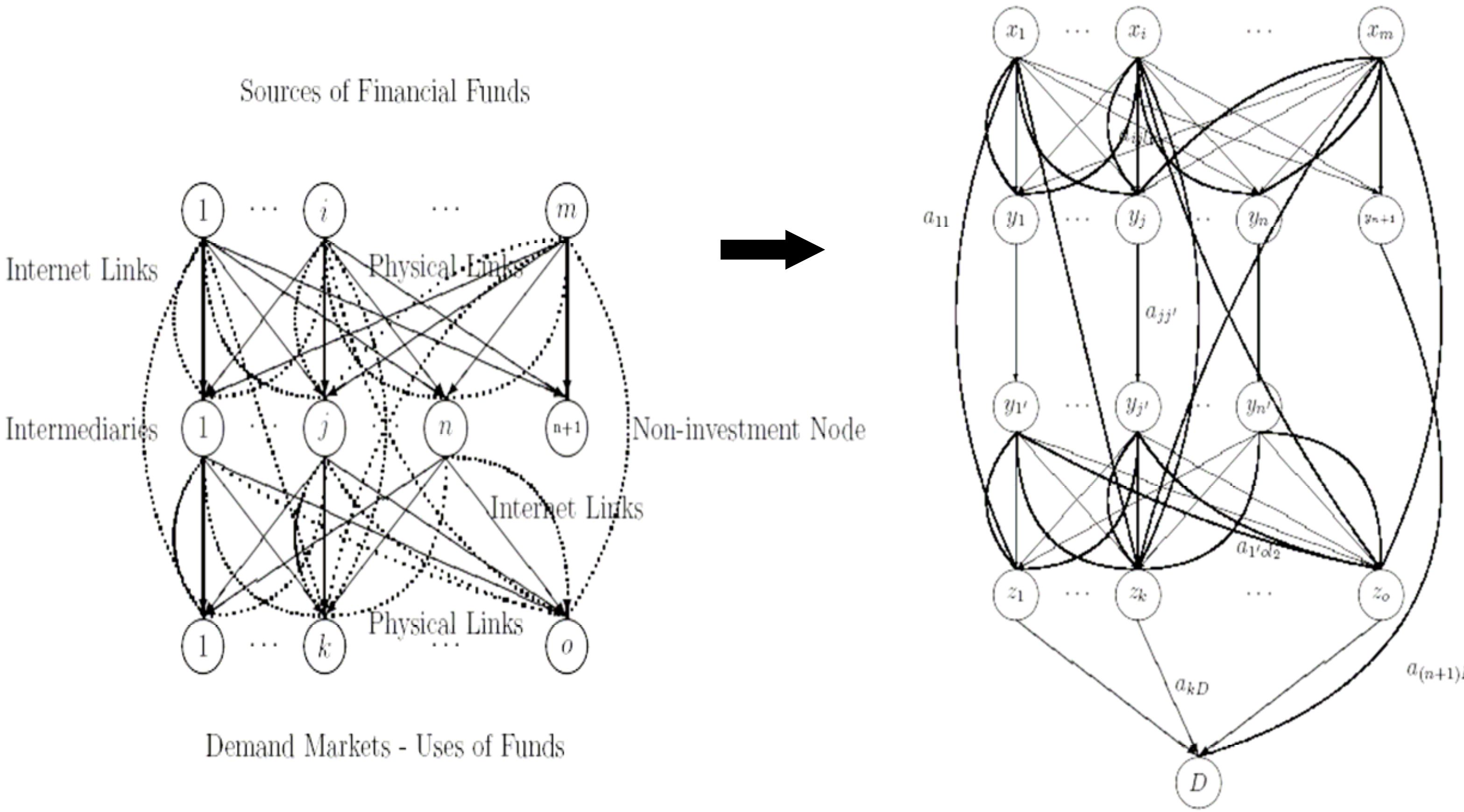


The Supernetwork Equivalence of the Financial Networks and the Transportation Networks

- Copeland in 1952 wondered whether money flows like water or electricity. We have showed that money and electricity flow like transportation flows!

- Liu, Z. and Nagurney, A. (2005), Financial Networks with Intermediation and Transportation Network Equilibria: A Supernetwork Equivalence and Reinterpretation of the Equilibrium Conditions with Computations
 (To appear in *Computational Management Science*.)

Transportation Network Equilibrium Reformulation of the Financial Network Model with Intermediation



Finite-Dimentional Variational Inequalities and Projected Dynamical Systems Literature

- Dupuis, P., Nagurney, A., (1993). Dynamical systems and variational inequalities. *Annals of Operations Research* 44, 9-42.
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- Nagurney, A., Zhang, D., (1997). Projected dynamical systems in the formulation, stability analysis, and computation of fixed demand traffic network equilibria. *Transportation Science* 31, 147-158.

More Finite-Dimensional Variational Inequalities Literature

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- Cojocaru, M.-G., Daniele, P., Nagurney, A. (2005c). Projected dynamical systems, evolutionary variational inequalities, applications, and a computational procedure. *Pareto Optimality, Game Theory and Equilibria*. A. Migdalas, P. M. Pardalos, and L. Pitsoulis, editors, Springer Verlag.
- Barbagallo, A., (2005). Regularity results for time-dependent variational and quasivariational inequalities and computational procedures. *To appear in Mathematical Models and Methods in Applied Sciences*.

More Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Daniele, P., Maugeri, A., Oettli, W., (1998). Variational inequalities and time-dependent traffic equilibria. *Comptes Rendue Academie des Science*, Paris 326, serie I, 10591062.

- Daniele, P., Maugeri, A., Oettli, W., (1999). Time-dependent traffic equilibria. *Journal of Optimization Theory and its Applications* 103, 543-555.

Finite-Dimensional Projected Dynamical Systems

- Finite-Dimensional Projected Dynamical Systems (PDSs) (Dupuis and Nagurney (1993))
  - PDS_t describes how the state of the network system approaches an equilibrium point on the curve of equilibria at time t.
  - For almost every moment 't' on the equilibria curve, there is a PDS_t associated with it.
  - A PDS_t is usually applied to study small scale time dynamics, i.e $[t, t+\tau]$

Finite-Dimensional Projected Dynamical Systems

Definition: $$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))).$$

In this formulation, $\mathcal{K}$ is a convex polyhedral set in R^n, $F : \mathcal{K} \rightarrow R^n$ is a Lipschitz continuous function with linear growth and $\Pi_{\mathcal{K}} : R \times \mathcal{K} \rightarrow R^n$ is the Gateaux directional derivative

$$\Pi_{\mathcal{K}}(x, -F(x)) = \lim_{\delta \rightarrow 0+} \frac{P_{\mathcal{K}}(x - \delta F(x)) - x}{\delta}$$

of the projection operator $P_{\mathcal{K}} : R^n \rightarrow \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$

Projected Dynamical Systems and Finite-Dimensional Variational Inequalities

Theorem

The equilibria of a PDS:

$$\frac{\partial x(t)}{\partial t} = \Pi_{\mathcal{K}}(x(t), -F(x(t)))$$

that is, $x^* \in \mathcal{K}$ *such that*

$$\Pi_{\mathcal{K}}(x^*, -F(x^*)) = 0$$

are solutions to the $VI(F, K)$*: find* $x^* \in \mathcal{K}$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{K}$$

and vice versa.

Infinite-Dimensional Projected Dynamical Systems

Definition: $$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))), \quad x(t,0) \in \hat{\mathcal{K}}, \qquad (54)$$

where $$\Pi_{\hat{\mathcal{K}}}(y, -F(y)) = \lim_{\delta \to 0+} \frac{P_{\hat{\mathcal{K}}}((y - \delta F(y)) - y)}{\delta}, \quad \forall y \in \hat{\mathcal{K}}$$

with the projection operator $P_{\hat{\mathcal{K}}} : H \to \hat{\mathcal{K}}$ given by

$$\|P_{\hat{\mathcal{K}}}(z) - z\| = \inf_{y \in \hat{\mathcal{K}}} \|y - z\|,$$

The feasible set $\hat{\mathcal{K}}$ is defined as follows

$$\hat{\mathcal{K}} = \bigcup_{t \in [0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$$

Evolutionary Variational Inequalities

- Evolutionary Variational Inequalities (EVIs)
  - EVI provides a curve of equilibria of the network system over a finite time interval [0,T]
  - An EVI is usually used to model large scale time, i.e, [0, T]
  - EVIs have been applied to time-dependent equilibrium problems in transportation, and in economics and finance.

Evolutionary Variational Inequalities

Define $\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$

EVI:

determine $x \in \hat{\mathcal{K}} : \langle\langle F(x), z - x \rangle\rangle \geq 0, \quad \forall z \in \hat{\mathcal{K}}.$ (53)

where $\hat{\mathcal{K}} = \bigcup_{t\in[0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$

$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$

Projected Dynamical Systems and Evolutionary Variational Inequalities

- Cojocaru, Daniele, and Nagurney (2005b) showed the following:

Theorem

Assume that $\hat{\mathcal{K}} \subseteq H$ is non-empty, closed and convex and $F : \hat{\mathcal{K}} \to H$ is a pseudo-monotone Lipschitz continuous vector field, where H is a Hilbert space. Then the solutions of EVI (53) are the same as the critical points of the projected differential equation (54) that is, they are the functions $x \in \hat{\mathcal{K}}$ such that

$$\Pi_{\hat{\mathcal{K}}}(x(t), -F(x(t))) = 0,$$

and vice versa.

Projected Dynamical Systems and Evolutionary Variational Inequalities

The solutions to the evolutionary variational inequality:

$$\text{determine} \quad x \in \hat{\mathcal{K}}: \quad \int_0^T \langle F(x(t)), z(t) - x(t) \rangle dt \geq 0, \quad \forall z \in \hat{\mathcal{K}},$$

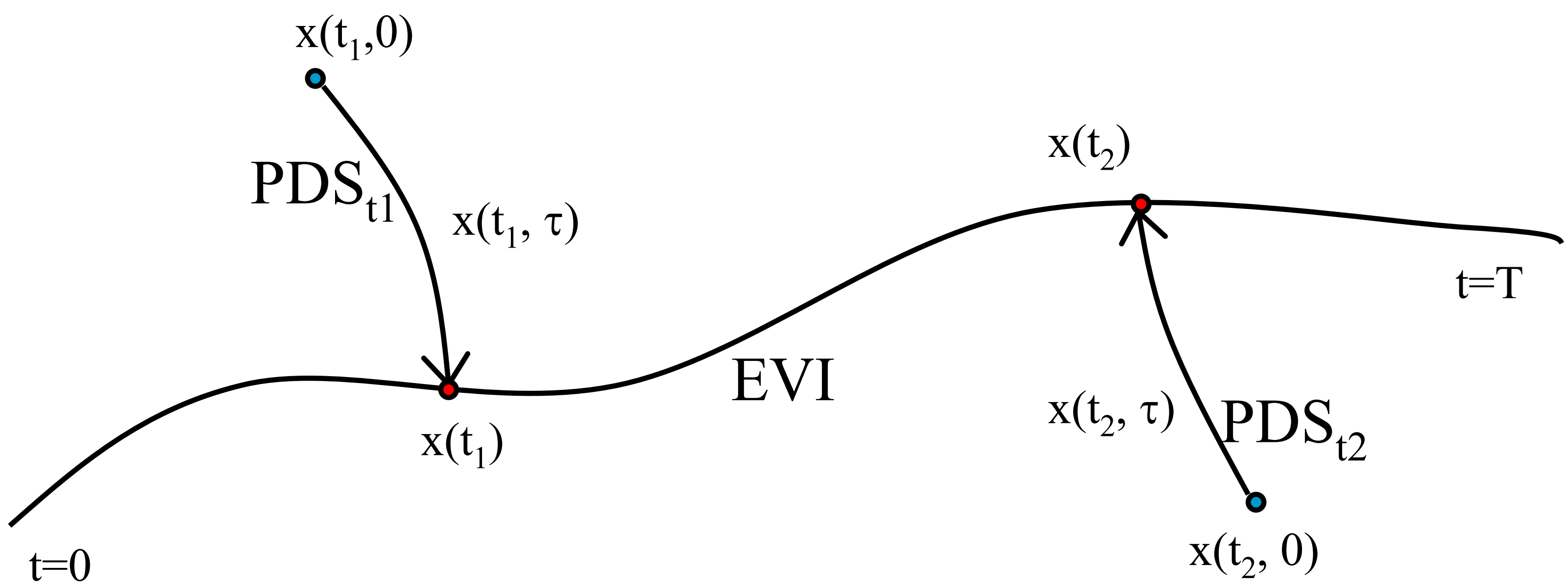
are the same as the critical points of the equation:

$$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))),$$

that is, the points such that

$$\Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))) \equiv 0 \text{ a.e. in } [0,T],$$

A Pictorial of EVIs and PDSs



The EVI Formulation of the Transportation Network Model with Time-Varying Demands

Define $\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$

EVI Formulation:

$$\text{determine} \quad x \in \hat{\mathcal{K}} : \langle\langle C(x), z - x \rangle\rangle \geq 0, \quad \forall z \in \hat{\mathcal{K}}, \tag{61}$$

where C is the vector of path costs.

Feasible set

$$\hat{\mathcal{K}} = \left\{ x \in L^2([0,T], R^Q) : 0 \leq x(t) \leq \mu \text{ a.e. in } [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0,T] \right\}.$$

The Numerical Solution of Evolutionary Variational Inequalities
(Cojocaru, Daniele, and Nagurney (2005 a, b, c))

- The vector field *F* satisfies the requirement in the preceding Theorem.
- We first discretize time horizon T. (Barbagallo, A., (2005))
- At each fixed time point, we solve the associated finite dimensional projected dynamical system PDS_t
- We use the Euler method to solve the finite dimensional projected dynamical system PDS_t.

The Euler Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$ and set $T = 0$. T is an iteration counter which may also be interpreted as a time period.

Step 1: Computation

Compute X^{T+1} by solving the variational inequality problem:

$$X^{T+1} = P_{\mathcal{K}}(X^T - \alpha_T F(X^T)),$$

where $\{\alpha_T\}$ is a sequence of positive scalers satisfying: $\sum_{T=0}^{\infty} \alpha_T = \infty$, $\alpha_T \to 0$ as $T \to \infty$. and $P_{\mathcal{K}}$ is the projection of X on the set $\mathcal{K}$ defined as:

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \| X - z \| .$$

Step 2: Convergence Verification

If $\| X^{T+1} - X^T \| \leq \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; else, set $T = T + 1$, and go to Step 1.

The EVI Formulation of the Supply Chain Network Model with Time-Varying Demands

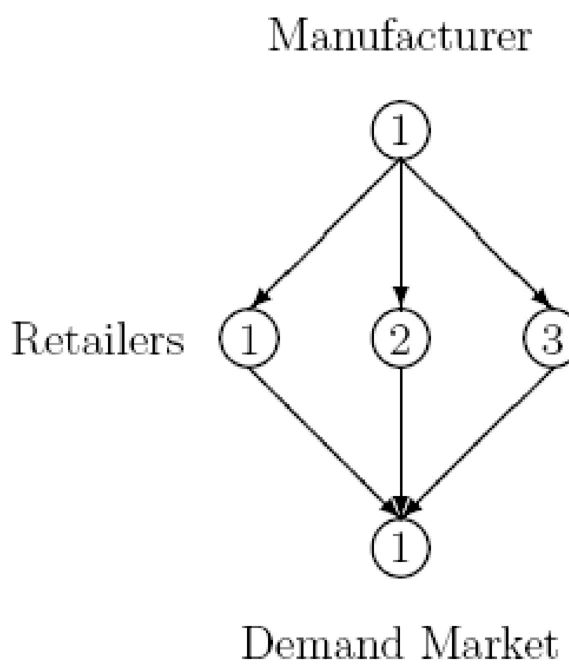
- We know that the supply chain network equilibrium problem with fixed demands can be reformulated as a fixed demand transportation network equilibrium problem in path flows over the equivalent transportation network.
- Evolutionary variational inequality (61) provides us with a dynamic version of the supply chain network problem in which the demands vary over time.

Solving the Supply Chain Network Model with Time-Varying Demands

- First, construct the equivalent transportation network equilibrium model
- Solve the transportation network equilibrium model with time-varying demands
- Convert the solution of the transportation network into the time-dependent supply chain network equilibrium model

Dynamic Supply Chain Network Examples with Computations

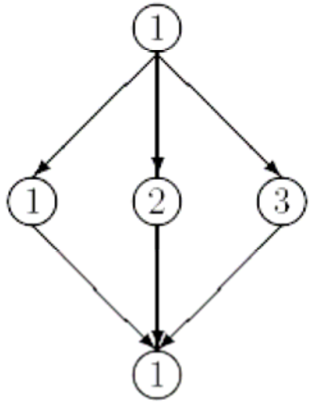
- Example 1



Numerical Example 1

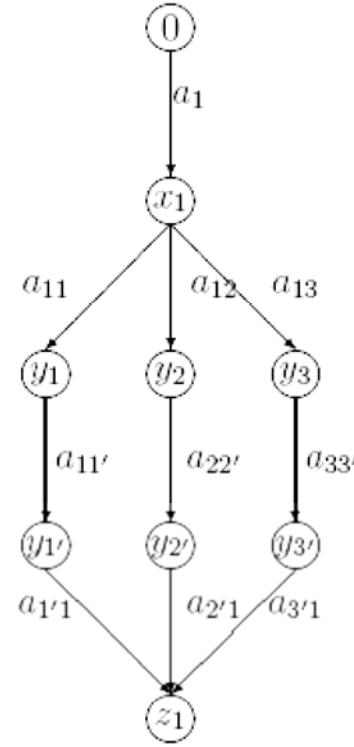
Manufacturer

Retailers



Demand Market

$\Longrightarrow$



The Equivalent Transportation Network

Numerical Example 1

- Production cost functions

$$f_1(q_1(t)) = 2.5(q_1(t))^2 + 2q_1(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5(q_{12}(t))^2 + 2.5q_{12}(t),$$

$$c_{13}(q_{13}(t)) = .5(q_{13}(t))^2 + 1.5q_{13}(t).$$

Numerical Example 1

- Handling cost functions of the retailers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

- Unit transaction cost between the retailers and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{21}(Q^2(t)) = q^1_{21}(t) + 5, \quad \hat{c}^1_{31}(Q^2(t)) = q^1_{31}(t) + 10.$$

Numerical Example 1

- Three paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

- The time-varying demand function

$$d_{w_1}(t) = d_1(t) = 41 + 10t.$$

The Solution of Numerical Example 1

- Explicit Solution
  - Path flows

$$x^*_{p_1}(t) = 3.33t + 14.78,$$

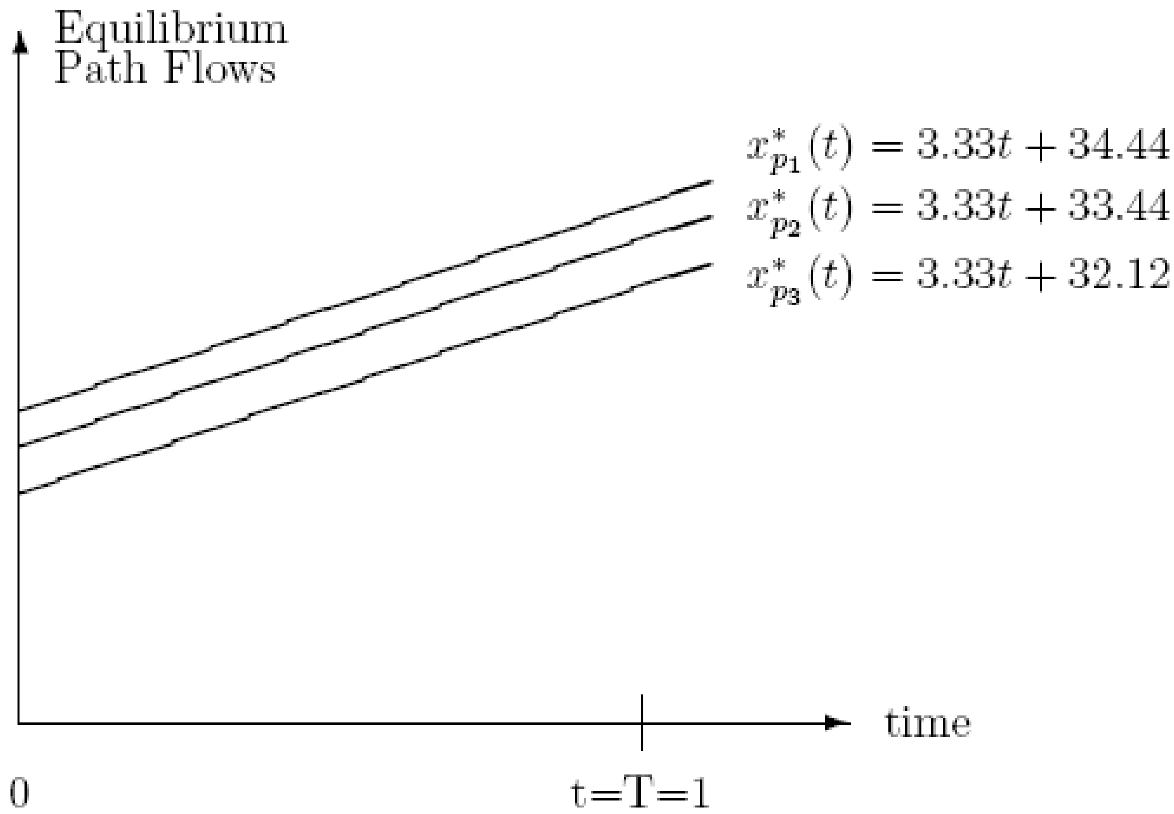
$$x^*_{p_2}(t) = 3.33t + 13.78,$$

$$x^*_{p_3}(t) = 3.33t + 12.45,$$

  - Travel disutility

$$\lambda^*_{w_1}(t) = 60t + 255.83, \quad \text{for} \quad t \in [0, T].$$

Time-Dependent Equilibrium Path Flows for Numerical Example 1

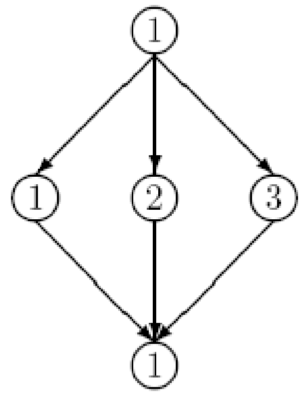


The Solution of Numerical Example 1

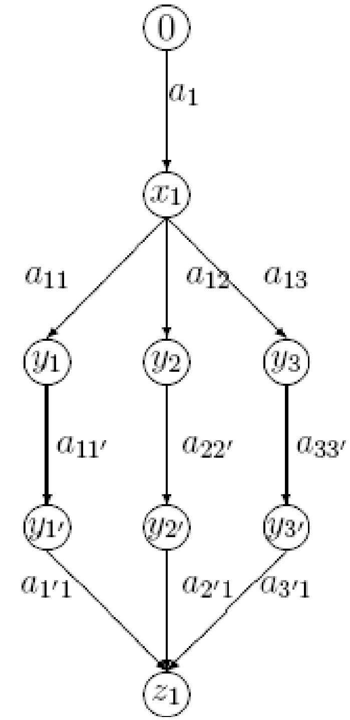
t=0

Manufacturer

Retailers



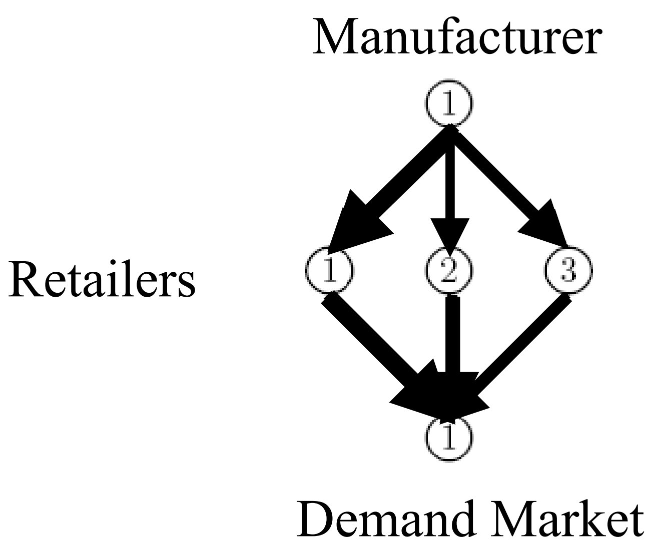
Demand Market

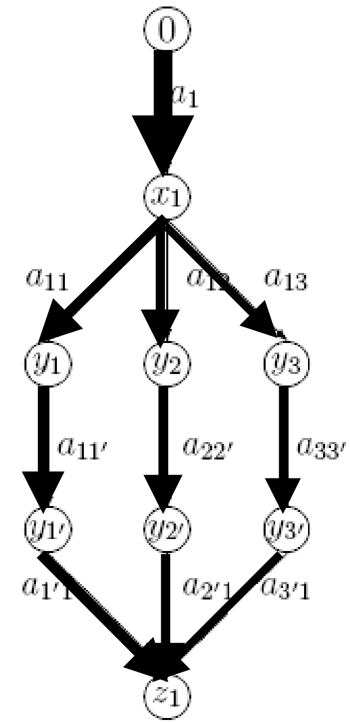


The Equivalent Transportation Network

The Solution of Numerical Example 1

t=1/2





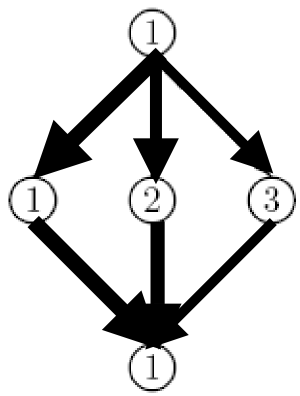
The Equivalent Transportation Network

The Solution of Numerical Example 1

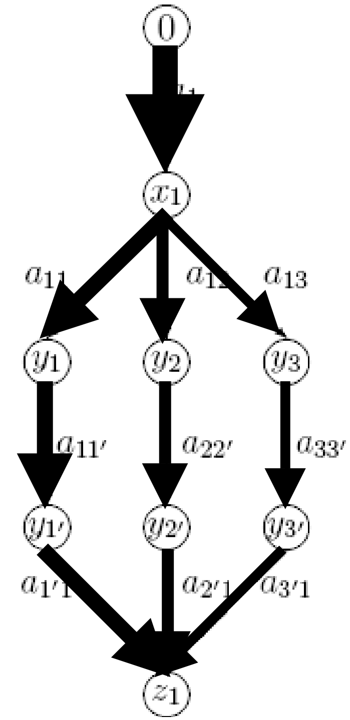
t=1

Manufacturer

Retailers



Demand Market



The Equivalent Transportation Network

Numerical Example 2

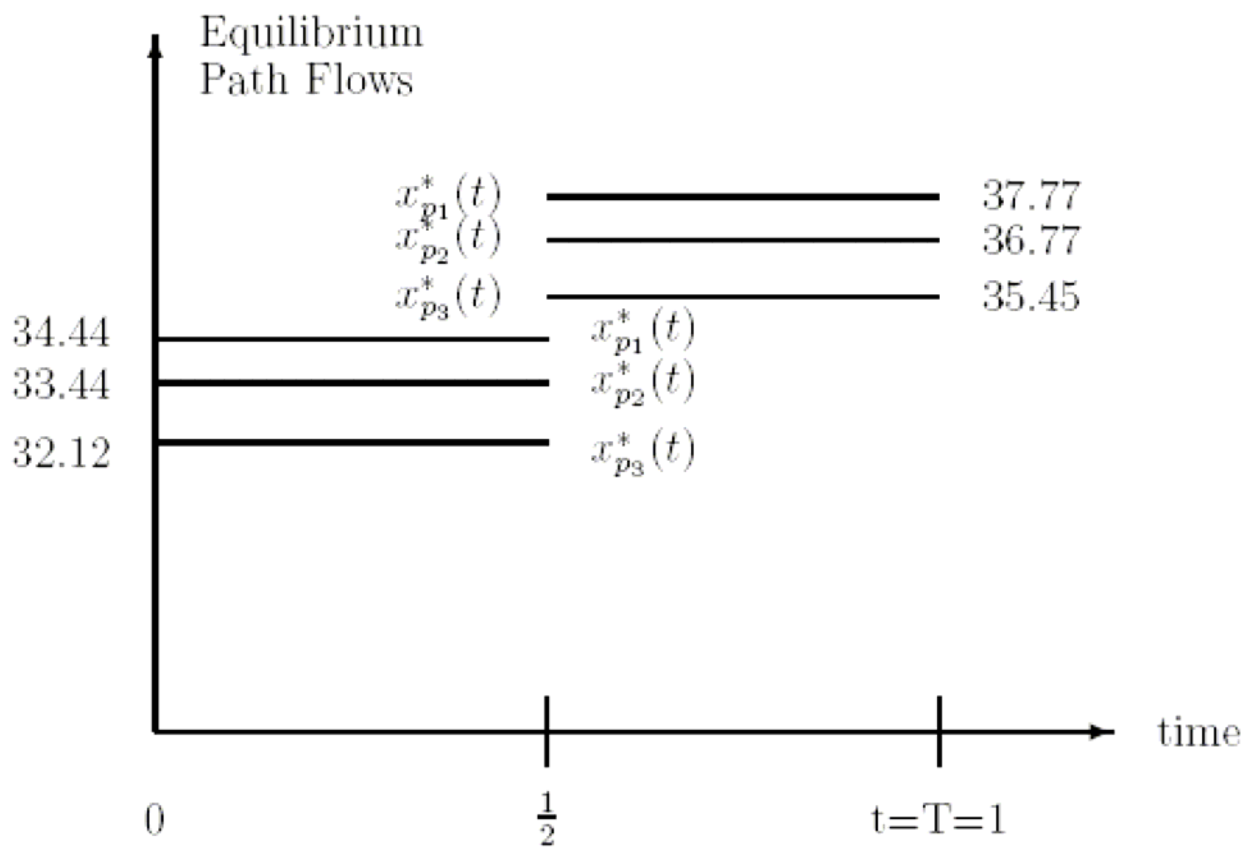
- The network structure and the cost functions are the same as the first example.
- The demand function is the step function:

$$d_1(t) = \begin{cases} 100, & \text{if} \quad 0 < t \le t_1 = \frac{1}{2}, \\ 110, & \text{if} \quad t_1 < t \le t_2 = T = 1. \end{cases}$$

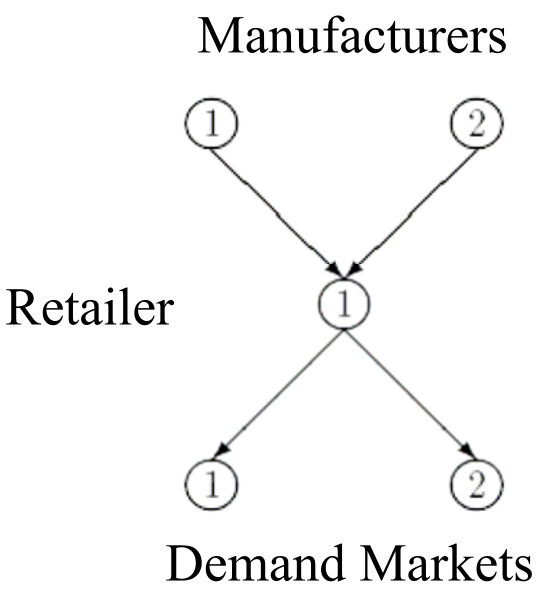
- The explicit solution:

$$x^*(t) = (x^*_{p_1}(t), x^*_{p_2}(t), x^*_{p_3}(t)) = \begin{cases} (34.44, 33.44, 32.12), & \text{if} \quad 0 < t \le t_1 = \frac{1}{2}, \\ (37.77, 36.77, 35.45), & \text{if} \quad t_1 < t \le t_2 = 1 = T. \end{cases}$$

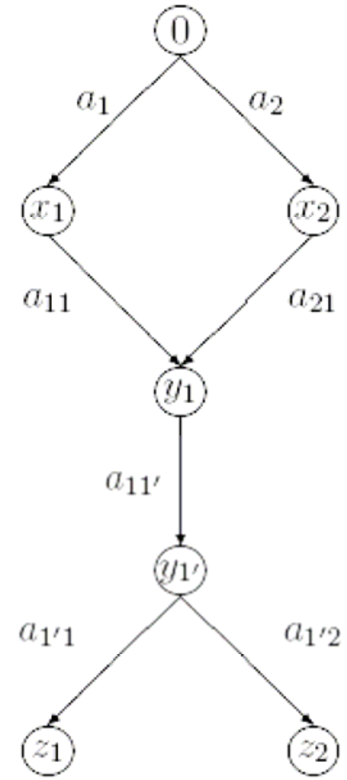
Time-Dependent Equilibrium Path Flows for Numerical Example 2



Numerical Example 3



$\Longrightarrow$



The Equivalent Transportation Network

Numerical Example 3

- Production cost functions

$$f_1(q(t)) = 2.5(q_1(t))^2 + q_1(t)q_2(t) + 2q_1(t), \quad f_2(q(t)) = 2.5(q_2(t))^2 + q_2(t)q_1(t) + 2q_2(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{21}(q_{21}(t)) = .5(q_{21}(t))^2 + 1.5q_{21}(t).$$

Numerical Example 3

- Handling cost function of the retailer

$$c_1(Q^1(t)) = .5(q_{11}(t))^2.$$

- Unit transaction costs between the retailer and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{12}(Q^2(t)) = q^1_{12}(t) + 1,$$

Numerical Example 3

- Four paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}),\ p_2 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$

$$p_3 = (a_1, a_{11}, a_{11'}, a_{1'2}),\ p_4 = (a_2, a_{21}, a_{11'}, a_{1'2})$$

- The time-varying demand functions

$$d_{w_1}(t) = d_1(t) = 100 + 5t, \quad d_{w_2}(t) = d_2(t) = 80 + 4t.$$

The Solution of Numerical Example 3

- Numerical Solution
  - t=0

    $$x^*_{p_1} = 49.90, \quad x^*_{p_2} = 50.10, \quad x^*_{p_3} = 39.90, \quad x^*_{p_4} = 40.10.$$

    $$\lambda^*_{w_1}(t_0) = 915.50 \quad \lambda^*_{w_2}(t_0) = 895.50.$$

  - t=1/2

    $$x^*_{p_1} = 51.15, \quad x^*_{p_2} = 51.35, \quad x^*_{p_3} = 40.90, \quad x^*_{p_4} = 41.10.$$

    $$\lambda^*_{w_1}(t_1) = 938.25. \quad \lambda^*_{w_2}(t_1) = 917.75$$

  - t=1

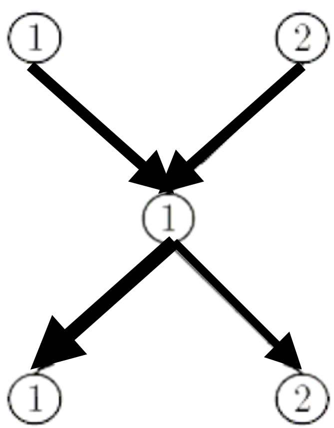
    $$x^*_{p_1} = 52.40, \quad x^*_{p_2} = 52.60, \quad x^*_{p_3} = 41.90, \quad x^*_{p_4} = 42.10.$$

    $$\lambda^*_{w_1}(T) = 961.00, \quad \lambda^*_{w_2}(T) = 940.00.$$

The Solution of Numerical Example 3

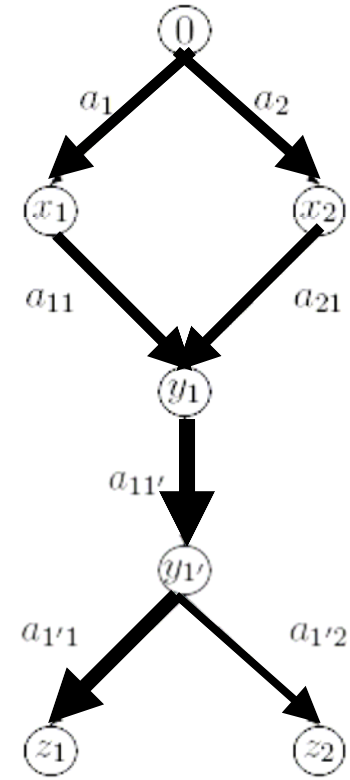
t=0

Manufacturers



Retailer

Demand Markets

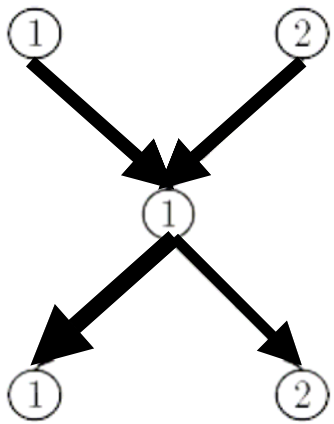


The Equivalent Transportation Network

The Solution of Numerical Example 3

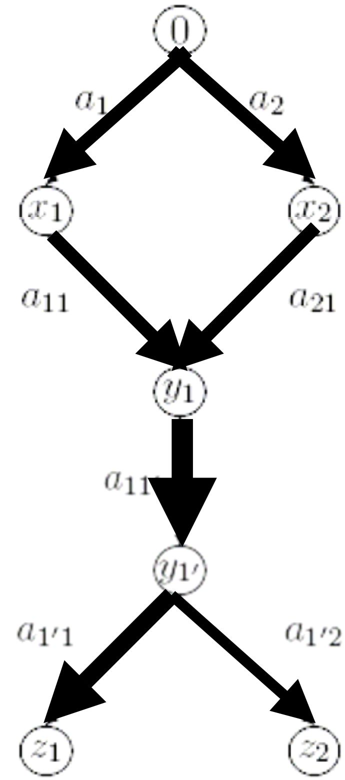
t=1/2

Manufacturers



Retailer

Demand Markets

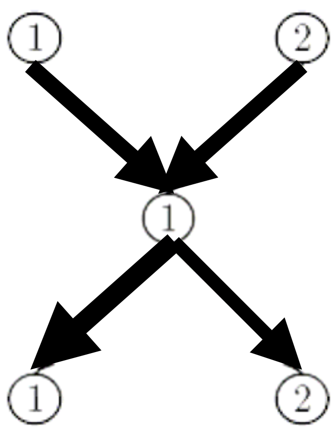


The Equivalent Transportation Network

The Solution of Numerical Example 3

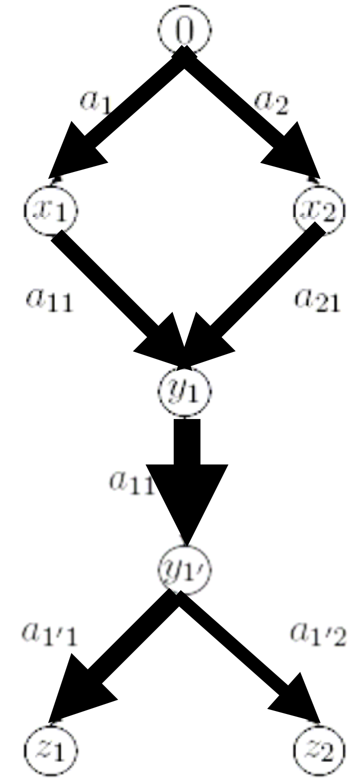
t=1

Manufacturers



Retailer

Demand Markets



The Equivalent Transportation Network

Conclusions

- We established the supernetwork equivalence of the supply chain networks, the electric power networks and the financial networks with transportation networks with fixed demands.
- This identification provided a new interpretation of equilibrium in multi-tiered networks in terms of path flows.
- We utilized this isomorphism in the computation of the supply chain network equilibrium and the electric power network equilibrium with time-varying demands.
- We are also investigating the dynamic financial network with time-varying sources of funds.

Thank You!

For more information, please see:

The Virtual Center for Supernetworks

http://supernet.som.umass.edu