Professors Parkes, Daniele, and Nagurney at Harvard.

Professors Nagurney and Daniele acknowledge support from the Radcliffe Institute for Advanced Study at Harvard University while Nagurney was a 2005-2006 Science Fellow and Daniele was a Visiting Scholar March – May, 2006.
Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Evolutionary Variational Inequalities and the Internet

Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Outline

1. Some References on Evolutionary Variational Inequalities
2. The Internet as a Dynamic Network
3. A Multiclass Numerical Example
4. The Time-dependent Braess Paradox
5. The Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities
6. Identification of the Importance of Nodes and Links and their Rankings for the Time-dependent Braess Network
Static Variational Inequalities – Classical References

- G. Stampacchia (1964): first theorem of existence and uniqueness of the solution of variational inequalities;
- J.L. Lions - G. Stampacchia (1967): second proof of the same theorem and introduction of evolutionary variational inequalities;
Evolutionary Variational Inequalities and the Internet

Some References on Evolutionary Variational Inequalities

Problem in $\mathbb{R}^n$:

$K \subseteq \mathbb{R}^n$, $F : K \rightarrow \mathbb{R}^n$,

Find $x \in K$ such that $\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K$,

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. 
Problem in a Banach space $E$:

$$K \subseteq E, \quad F : K \rightarrow E^*,$$

Find $x \in K$ such that $\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K,$

where $\langle \cdot, \cdot \rangle : E^* \times E \rightarrow \mathbb{R}$ is the duality pairing.

Evolutionary Variational Inequalities


F. Raciti (2001): time-dependent traffic networks with delay;


P. Daniele (2003): spatial price equilibrium problem with price and bounds depending on time;


P. Daniele, S. Giuffre’, S. Pia (2005): financial network problem with policy interventions;

A. Nagurney, Z. Liu, M.G. Cojocaru, P. Daniele (2007): dynamic electric power supply chains;

A. Nagurney, D. Parkes, P. Daniele (2007): the Internet, evolutionary variational inequalities and the time-dependent Braess paradox;


A. Nagurney, Q. Qiang (2008): Efficiency measure for dynamic networks with application to the Internet and vulnerability analysis.
Two books with results on and applications of Evolutionary Variational Inequalities.
Time-dependent Variational Inequality:

\[ \mathcal{K} \subseteq \mathcal{L} = L^p([0, T], \mathbb{R}^n), \quad F : \mathcal{K} \to \mathcal{L}^*, \]

Find \( x \in \mathcal{K} \) such that \( \ll F(x), y - x \gg \geq 0 \quad \forall y \in \mathcal{K}, \)

where \( \ll G, H \gg = \int_0^T \langle G(t), H(t) \rangle \, dt, \ G \in \mathcal{L}^*, \ H \in \mathcal{L}. \)
Roughgarden in his (2005) book *Selfish Routing and the Price of Anarchy* states that:

A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically ... The assumption of a static model is therefore particularly suspect in such networks.
The Internet as a Dynamic Network

- $G = [N, L]$ : network;
- $W$ : set of origin/destination (O/D) pairs of nodes;
- $P_w$ : set of routes joining $w$;
- $P$ : set of all routes connecting all the O/D pairs;
- $d^k_w(t)$ : demand between $w$ at time $t$ by job class $k$;
- $x^k_r(t)$ : flow on route $r$ at time $t$ of class $k$;
- $f^k_a(t)$ : flow on link $a$ of class $k$ at time $t$;
- $C^k_r(t)$ : cost on route $r$ at time $t$ of class $k$;
- $c^k_a(f(t))$ : cost on a link $a$ of class $k$ at time $t$. 
Conservation of flow equations:

\[ d^k_w(t) = \sum_{r \in P_w} x^k_r(t), \quad \forall w \in W, \forall k \]

\[ f^k_a(t) = \sum_{r \in P} x^k_r(t)\delta_{ar}, \quad \forall a \in L \]

Capacity Constraints:

\[ 0 \leq x^k_r(t) \leq \mu^k_r(t), \quad \forall r \in P, \forall k \]
Connection among costs:

\[ C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t)) \delta_{ar}, \quad \forall r \in P, \forall k \]

Feasible Set:

\[ \mathcal{K} = \left\{ x \in L^2([0, T], R^{KnP}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T]; \right\} \]

\[ \sum_{p \in P_w} x_p^k(t) = d_w^k(t), \, \forall w, \forall k \text{ a.e. in } [0, T] \]
Additional Assumptions:

- $0 \leq d(t) \leq \Phi \mu(t)$, a.e. on $[0, T] \Rightarrow K \neq \emptyset$
- All classes can use all the routes

Canonical Bilinear Form on $\mathcal{L}^* \times \mathcal{L}$:

$$
\langle \langle G, x \rangle \rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}
$$
**Definition (Dynamic Multiclass Network Equilibrium)**

$x^* \in \mathcal{K}$ is said to be a **dynamic network equilibrium** if $\forall w \in W$, $\forall r \in P_w$, $\forall k; k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C^k_r(x^*(t)) - \lambda^k_w(t) \begin{cases} 
\leq 0, & \text{if } x^k_r(t) = \mu^k_r(t), \\
= 0, & \text{if } 0 < x^k_r(t) < \mu^k_r(t), \\
\geq 0, & \text{if } x^k_r(t) = 0.
\end{cases}$$

**Theorem (Variational Formulation)**

$x^* \in \mathcal{K}$ is an equilibrium flow if and only if it satisfies the **evolutionary variational inequality**:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}$$
Preliminary Definitions

\[ C : \mathcal{K} \to \mathcal{L}^* \] is said to be

1. \textit{pseudomonotone} if and only if, \( \forall x, y \in \mathcal{K} \)
   \[ \langle \langle C(x), y - x \rangle \rangle \geq 0 \implies \langle \langle C(y), x - y \rangle \rangle \leq 0; \]

2. \textit{Fan-hemicontinuous} if and only if, \( \forall y \in \mathcal{K}, \xi \mapsto \langle \langle C(\xi), y - \xi \rangle \rangle \) is upper semicontinuous on \( \mathcal{K} \);

3. \textit{hemicontinuous along line segments} if and only if, \( \forall x, y \in \mathcal{K}, \xi \mapsto \langle \langle C(\xi), y - x \rangle \rangle \) is upper semicontinuous on the line segment \([x, y]\).
Theorem (Existence)

If $C$ satisfies any of the following conditions:

1. $C$ is Fan-hemicontinuous with respect to the strong topology on $\mathcal{K}$, and $\exists A \subseteq \mathcal{K}$ nonempty, compact, and $\exists B \subseteq \mathcal{K}$ compact such that, $\forall y \in \mathcal{K} \setminus A$, $\exists x \in B$ with $\langle \langle C(x), y - x \rangle \rangle < 0$;

2. $C$ is Fan-hemicontinuous with respect to the weak topology on $\mathcal{K}$;

3. $C$ is pseudomonotone and hemicontinuous along line segments,

then the EVI problem admits a solution in $\mathcal{K}$. 
A Multiclass Numerical Example

Figure: Network Structure of the Multiclass Numerical Example

Time horizon: [0, 10]
Costs for Class 1 and Class 2:

\[
C_{r_1}^1(x(t)) = 2x_{r_1}^1(t) + x_{r_1}^2(t) + 5, \quad C_{r_2}^1(x(t)) = 2x_{r_2}^2(t) + 2x_{r_2}^1(t) + 10,
\]
\[
C_{r_1}^2(x(t)) = x_{r_1}^2(t) + x_{r_1}^1(t) + 5, \quad C_{r_2}^2(x(t)) = x_{r_2}^1(t) + 2x_{r_2}^2(t) + 5.
\]

Demands for the O/D pair:

\[
d_{w}^1(t) = 10 - t, \quad d_{w}^2(t) = t
\]

Upper bounds:

\[
\mu_{r_1}^1 = \mu_{r_2}^1 = \mu_{r_1}^1 = \mu_{r_2}^2 = \infty
\]
A Multiclass Numerical Example

**Equilibrium Multiclass Route Flows at time** \( t \)

<table>
<thead>
<tr>
<th>Flow</th>
<th>( t = 0 )</th>
<th>( t = 2.5 )</th>
<th>( t = 5 )</th>
<th>( t = 7.5 )</th>
<th>( t = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{r_1}^1(t) )</td>
<td>6.25</td>
<td>6.25</td>
<td>5.00</td>
<td>2.50</td>
<td>0.00</td>
</tr>
<tr>
<td>( x_{r_2}^1(t) )</td>
<td>3.75</td>
<td>1.25</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>( x_{r_1}^2(t) )</td>
<td>0.00</td>
<td>0.00</td>
<td>1.66</td>
<td>4.166</td>
<td>6.66</td>
</tr>
<tr>
<td>( x_{r_2}^2(t) )</td>
<td>0.00</td>
<td>2.50</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
</tr>
</tbody>
</table>

**Table:** Equilibrium Route Flows for the Multiclass Numerical Example
Figure: Equilibrium Trajectories for the Multiclass Numerical Example
The Time-dependent Braess Paradox

Figure: The Time-dependent Braess Network Example with Relevance to the Internet
Capacities:

\[ \mu_{r_1}(t) = \mu_{r_2}(t) = \infty, \quad \forall t \in [0, T] \]

Link cost functions:

\[ c_a(f_a(t)) = 10f_a(t) \quad c_b(f_b(t)) = f_b(t) + 50 \]
\[ c_c(f_c(t)) = f_c(t) + 50 \quad c_d(f_d(t)) = 10f_d(t) \]

Time-varying demand:

\[ d_w(t) = t, \quad t \in [0, T] \]
Remark

At time $t = 6$, $d_w(6) = 6$, and it is easy to verify that the equilibrium route flows at time $t = 6$ are:

$$x_{r_1}^*(6) = 3, \quad x_{r_2}^*(6) = 3,$$

the equilibrium link flows are:

$$f_a^*(6) = 3, \quad f_b^*(6) = 3, \quad f_c^*(6) = 3, \quad f_d^*(6) = 3,$$

with associated equilibrium route costs:

$$C_{r_1}(6) = c_a(6) + c_c(6) = 83, \quad C_{r_2} = c_b(6) + c_d(6) = 83.$$

This is the solution to the classical (static) Braess (1968) network without the route addition.
EVI for the Dynamic Network Equilibrium Problem

Route costs in terms of route flows:

\[ f_a(t) = f_c(t) = x_{r_1}(t), \quad f_b(t) = f_d(t) = x_{r_2}(t) \]

\[ \Downarrow \]

\[ C_{r_1}(t) = 11x_{r_1}(t) + 50, \quad C_{r_2}(t) = 11x_{r_2}(t) + 50 \]

Route conservation of flow equations:

\[ d_w(t) = t = x_{r_1}(t) + x_{r_2}(t), \]

\[ \Downarrow \]

\[ x_{r_2}(t) = t - x_{r_1}(t) \]
EVI Problem

Find $x^* \in \mathcal{K}$:

$$\int_0^T (11x_{r_1}^*(t) + 50) \times (x_{r_1}(t) - x_{r_1}^*(t)) + (11x_{r_2}^*(t) + 50) \times (x_{r_2}(t) - x_{r_2}^*(t)) dt \geq 0, \quad \forall x \in \mathcal{K}$$

$$\updownarrow$$

$$\int_0^T (22x_{r_1}^*(t) - 11t) \times (x_{r_1}(t) - x_{r_1}^*(t)) dt \geq 0, \quad \forall x \in \mathcal{K}$$
Equilibrium Flows:

\[ x_{r_1}^*(t) = \frac{t}{2}, \quad x_{r_2}^*(t) = \frac{t}{2} \]

Equilibrium route costs:

\[ C_{r_1}(x_{r_1}^*(t)) = 5 \frac{1}{2} t + 50 = C_{r_2}(x_{r_2}^*(t)) = 5 \frac{1}{2} t + 50 \]
Cost on the new link “e”:

\[ c_e(f_e(t)) = f_e(t) + 10, \quad t \in [0, T] \]

Remark

For \( t = 6 \), equilibrium flows: \( x_{r_1}^*(6) = x_{r_2}^*(6) = x_{r_3}^*(6) = 2 \), equilibrium link flows:

\[ f_a^*(6) = 4, \quad f_b^*(6) = 2, \quad f_c^*(6) = 2, \quad f_e^*(6) = 2, \quad f_d^*(6) = 4, \]

equilibrium route costs:

\[ C_{r_1}(6) = C_{r_2}(6) = C_{r_3}(6) = 92. \]
New EVI Problem

Find $x^* \in \mathcal{K}$:

$$\int_0^T (11x_{r_1}^*(t) + 10x_{r_3}^*(t) + 50) \times (x_{r_1}(t) - x_{r_1}^*(t)) \, dt \geq 0, \quad \forall x \in \mathcal{K}$$

$$+ (11x_{r_2}^*(t) + 10x_{r_3}^*(t) + 50) \times (x_{r_2}(t) - x_{r_2}^*(t)) \, dt \geq 0, \quad \forall x \in \mathcal{K}$$

$$+ (10x_{r_1}^*(t) + 21x_{r_3}^*(t) + 10x_{r_2}^*(t) + 10) \times (x_{r_3} - x_{r_3}^*(t)) \, dt \geq 0, \quad \forall x \in \mathcal{K}$$
Evolutionary Variational Inequalities and the Internet

The Time-dependent Braess Paradox

Equilibrium Distribution

- for $d_w(t) = t \in \left[0, t_1 = 3 \frac{7}{11}\right]$ (Regime I):
  \[ x_{r_1}^*(t) = x_{r_2}^*(t) = 0, \quad x_{r_3}^*(t) = d_w(t) = t; \]

- for $d_w(t) = t \in \left(t_1 = 3 \frac{7}{11}, 8 \frac{8}{9}\right]$ (Regime II):
  \[ x_{r_1}^*(t) = x_{r_2}^*(t) = \frac{11}{13} t - \frac{40}{13}, \quad x_{r_3}^*(t) = -\frac{9}{16} t + \frac{43}{8}. \]

- for $d_w(t) = t \in \left(t_2 = 8 \frac{8}{9}, T < \infty\right]$ (Regime III):
  \[ x_{r_1}^*(t) = x_{r_2}^*(t) = \frac{d_{r_1}(t)}{2} = \frac{t}{2}, \quad x_{r_3}^*(t) = 0. \]
Evolutionary Variational Inequalities and the Internet

The Time-dependent Braess Paradox

Evolutionary Variational Inequalities and the Internet

The Time-dependent Braess Paradox

Evolutionary Variational Inequalities and the Internet

The Time-dependent Braess Paradox

Figure: Equilibrium Trajectories of the Braess Network with Time-Dependent Demands
Nagurney and Qiang (2008) proposed an efficiency measure for dynamic networks, modeled as evolutionary variational inequalities, denoted by $E(G, d, T)$.

**Definition (Dynamic Network Efficiency: Continuous Time Version)**

The network efficiency for the network $G$ with time-varying demand $d$ for $t \in [0, T]$, denoted by $E(G, d, T)$, is defined as follows:

$$E(G, d, T) = \frac{\int_0^T \left[ \sum_{w \in W} \frac{d_w(t)}{x_w(t)} \right] / n_W}{T} dt.$$
The dynamic network efficiency measure $\mathcal{E}$ defined above is actually the average demand to price ratio over time.

It measures the overall (economic) functionality of the network with time-varying demands.

When the network topology $\mathcal{G}$, the demand pattern over time and the time span are given, a network is considered to be more efficient if it can satisfy higher demands at lower costs over time.
The network efficiency measure above can be easily adapted to dynamic networks in which the demands change at discrete points in time, as we now demonstrate. Let \( d_1^w, d_2^w, \ldots, d_H^w \) denote demands for O/D pair \( w \) in \( H \) discrete time intervals, given, respectively, by: \([t_0, t_1], (t_1, t_2), \ldots, (t_{H-1}, t_H]\), where \( t_H \equiv T \).

We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair \( w \) at the \( H \) different time intervals by: \( \lambda_1^w, \lambda_2^w, \ldots, \lambda_H^w \). The demand vector \( d \), in this special discrete case, is a vector in \( R^{n_W \times H} \). The dynamic network efficiency measure in this case is as follows.
Definition (Dynamic Network Efficiency: Discrete Time Version)

The network efficiency for the network \((G, d)\) over \(H\) discrete time intervals: \([t_0, t_1], (t_1, t_2], \ldots, (t_{H-1}, t_H]\), where \(t_H \equiv T\), and with the respective constant demands: \(d^1_w, d^2_w, \ldots, d^H_w\) for all \(w \in W\) is defined as:

\[
E(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d^i_w}{\lambda^i_w})(t_i - t_{i-1})/n_w]}{t_H}.
\]
We now provide the relationship between the dynamic network efficiency measure and the network efficiency measure proposed by Nagurney and Qiang (2007a,b) for static transportation (or congested) networks with fixed demands.

**Theorem**

Assume that \( d_w(t) = d_w \), for all O/D pairs \( w \in W \) and for \( t \in [0, T] \). Then, the dynamic network efficiency measure collapses to the Nagurney and Qiang (2007a, b) measure:

\[
\mathcal{E} = \frac{1}{n_W} \sum_{w \in W} \frac{d_w}{\lambda_w}.
\]
The importance of a network component in the dynamic network case is the same as that defined in Nagurney and Qiang (2007a, b), but with the static efficiency measure now replaced by the dynamic network efficiency measure in the continuous case and by the discrete version in the discrete case. Hence, we have the following:

**Definition (Importance of a Network Component)**

The importance of network component \( g \) of network \( \mathcal{G} \) with demand \( d \) over time horizon \( T \) is defined as follows:

\[
I(g, d, T) = \frac{\mathcal{E}(\mathcal{G}, d, T) - \mathcal{E}(\mathcal{G} - g, d, T)}{\mathcal{E}(\mathcal{G}, d, T)}
\]

where \( \mathcal{E}(\mathcal{G} - g, d, T) \) is the dynamic network efficiency after component \( g \) is removed.
In studying the importance of a network component, the elimination of a link is treated in the above measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path/route connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Thus, our measure is well-defined even in the case of disconnected networks; see also Nagurney and Qiang (2007a, b). Additional theoretical properties of the static measure can be found in Qiang and Nagurney (2008).
An Application to the Time-dependent Braess Network

We now apply the above proposed dynamic network measure to the time-dependent Braess example (cf. Nagurney and Qiang (2008), Nagurney, Parkes, and Daniele (2007), Nagurney (2006), and, also, Pas and Principio (1997)).
Let us now consider the dynamic Braess network (with three paths) in for $t \in [0, 10]$. As shown in Nagurney, Parkes, and Daniele (2007) and recalled above, different routes (and links) are used in different demand ranges. Therefore, it is interesting and relevant to study the network efficiency and the importance of the network components over the time horizon.

The network efficiency $\mathcal{E}(G, d, 10)$ for this dynamic network is 0.5793. The importance and the rankings of the links and the nodes are summarized in the Tables below.
**Table: Importance and Ranking of Links in the Dynamic Braess Network**

<table>
<thead>
<tr>
<th>Link</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$c$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$d$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$e$</td>
<td>-0.1341</td>
<td>3</td>
</tr>
</tbody>
</table>
Table: Importance and Ranking of Nodes in the Dynamic Braess Network

<table>
<thead>
<tr>
<th>Node</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.2604</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0.2604</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1.0000</td>
<td>1</td>
</tr>
</tbody>
</table>
From the above analysis, it is clear which nodes and links are more important in the dynamic Braess network, and, hence, should, in effect, be better protected and secured, in practice, since their elimination results in a more significant drop in network efficiency or performance.

Indeed, link $e$ after $t = 8\frac{8}{9}$ is never used and in the range $t \in [3\frac{7}{11}, 8\frac{8}{9}]$ increases the cost, so the fact that link $e$ has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency! This analysis also has implications for network design since, over the time horizon, adding/building link $e$ does not make sense.
For further reading, please see:


Thank you!