Evolutionary Variational Inequalities and the Internet

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20th International Symposium on Mathematical Programming
Chicago, August 23-28, 2009

Professors Parkes, Daniele, and Nagurney at Harvard.

Professors Nagurney and Daniele acknowledge support from the Radcliffe Institute for Advanced Study at Harvard University while Nagurney was a 2005-2006 Science Fellow and Daniele was a Visiting Scholar March – May, 2006.

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Static Variational Inequalities – Classical References

- G. Stampacchia (1964): first theorem of existence and uniqueness of the solution of variational inequalities;
- G. Fichera (1964, 1972): another founder of the variational inequality theory;
- P. Hartman - G. Stampacchia (1966): study of partial differential equations;
- J.L. Lions - G. Stampacchia (1967): second proof of the same theorem and introduction of evolutionary variational inequalities;
- H. Brezis (1968): study of evolutionary variational inequalities.

Problem in $\mathbb{R}^n$:

$$\mathbb{K} \subseteq \mathbb{R}^n, \quad F : \mathbb{K} \to \mathbb{R}^n,$$

$$\text{Find } x \in \mathbb{K} \text{ such that } \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathbb{K},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

Problem in a Banach space E:

$$\mathbb{K} \subseteq E, \quad F : \mathbb{K} \to E^*,$$

$$\text{Find } x \in \mathbb{K} \text{ such that } \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathbb{K},$$

where $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is the duality pairing.

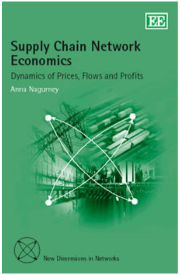
M.J. Smith (1979) and S. Dafermos (1980): formulation of a traffic network equilibrium problem in terms of a finite-dimensional variational inequality

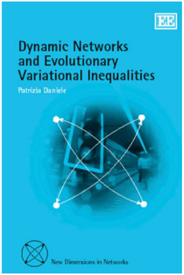
Evolutionary Variational Inequalties

- P. Daniele, A. Maugeri, W. Oettli (1998 - 1999): traffic network equilibrium problem with time-dependent capacity constraints and demands;
- F. Raciti (2001): time-dependent traffic networks with delay;
- L. Scrimali (2004): quasi-variational inequalities in transportation networks;
- P. Daniele (2003): spatial price equilibrium problem with price and bounds depending on time;
- P. Daniele (2003): time-dependent financial network problem;
- P. Daniele, S. Giuffre', S. Pia (2005): financial network problem with policy interventions;

- M.G. Cojocaru, P. Daniele, A. Nagurney (2005 - 2006 - 2007): connection between time-dependent variational inequalities projected dynamical systems;
- A. Nagurney, Z. Liu, M.G. Cojocaru, P. Daniele (2007): dynamic electric power supply chains;
- A. Nagurney, D. Parkes, P. Daniele (2007): the Internet, evolutionary variational inequalities and the time-dependent Braess paradox;
- P. Daniele, S. Giuffre', G. Idone, A. Maugeri (2007): infinite-dimensional duality theory;

- A. Nagurney, Z. Liu (2007): supply chain networks with time-varying demands;
- M.B. Donato, M. Milasi, C. Vitanza (2008): Walrasian price equilibrium problem with time-dependent data;
- A. Nagurney, Q. Qiang (2008): Efficiency measure for dynamic networks with application to the Internat and vulnerability analysis.





Two books with results on and applications of Evolutionary Variational Inequalities.

Time-dependent Variational Inequality:

$$\mathbb{K} \subseteq \mathcal{L} = L^p([0,T],\mathbb{R}^n), \quad F : \mathbb{K} \to \mathcal{L}^*,$$

$$\text{Find } x \in \mathbb{K} \text{ such that } \ll F(x), y - x \gg \geq 0 \quad \forall y \in \mathbb{K},$$

$$\text{where } \ll G, H \gg = \int_0^T \langle G(t), H(t) \rangle \, dt, \; G \in \mathcal{L}^*, \; H \in \mathcal{L}.$$

Roughgarden in his (2005) book *Selfish Routing and the Price of Anarchy* states that:

> *A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically ... The assumption of a static model is therefore particularly suspect in such networks.*

The Internet as a Dynamic Network

- $\mathcal{G} = [N, L]$: network;
- W : set of origin/destination (O/D) pairs of nodes;
- P_w : set of routes joining w;
- P : set of all routes connecting all the O/D pairs;
- $d_w^k(t)$: demand between w at time t by job class k;
- $x_r^k(t)$: flow on route r at time t of class k;
- $f_a^k(t)$: flow on link a of class k at time t;
- $C_r^k(t)$: cost on route r at time t of class k;
- $c_a^k(f(t))$: cost on a link a of class k at time t.

Conservation of flow equations:

$$d_w^k(t) = \sum_{r \in P_w} x_r^k(t), \quad \forall w \in W, \forall k$$

$$f_a^k(t) = \sum_{r \in P} x_r^k(t)\delta_{ar}, \quad \forall a \in L$$

Capacity Constraints:

$$0 \leq x_r^k(t) \leq \mu_r^k(t), \quad \forall r \in P, \forall k$$

Connection among costs:

$$C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t))\delta_{ar}, \quad \forall r \in P, \forall k$$

Feasible Set:

$$\mathcal{K} = \Big\{ x \in L^2([0, T], R^{Kn_P}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T];$$

$$\sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k \text{ a.e. in } [0, T] \Big\}$$

Additional Assumptions:

- $0 \leq d(t) \leq \Phi\mu(t)$, a.e. on $[0, T] \Rightarrow \mathcal{K} \neq \emptyset$
- All classes can use all the routes

Canonical Bilinear Form on $\mathcal{L}^* \times \mathcal{L}$:

$$\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}$$

Definition (Dynamic Multiclass Network Equilibrium)

$x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium if $\forall w \in W$, $\forall r \in P_w$, $\forall k$; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if} \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if} \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if} \quad x_r^{k*}(t) = 0. \end{cases}$$

Theorem (Variational Formulation)

$x^* \in \mathcal{K}$ is an equilibrium flow if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}$$

Preliminary Definitions

$C : \mathcal{K} \to \mathcal{L}^*$ is said to be

1. *pseudomonotone* if and only if, $\forall x, y \in \mathcal{K}$

$$\langle\langle C(x), y - x \rangle\rangle \geq 0 \Rightarrow \langle\langle C(y), x - y \rangle\rangle \leq 0;$$

2. *Fan-hemicontinuous* if and only if, $\forall y \in \mathcal{K}$, $\xi \mapsto \langle\langle C(\xi), y - \xi \rangle\rangle$ is upper semicontinuous on $\mathcal{K}$;
3. *hemicontinuous along line segments* if and only if, $\forall x, y \in \mathcal{K}$, $\xi \mapsto \langle\langle C(\xi), y - x \rangle\rangle$ is upper semicontinuous on the line segment $[x, y]$.

Theorem (Existence)

If C satisfies any of the following conditions:

1. *C is Fan-hemicontinuous with respect to the strong topology on $\mathcal{K}$, and $\exists A \subseteq \mathcal{K}$ nonempty, compact, and $\exists B \subseteq \mathcal{K}$ compact such that, $\forall y \in \mathcal{K} \setminus A$, $\exists x \in B$ with $\langle\langle C(x), y - x\rangle\rangle < 0$;*
2. *C is Fan-hemicontinuous with respect to the weak topology on $\mathcal{K}$;*
3. *C is pseudomonotone and hemicontinuous along line segments,*

then the EVI problem admits a solution in $\mathcal{K}$.

A Multiclass Numerical Example

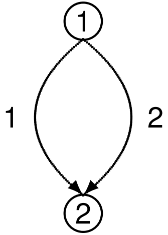


Figure: Network Structure of the Multiclass Numerical Example

Time horizon:

$$[0, 10]$$

Costs for Class 1 and Class 2:

$$C^1_{r_1}(x(t)) = 2x^1_{r_1}(t)+x^2_{r_1}(t)+5, \quad C^1_{r_2}(x(t)) = 2x^2_{r_2}(t)+2x^1_{r_2}(t)+10,$$

$$C^2_{r_1}(x(t)) = x^2_{r_1}(t) + x^1_{r_1}(t) + 5, \quad C^2_{r_2}(x(t)) = x^1_{r_2}(t) + 2x^2_{r_2}(t) + 5.$$

Demands for the O/D pair:

$$d^1_w(t) = 10 - t, \quad d^2_w(t) = t$$

Upper bounds:

$$\mu^1_{r_1} = \mu^1_{r_2} = \mu^1_{r_1} = \mu^2_{r_2} = \infty$$

Equilibrium Multiclass Route Flows at time t					
Flow	$t=0$	$t=2.5$	$t=5$	$t=7.5$	$t=10$
$x_{r_1}^{1*}(t)$	6.25	6.25	5.00	2.50	0.00
$x_{r_2}^{1*}(t)$	3.75	1.25	0.00	0.00	0.00
$x_{r_1}^{2*}(t)$	0.00	0.00	$1.\bar{6}6$	$4.1\bar{6}6$	$6.\bar{6}6$
$x_{r_2}^{2*}(t)$	0.00	2.50	$3.\bar{3}3$	$3.\bar{3}3$	$3.\bar{3}3$

Table: Equilibrium Route Flows for the Multiclass Numerical Example

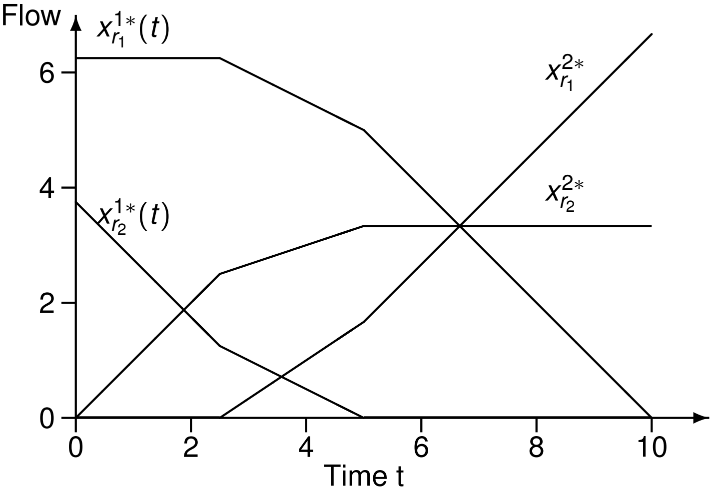


Figure: Equilibrium Trajectories for the Multiclass Numerical Example

The Time-dependent Braess Paradox

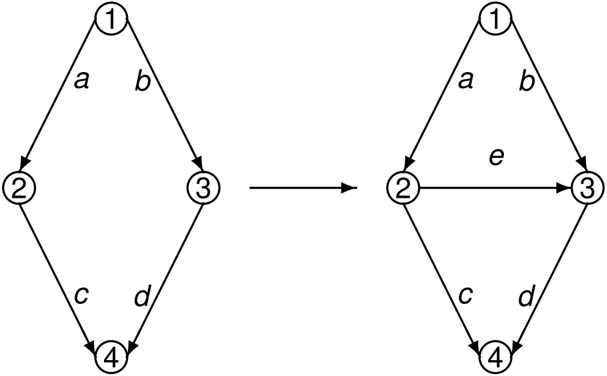


Figure: The Time-dependent Braess Network Example with Relevance to the Internet

Capacities:

$$\mu_{r_1}(t) = \mu_{r_2}(t) = \infty, \quad \forall t \in [0, T]$$

Link cost functions:

$$c_a(f_a(t)) = 10 f_a(t) \qquad c_b(f_b(t)) = f_b(t) + 50$$

$$c_c(f_c(t)) = f_c(t) + 50 \qquad c_d(f_d(t)) = 10 f_d(t)$$

Time-varying demand:

$$d_w(t) = t, \quad t \in [0, T]$$

Remark

At time $t = 6$, $d_w(6) = 6$, and it is easy to verify that the equilibrium route flows at time $t = 6$ are:

$$x^*_{r_1}(6) = 3, \quad x^*_{r_2}(6) = 3,$$

the equilibrium link flows are:

$$f^*_a(6) = 3, \quad f^*_b(6) = 3, \quad f^*_c(6) = 3, \quad f^*_d(6) = 3,$$

with associated equilibrium route costs:

$$C_{r_1}(6) = c_a(6) + c_c(6) = 83, \quad C_{r_2} = c_b(6) + c_d(6) = 83.$$

This is the solution to the classical (static) Braess (1968) network without the route addition.

EVI for the Dynamic Network Equilibrium Problem

Route costs in terms of route flows:

$$f_a(t) = f_c(t) = x_{r_1}(t), \quad f_b(t) = f_d(t) = x_{r_2}(t)$$

$$\Downarrow$$

$$C_{r_1}(t) = 11x_{r_1}(t) + 50, \quad C_{r_2}(t) = 11x_{r_2}(t) + 50$$

Route conservation of flow equations:

$$d_w(t) = t = x_{r_1}(t) + x_{r_2}(t),$$

$$\Downarrow$$

$$x_{r_2}(t) = t - x_{r_1}(t)$$

EVI Problem

$$\text{Find } x^* \in \mathcal{K}:$$

$$\int_0^T (11x^*_{r_1}(t) + 50) \times (x_{r_1}(t) - x^*_{r_1}(t)) + (11x^*_{r_2}(t) + 50)$$

$$\times (x_{r_2}(t) - x^*_{r_2}(t))dt \geq 0, \quad \forall x \in \mathcal{K}$$

$$\Updownarrow$$

$$\int_0^T (22x^*_{r_1}(t) - 11t) \times (x_{r_1}(t) - x^*_{r_1}(t))dt \geq 0, \quad \forall x \in \mathcal{K}$$

Equilibrium Flows:

$$x^*_{r_1}(t) = \frac{t}{2} \qquad x^*_{r_2}(t) = \frac{t}{2}$$

Equilibrium route costs:

$$C_{r_1}(x^*_{r_1}(t)) = 5\frac{1}{2}t + 50 = C_{r_2}(x^*_{r_2}(t)) = 5\frac{1}{2}t + 50$$

Cost on the new link "*e*":

$$c_e(f_e(t)) = f_e(t) + 10, \quad t \in [0, T]$$

Remark

For $t = 6$, equilibrium flows: $x^*_{r_1}(6) = x^*_{r_2}(6) = x^*_{r_3}(6) = 2$,
equilibrium link flows:

$$f^*_a(6) = 4, \quad f^*_b(6) = 2, \quad f^*_c(6) = 2, \quad f^*_e(6) = 2, \quad f^*_d(6) = 4,$$

equilibrium route costs:

$$C_{r_1}(6) = C_{r_2}(6) = C_{r_3}(6) = 92.$$

New EVI Problem

$$\text{Find } x^* \in \mathcal{K} :$$

$$\int_0^T (11x^*_{r_1}(t) + 10x^*_{r_3}(t) + 50) \times (x_{r_1}(t) - x^*_{r_1}(t))$$

$$+(11x^*_{r_2}(t) + 10x^*_{r_3}(t) + 50) \times (x_{r_2}(t) - x^*_{r_2}(t))$$

$$+(10x^*_{r_1}(t)+21x^*_{r_3}(t)+10x^*_{r_2}(t)+10)\times(x_{r_3}-x^*_{r_3}(t))dt \geq 0, \quad \forall x \in \mathcal{K}$$

$$\Updownarrow$$

$$\int_0^T (13x^*_{r_1}(t)-11t+40)\times((x_{r_1}(t)+x_{r_2}(t))-2x^*_{r_1}(t))dt \geq 0, \quad \forall x \in \mathcal{K}$$

Equilibrium Distribution

- for $d_w(t) = t \in \left[0, t_1 = 3\frac{7}{11}\right]$ (Regime I):

$$x^*_{r_1}(t) = x^*_{r_2}(t) = 0, \quad x^*_{r_3}(t) = d_w(t) = t;$$

- for $d_w(t) = t \in \left(t_1 = 3\frac{7}{11}, 8\frac{8}{9}\right]$ (Regime II):

$$x^*_{r_1}(t) = x^*_{r_2}(t) = \frac{11}{13}t - \frac{40}{13}, \quad x^*_{r_3}(t) = -\frac{9}{16}t + \frac{43}{8}.$$

- for $d_w(t) = t \in \left(t_2 = 8\frac{8}{9}, T < \infty\right]$ (Regime III):

$$x^*_{r_1}(t) = x^*_{r_2}(t) = \frac{d_{r_1}(t)}{2} = \frac{t}{2}, \quad x^*_{r_3}(t) = 0.$$

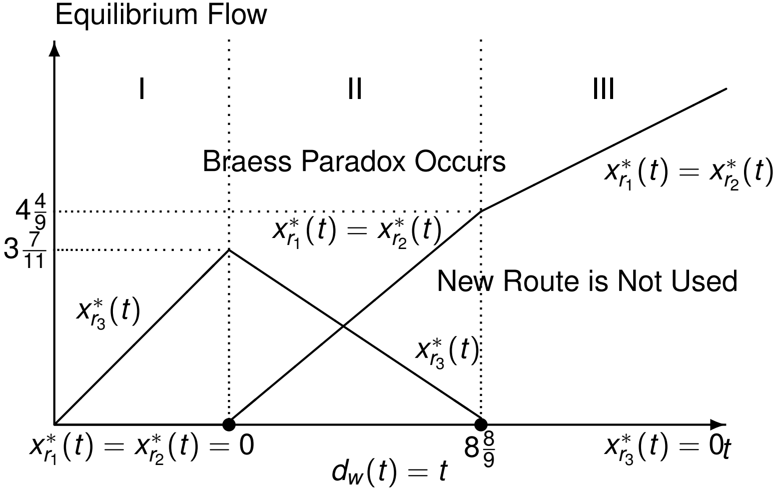


Figure: Equilibrium Trajectories of the Braess Network with Time-Dependent Demands

Efficiency Measure

Nagurney and Qiang (2008) proposed an efficiency measure for dynamic networks, modeled as evolutionary variational inequalities, denoted by $\mathcal{E}(\mathcal{G}, d, T)$.

Definition (Dynamic Network Efficiency: Continuous Time Version)

The network efficiency for the network $\mathcal{G}$ with time-varying demand d for $t \in [0, T]$, denoted by $\mathcal{E}(\mathcal{G}, d, T)$, is defined as follows:

$$\mathcal{E}(\mathcal{G}, d, T) = \frac{\int_0^T [\sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)}]/n_W \, dt}{T}.$$

The dynamic network efficiency measure $\mathcal{E}$ defined above is actually the average demand to price ratio over time.

It measures the overall (economic) functionality of the network with time-varying demands.

When the network topology $\mathcal{G}$, the demand pattern over time and the time span are given, a network is considered to be more efficient if it can satisfy higher demands at lower costs over time.

The network efficiency measure above can be easily adapted to dynamic networks in which the demands change at discrete points in time, as we now demonstrate. Let $d_w^1, d_w^2, ..., d_w^H$ denote demands for O/D pair w in H discrete time intervals, given, respectively, by: $[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$.

We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair w at the H different time intervals by: $\lambda_w^1, \lambda_w^2, ..., \lambda_w^H$. The demand vector d, in this special discrete case, is a vector in $R^{n_W \times H}$. The dynamic network efficiency measure in this case is as follows.

Definition (Dynamic Network Efficiency: Discrete Time Version)

The network efficiency for the network $(\mathcal{G}, d)$ *over* H *discrete time intervals:* $[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$*, where* $t_H \equiv T$*, and with the respective constant demands:* d_w^1*,* $d_w^2, ..., d_w^H$ *for all* $w \in W$ *is defined as:*

$$\mathcal{E}(\mathcal{G}, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.$$

We now provide the relationship between the dynamic network efficiency measure and the network efficiency measure proposed by Nagurney and Qiang (2007a,b) for static transportation (or congested) networks with fixed demands.

Theorem

Assume that $d_w(t) = d_w$, for all O/D pairs $w \in W$ and for $t \in [0, T]$. Then, the dynamic network efficiency measure collapses to the Nagurney and Qiang (2007a, b) measure:

$$\mathcal{E} = \frac{1}{n_W} \sum_{w \in W} \frac{d_w}{\lambda_w}.$$

The importance of a network component in the dynamic network case is the same as that defined in Nagurney and Qiang (2007a, b), but with the static efficiency measure now replaced by the dynamic network efficiency measure in the continuous case and by the discrete version in the discrete case. Hence, we have the following:

Definition (Importance of a Network Component)

The importance of network component g of network $\mathcal{G}$ with demand d over time horizon T is defined as follows:

$$I(g,d,T) = \frac{\mathcal{E}(\mathcal{G},d,T) - \mathcal{E}(\mathcal{G}-g,d,T)}{\mathcal{E}(\mathcal{G},d,T)}$$

where $\mathcal{E}(\mathcal{G}-g,d,T)$ is the dynamic network efficiency after component g is removed.

In studying the importance of a network component, the elimination of a link is treated in the above measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path/route connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Thus, our measure is well-defined even in the case of disconnected networks; see also Nagurney and Qiang (2007a, b). Additional theoretical properties of the static measure can be found in Qiang and Nagurney (2008).

An Application to the Time-dependent Braess Network

We now apply the above proposed dynamic network measure to the time-dependent Braess example (cf. Nagurney and Qiang (2008), Nagurney, Parkes, and Daniele (2007), Nagurney (2006), and, also, Pas and Principio (1997)).

Network Efficiency of the Dynamic Braess Network and Importance Rankings of Nodes and Links Over the Time Horizon

Let us now consider the dynamic Braess network (with three paths) in for $t \in [0, 10]$. As shown in Nagurney, Parkes, and Daniele (2007) and recalled above, different routes (and links) are used in different demand ranges. Therefore, it is interesting and relevant to study the network efficiency and the importance of the network components over the time horizon.

The network efficiency $\mathcal{E}(\mathcal{G}, d, 10)$ for this dynamic network is 0.5793. The importance and the rankings of the links and the nodes are summarized in the Tables below.

Table: Importance and Ranking of Links in the Dynamic Braess Network

Link	Importance Value	Importance Ranking
a	0.2604	1
b	0.1784	2
c	0.1784	2
d	0.2604	1
e	-0.1341	3

Table: Importance and Ranking of Nodes in the Dynamic Braess Network

Node	Importance Value	Importance Ranking
1	1.0000	1
2	0.2604	2
3	0.2604	2
4	1.0000	1

From the above analysis, it is clear which nodes and links are more important in the dynamic Braess network, and, hence, should, in effect, be better protected and secured, in practice, since their elimination results in a more significant drop in network efficiency or performance.

Indeed, link e after $t = 8\frac{8}{9}$ is never used and in the range $t \in [3\frac{7}{11}, 8\frac{8}{9}]$ increases the cost, so the fact that link e has a *negative* importance value makes sense; over time, its removal would, on the average, improve the network efficiency! This analysis also has implications for network design since, over the time horizon, adding/building link e does not make sense.

For further reading, please see:

A. Nagurney, D. Parkes, P. Daniele, “The Internet, evolutionary variational inequalities, and the time-dependent Braess paradox,” *Computational Management Science* 4 (2007), 355-375.

A. Nagurney, Q. Qiang, “An efficiency measure for dynamic networks with application to the Internet and vulnerability analysis,” *Netnomics* 9 (2008), 1-20.

A. Nagurney, Q. Qiang, *Fragile Networks; Identifying Vulnerabilities and Synergies in an Uncertain World*, John Wiley & Sons, Hoboken, New Jersey (2009).

Thank you!