Physical Proof of the Occurrence of the Braess Paradox in Electrical Circuits

Ladimer S. Nagurney[1] and Anna Nagurney[2]

[1]Department of Electrical and Computer Engineering
University of Hartford, West Hartford, CT 06117
and
[2] Department of Operations and Information Management
University of Massachusetts, Amherst, MA 01003

July 5, 2016

Acknowledgments

The first author thanks the Department of Electrical and Computer Engineering at the University of Massachusetts Amherst for hosting him during his sabbatical and providing the facilities for this work.

The second author was supported by the National Science Foundation under *EAGER: Collaborative Research: Enabling Economic Policies in Software-Defined Internet Exchange Points*, Award Number:1551444 and also by the Visiting Fellowship program of All Souls College at Oxford University inn England.

This support is gratefully acknowledged.

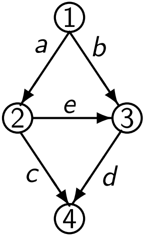
Braess Paradox in Transportation Networks

- First noted by Dietrich Braess in 1968
- In a user-optimized transportation network, when a new link (road) is added, the change in equilibrium flows may result in a higher cost (travel time) to all travelers in the network, implying that users were better off without that link.

Examples of Braess Paradox

- Stuttgart Germany - In 1969 a newly constructed road worsened traffic. Traffic improved when the road was closed.
- New York City - Earth Day 1990 Traffic improved when 42nd St was closed
- Seoul, Korea - A 6 lane road that was perpetually jammed was closed and removed, traffic improved.

Classical Braess Paradox (1968) Transportation Network



Link cost functions

$c_a(f_a) = 10f_a$

$c_b(f_b) = f_b + 50$

$c_c(f_c) = f_c + 50$

$c_d(f_d) = 10f_d$

$c_e(f_e) = f_e + 10$

f_a - Flow on link a

O/D Pair (1,4)

Demand $= 6$

3 paths $p_1 = (a, c)$

$p_2 = (b, d)$ $p_3 = (a, e, d)$

With link e, flows on the paths 1, 2, and 3 are 2 and the user costs are 92.

No user has any incentive to switch, since switching would result in a higher path cost.

Without link e, the path flow pattern on the 2 paths is 3 for each path and the user costs are 83.

Hence, the addition of link e makes all users of the network worse-off since the cost increases from 83 to 92!

Translation of the 1968 Braess article from German into English

On a Paradox of Traffic Planning, D. Braess, A. Nagurney, and T. Wakolbinger (2005) *Transportation Science* **39**, pp 446-450.


Über ein Paradoxon aus der Verkehrsplanung

Von D. BRAESS, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

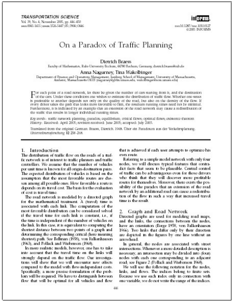
1. Einleitung

Für die Verkehrsplanung und Verkehrssteuerung interessiert, wie sich der Fahrzeugstrom auf die einzelnen Straßen des Verkehrsnetzes verteilt. Bekannt sei dabei die Anzahl der Fahrzeuge für alle Ausgangs- und Zielpunkte. Bei der Berechnung wird davon ausgegangen, daß von den möglichen Wegen jeweils der günstigste gewählt wird. Wie günstig ein Weg ist, richtet sich nach dem Aufwand, der zum Durchfahren nötig ist. Die Grundlage für die Bewertung des Aufwandes bildet die Fahrzeit.

Für die mathematische Behandlung wird das Straßennetz durch einen gerichteten Graphen beschrieben. Zur Charakterisierung der Bögen gehört die Angabe des Zeitaufwandes. Die Bestimmung der günstigen Stromverteilungen kann als gelöst betrachtet werden, wenn die Bewertung konstant ist, d. h., wenn die Fahrzeiten unabhängig von der Größe des Verkehrsflusses sind. Sie ist dann äquivalent mit der bekannten Aufgabe, den kürzesten Abstand zweier Punkte eines Graphen und den zugehörigen kritischen Pfad zu bestimmen [1], [5], [7].

Will man das Modell aber realistischer gestalten, ist zu berücksichtigen, daß die benötigte Zeit stark von der Stärke des Verkehrs abhängt. Wie die folgenden Untersuchungen zeigen, ergeben sich dann gegenüber dem Modell mit konstanter (belastungsunabhängiger) Bewertung z. T. völlig neue Aspekte. Dabei erweist sich schon eine Präzisierung der Problemstellung als notwendig; denn es ist zwischen dem Strom zu unterscheiden, der für alle am günstigsten ist,, und dem, der sich einstellt, wenn jeder Fahrer nur seinen eigenen Weg optimalisiert.

[1]) Priv.-Doz. Dr. DIETRICH BRAESS, Institut für numerische und instrumentelle Mathematik, 44 Münster, Hüfferstr. 3 a.


TRANSPORTATION SCIENCE
Vol. 39, No. 4, November 2005, pp. 446–450

On a Paradox of Traffic Planning

Dietrich Braess

Anna Nagurney, Tina Wakolbinger

1. Introduction

Ladimer S. Nagurney and Anna Nagurney

Braess Paradox in Other Network Systems

The Braess paradox is also relevant to other network systems in which the users operate under decentralized (selfish) decision-making behavior rather than centralized or system-optimizing behavior.

- Spring Systems - Cohen and Horowitz (1991) - Penchina and Penchina (2003)
- The Internet - Nagurney, Parkes, and Daniele (2007)
- Electric power generation and distribution networks - Bjorndal and Jornsten (2008) - Witthaut and Timme Phys.org (2012)
- Biology - Motter (2014)
- Nanoscale Systems - Pala et al (2012)
- Wireless Systems - Altman et al (2008)

Can Braess Paradox exist in a Circuit consisting only of Passive Electrical Components?

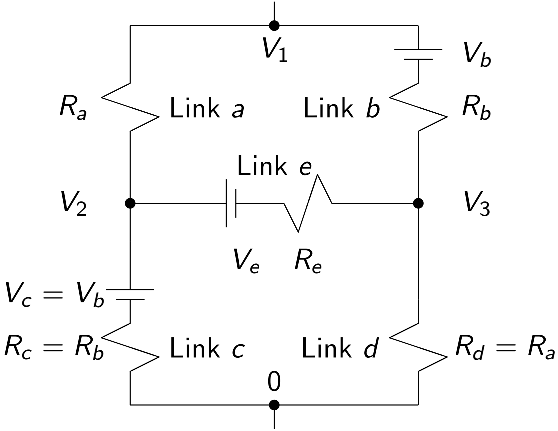
- In a passive circuit, the conventional wisdom is that by adding a link (branch in EE terminology), the resistance of such a circuit will decrease.
- If the Braess Paradox would occur the *equivalent resistance* of the circuit would increase.
- In terms of flow, for a fixed flow through the circuit, the voltage would rise (rather than decrease) when a branch was added.
- This is analogous to the increase of cost in a transportation network.
- Because all electrons move through the network with the same voltage drop, an electrical network is an example of a user-optimized network.

Electrical Circuit Equivalent of the Braess Paradox

An idealized electrical circuit that exhibits the Braess Paradox behavior is developed by mapping the cost functions on links into the voltage drops of electrical components.

- The demand on the Braess network is analogous to the current in the electrical circuit.
- The cost on a link is analogous to the voltage drop on the corresponding link of the circuit.
- Ohm's Law states that in a resistor the voltage drop is proportional to the current through that resistor. Thus, a term that has cost proportional to flow is the equivalent of a resistor.
- Fixed cost terms in a link cost function can be modeled as a constant voltage drop which can symbolically represented as a battery.

Electrical Circuit Analogue for the Classical Braess Paradox



Because of the symmetry of the Braess Paradox example

$$R_d = R_a \qquad V_c = V_b \qquad R_c = R_b$$

Let V_i; $i = 1, \ldots, n$, be the voltage at node i referenced to the reference/ground node of the circuit.

Let the demand through the electrical network be I and the flow through a link i be I_i.

In the electrical circuit, the voltage, V_1, is the equivalent of the cost for a user (electron) to flow through the circuit.

Thus, the Braess Paradox occurs if, by adding link e, the voltage V_1 increases.

Nodal Analysis for the Braess Paradox Circuit

Using Kirchhoff's current law at each node i; $i = 1, 2, 3$, the nodal equations can be written in matrix notation as:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = G^{-1} \begin{bmatrix} I + \frac{V_b}{R_b} \\ -\left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \\ \left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \end{bmatrix}$$

where G is the Conductance Matrix

$$G = \begin{bmatrix} \frac{1}{R_a} + \frac{1}{R_b} & -\frac{1}{R_a} & -\frac{1}{R_b}, \\ \frac{1}{R_a} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} & \frac{1}{R_e}, \\ \frac{1}{R_a} & \frac{1}{R_e} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} \end{bmatrix}$$

For the case when link e is not in the circuit, by symmetry, the voltage drop on each of the two paths is the same, since the current splits equally at the top node.

$$I_a = I_b = I_c = I_d = I/2$$

By straightforward algebra, the node voltages become

$$\begin{aligned} V_1 = &\quad V_b + (R_a + R_b)\tfrac{I}{2} \\ V_2 = &\quad V_b + R_b\tfrac{I}{2} \\ V_3 = &\quad R_a\tfrac{I}{2}. \end{aligned}$$

In this Kirchhoff's nodal analysis, it is not explicit that the batteries corresponding to V_b and V_e are passive voltage drops and are not generating current.

Solutions of the matrix equation might exist where the flow $I_a > I$ since if the battery supplies current I_b would not be in the direction from node 1 to node 3.

While this type of solution is mathematically possible, it does not correspond to a passive electrical circuit Braess Paradox example.

To verify that the batteries, V_b and V_e, are passive voltage drops, after computing the solutions for V_1, V_2, and V_3 by matrix inversion, the flows on links b and c must be calculated.

If the two inequalities below are satisfied, then the batteries are acting as passive voltage drops corresponding to a passive electrical circuit:

$$\frac{V_1 - V_b - V_3}{R_b} \geq 0 \qquad \text{and} \qquad \frac{V_2 - V_b}{R_b} \geq 0$$

Classical Braess Paradox Example

The classical Braess Paradox example from his 1968 paper (in terms of voltages and currents) has

$$V_b = 50V \quad V_e = 10V \quad R_a = 10\Omega \quad R_b = R_e = 1\Omega \quad \text{and} \quad I = 6$$

With link e in the circuit, the matrix equation becomes

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1.1 & -.1 & -1 \\ .1 & -2.1 & 1 \\ .1 & 1 & -2.1 \end{bmatrix}^{-1} \begin{bmatrix} 56 \\ -60 \\ 60 \end{bmatrix} = \begin{bmatrix} 92 \\ 52 \\ 40 \end{bmatrix}$$

Without link e, the equation becomes

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 50 + 11 \cdot \frac{6}{2} \\ 50 + \frac{6}{2} \\ 10 \cdot \frac{6}{2} \end{bmatrix} = \begin{bmatrix} 83 \\ 53 \\ 30 \end{bmatrix}$$

$V_1 = 83V$ without link e in the circuit and $V_1 = 92V$ when link e is in the network.

Voltage increases when link e is added.

This reproduces the transportation network example in the original Braess article.

Additional Insights from Kirchoff's Formulation

From the right-hand-side of nodal equations with link e in the circuit, one notes that V_e only occurs in the sum

$$V_b + V_e \frac{R_b}{R_e}$$

This indicates that there might be networks that exhibit the Braess Paradox behavior without a fixed cost term in the added link e. This motivates some experimental work described later in this talk.

Zener Diode Formulation

In 1991 Cohen and Horowitz proposed that a Wheatstone Bridge topology circuit consisting of Zener diodes and resistors could exhibit the Braess Paradox.

Their circuit had *unrealistic values in practice, but convenient for illustration*.

Calculation of Component Values for a BP Electrical Circuit

To construct a circuit with values that could be realized in practice, consider the following.
Make the conductance matrix, G, dimensionless by factoring out R_b^{-1} to become

$$G = \hat{G} R_b^{-1},$$

the nodal equations become

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \hat{G}^{-1} \begin{bmatrix} IR_b + V_b \\ -\left(V_b + V_e \frac{R_b}{R_e}\right) \\ \left(V_b + V_e \frac{R_b}{R_e}\right) \end{bmatrix}.$$

Calculation of Component Values for a BP Electrical Circuit - II

- $\hat{G}$ depends only on the ratios $\frac{R_b}{R_a}$ and $\frac{R_b}{R_e}$, which in the classical Braess example are .1 and 1, respectively.
- Because multiplying both sides of a matrix equation by a constant does not change the equation, we can scale V_b, V_e, and the quantity IR_b by the same factor and still have an electrical circuit that exhibits the Braess Paradox behavior.
- This allows the choice of realistic component values, and current, I.
- The batteries can then be replaced by Zener diodes, which, to a good approximation, can be modeled as voltage drops.

Zener Diode Circuit Development

- Scaling the nodal equation by .1, a circuit consisting of

$$V_b = 5V, \qquad V_e = 1V, \qquad I = 6mA,$$

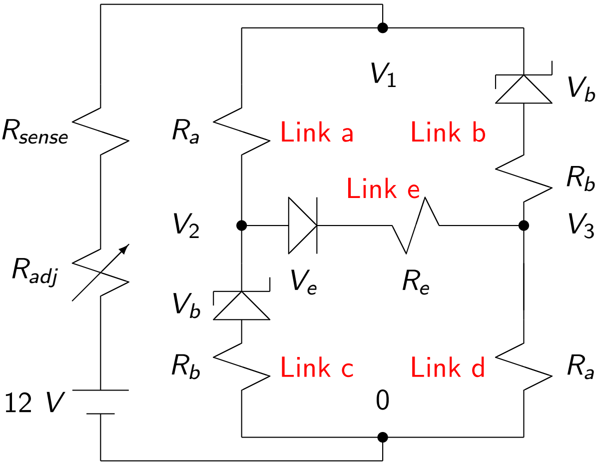
$$R_b = R_e = 100\Omega, \quad \text{and} \quad R_a = 1000\Omega,$$

  exhibits the Braess Paradox with the voltage $V_1 = 8.3V$ with link e removed and $V_1 = 9.2V$ with link e in the circuit.
- To construct a circuit with real components inspired by the classical Braess example, the scaled V_b should be near $5V$.
- The closest standard Zener diode (1N4733A) has Zener voltage 5.1V.
- Since the ratio of V_b/V_e is 10 in the ideal electrical analogue of the classical Braess example, we select the diode in link e to be a forward biased silicon diode (1N4002) that has a forward voltage drop of approximately .6 V.

Zener Diode Circuit Development - II

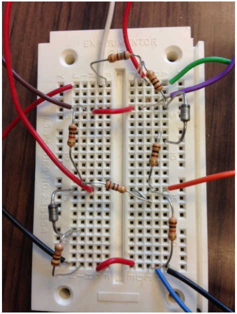
- To choose the value for R_b, note that the quantity IR_b should equal .6 V. For convenience, select $R_b = 100\Omega$ and $I = 6\ mA$.
- Since the ratios of R_b/R_a and R_b/R_e in the classical Braess example are .1 and 10, respectively, select $R_e = R_b = 100\ \Omega$ and $R_a = 10R_b = 1000\ \Omega$.
- It was verified computationally that for these choices of component values, the Braess Paradox is observed in the circuit.

Electrical Circuit using Zener Diodes



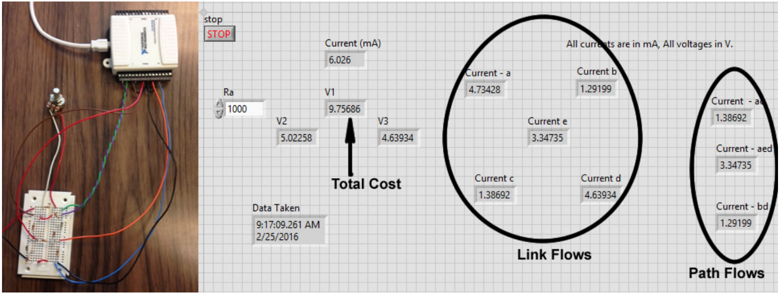
Circuit Details

- The current, I, is generated by a 12 V power supply.
- Two additional resistors, R_{sense} and R_{adj} are used to sense and adjust the current, I.
- R_{adj} is adjusted during the measurement to set the current I, the demand, to 6 mA.
- R_{sense} is used to measure the current. For the desired current of 6 mA, the voltage across V_{sense} would be .6 V.



[1] Department of Electrical and Computer Engineering University of Hartford, West Hartford, CT 06117 and [2] Department of Operatio

The voltages at nodes V_1, V_2, and V_3, and the voltage across R_{sense} are measured using the 4 analog input channels of a National Instruments USB-6009 Multifunction I/O Data Acquisition system. The USB-6009 system was programmed using Labview running on a PC. From these measurements and the knowledge of the resistor values R_a, R_b, and R_e, the link and path flows are calculated.



Zener Diode Circuit Measurements

For this circuit, five measurements are made for cases corresponding, respectively, to:

- Case 1: link e absent
- Case 2: link e present with $V_e = .62\ V$ and $R_e = 100\ \Omega$ (analogous to the classical Braess example)
- Case 3: link e present with only $R_e = 100\Omega$
- Case 4: link e present with only $V_e = .62\ V$
- Case 5: link e is a short circuit, i.e., $R_e = 0$.

For all cases the cost functions on links $a - d$ are as below:

$$\begin{array}{llll}
\text{Cost on link } a & c_a: & 1000 f_a & = 1000 I_a \\
\text{Cost on link } b & c_b: & 5.1 + 100 f_b & = 5.1 + 100 I_b \\
\text{Cost on link } c & c_c: & 5.1 + 100 f_c & = 5.1 + 100 I_c \\
\text{Cost on link } d & c_d: & 1000 f_d & = 1000 I_d.
\end{array}$$

Measured Voltage Across an Electrical Circuit using Zener Diodes Exhibiting the Braess Paradox

Case	V_e	R_e	V_1	Form of Link e Cost Function
1	-	∞	8.13	Link e not in network
2	.62	100	9.21	$V_e + I_e R_e$
3	0	100	9.72	$I_e R_e$
4	.62	0	8.14	V_e
5	-	0	9.88	Link e is a short circuit

Interpretation of Zener Diode Results

- In all cases, when link e is added, the voltage at node 1, V_1, increases, showing that the Braess Paradox occurs in the circuit.
- In electrical circuits one would normally expect the voltage to drop when a link is added. These multiple examples prove that, in contrast, the opposite can happen.
- This is the first experimental observation of the Braess Paradox in electrical circuits.

Interpretation of Zener Diode Results II

- The first and second cases correspond to the Braess Paradox transportation example scaled for real components.
- The first case is the circuit with link e not present.
- The second case is the circuit with link e added with a voltage drop of $.62\ V + 100 I_e$ for flow from node 2 to node 3.
- If the voltage at node 2 was not greater than the voltage at node 3 by .62V, there would be no flow between these two nodes.
- The cost for the flow through the circuit is 8.13 V in the absence of link e and 9.21V in the presence of link e; thus, confirming the observation of the Braess Paradox in the circuit.

Interpretation of Zener Diode Results III

The last three cases correspond to other functional forms of the cost functions for link e.

- Case 3 corresponds to the case in which link e's cost is only proportional to the flow. From the nodal analysis, we note that if the Braess Paradox exists in a circuit for a set of values I, V_b, and V_e, one can choose another set of values, $V_b' = V_b + V_e$, $V_e' = 0$, and $I' = I - (V_e/R_e)$, without changing the RHS of the nodal equations. The Braess Paradox does occur in this modified circuit and is measured.

Interpretation of Zener Diode Results IV

- Case 4 has link e as a fixed cost link. However, the fixed voltage drop model for a diode is only an approximation to the behavior of the real diode. The voltage drop on a link will always depend at least slightly upon the current through that link. This case only marginally illustrates the Braess Paradox.
- Case 5 corresponds to the case of a zero cost link e. This was constructed by using a piece of wire for the link. This case may be analyzed as a circuit with a resistor in parallel with the series Zener diode-resistor combination. The measured V_1 in this case is less than either twice the Zener voltage $(10.2V)$ or the total current through the resistors, R_a $(12V)$, which may be interpreted by assuming non-ideal behavior of the reverse leakage current of a Zener diode.

Extension of the Braess Paradox analysis to other forms of cost functions

Mathematically investigated in Leblanc (1975), Frank (1981), Bloy (2007).

The driving force for these investigations has been that realistic travel cost functions are based upon the Bureau of Public Roads (BPR) travel cost functions which model the cost on a link as

$$c_a(f_a) = t_a^0 \left(1 + k \left(\frac{f_a}{u_a}\right)^{\beta}\right)$$

where t_a^0, k, u_a, and β are positive constants. Often $k = .15$, $\beta = 4$, and u_a is the practical capacity of link a.

A full interpretation of the BPR cost functions can be found in Sheffi (1985).

While it is impossible to find a passive electrical component whose $I - V$ characteristics are identical in form to the BPR cost functions, the $I - V$ characteristics of a forward biased diode have an exponential shape.

The Shockley diode model predicts that the voltage across any diode, V_D, may be modeled as

$$V_D = V_T \ln \left(\frac{I_D}{I_S} \right)$$

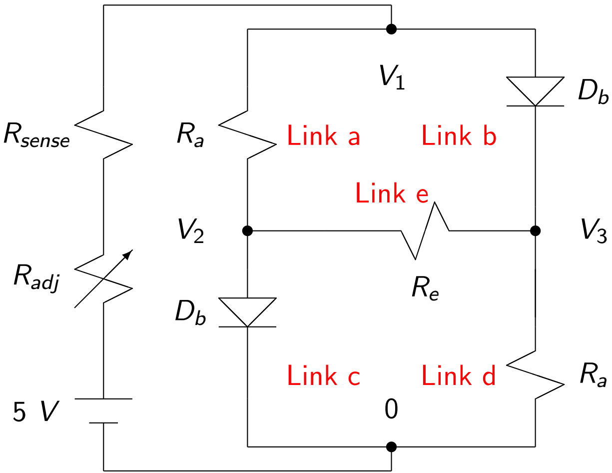
where V_T is the thermal voltage approximately 26 mV at room temperature and I_S is the saturation current approximately 1 pA for a silicon diode.

The first approximation to the Shockley model is the piecewise linear model which models the diode as a voltage source in series with a resistor. This model for the diode voltage drop is identical in form to the voltage drop on links b and c of our initial circuit. The Shockley can be expanded as a power series in I producing higher order terms similar to those suggested as more complicated transportation cost functions.

An electric circuit can be constructed with links b and c implemented by forward-biased silicon diodes. This topology was implied in Frank, Leblanc (1975) but in the context of transportation networks.

Unlike the earlier Zener diode electrical circuit example, it is not possible to write a direct matrix equation to analyze the circuit. The circuit can be analyzed by standard electrical circuit simulation software such as SPICE to predict the occurrence of the Braess paradox.

Diode Resistor Circuit for Braess Paradox Measurement



To verify the Braess Paradox in a diode resistor circuit was constructed.

Diodes D_b are 1N4148 silicon diodes. The two resistors labeled R_a are 330 Ω.

The current (demand), I, was chosen to be 1 mA.

The value of the fixed power supply voltage is 5 V and the value R_{sense} is set to 1000Ω.

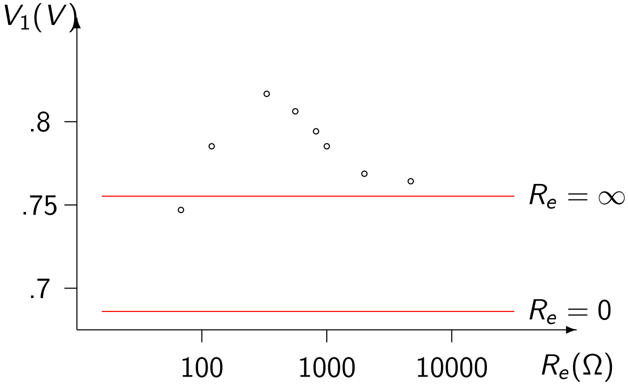
R_{adj} was adjusted to set the current in this case to 1 mA.

Diode Circuit Results

The value of R_e is varied from a short circuit, $R_e = 0$, to an open circuit, $R_e = \infty$, i.e., from a zero-cost link to when link e is not in the circuit. The upper horizontal line in the figure corresponds to the voltage at node 1, V_1, when link e is not included in the system. The lower horizontal line in the figure corresponds to V_1 when link e is a short circuit (wire). The circles are the measured values of V_1 at that value of resistance for link e.

If the Braess Paradox does not occur, the addition of a resistor as link e would lower the voltage at node 1, V_1, which would result in all measured values of V_1 being between the two horizontal lines. As can be seen from the figure, for link e resistances greater than 100 Ω, the results for the circuit illustrate that the equivalent resistance of the circuit increases when a resistor is added for link e; thus, confirming Braess Paradox behavior.

Measured Node 1 Voltage for a Diode Resistor Circuit



Summary

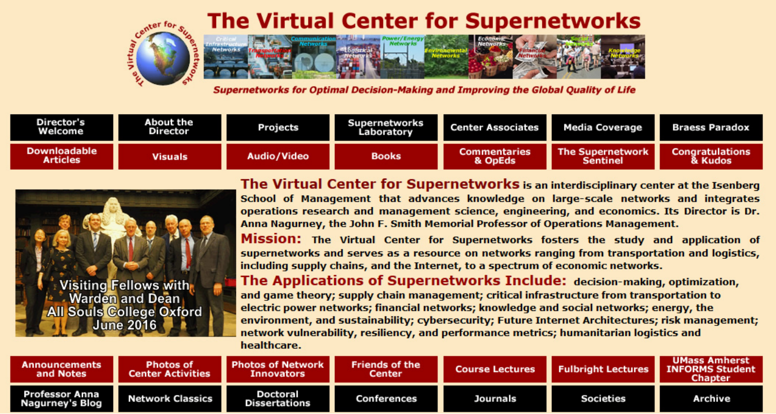
- We have explored the behavior of electrons flowing through an electrical circuit, which are governed by the same relationship that governs travelers driving in a road network and seeking their optimal routes of travel from origin nodes to destinations, acting independently.
- We proved that the Braess Paradox, originally proposed in user-optimized transportation networks, also can occur in electrical circuits, where the addition of a new link results in an increase in the voltage, rather than a decrease, as might be the expected.
- We provided examples in which cost functions are both linear as well as highly nonlinear and the same counterintuitive phenomenon is observed.

[1] Department of Electrical and Computer Engineering University of Hartford, West Hartford, CT 06117 and [2] Department of Operatio

Summary

- From an electrical circuit perspective, the circuits constructed and described demonstrate the development of a circuit structure where the current and voltage at a node may be independently controlled.
- This result enables the development of alternative circuit structures that can be exploited in constructing more complex circuits, which can be embedded in macro, micro, and mesoscale electrical circuit systems.
- Because of these results, appropriately designed electrical circuits can be used as testbeds to further explore the properties and range of occurrence of the Braess Paradox in a variety of network systems, including transportation.

Thank you



For more information see:

`http://supernet.isenberg.umass.edu`

References

Braess D., *Unternehmenforschung* **12** 1968 258.

Braess D., Nagurney A. and Wakolbinger T., *Transportation Science* **39** 2005 446.

Wardrop J. G., *Proceedings of the Institution of Civil Engineers* **Part II 1** 1952 325.

Beckmann M. J., McGuire C. B. and Winsten C. B., **Studies in the Economics of Transportation**, Yale University Press, New Haven, Connecticut 1956.

Sheffi Y., **Urban Transportation Networks – Equilibrium Analysis with Mathematical Programming Methods**, Prentice-Hall, Englewood Cliffs, New Jersey 1985.

Nagurney A., **Sustainable Transportation Networks.**, Edward Elgar Publishing, Cheltenham, England 2000.

References - II

Leblanc, L., *Transportation Science* **9** 1975 183.
Frank, M., *Mathematical Programmimg* **20** 1981 283.
Dafermos S. C. and Nagurney A., *Transportation Research B* **18** 1984 101.
Arnott, R. and K. Small, *American Scientist* **82** 1994 446.
Pas E. and Principio S., *Transportation Research B* **31** 1997 265.
Nagurney A., Parkes D. and Daniele P., **Computational Management Science 4** 2007 355.
Altman, E., V. Kambley and H. Kamedaz, *6th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks and Workshops, WiOPT 2008*, 2008 555.
Nagurney, A., *Europhsics Letters* **91** 2010 48002.

References - III

Motter, A., S. A. Myers, M. Anghel, and T. Nishikawa, *Nature Physics* **9** 2013 191.

Kolata, G., *The New York Times* 1990 December 25.

Baker L., *Scientific American* February 2009.

Cohen, J. and P. Horowitz, *Nature* **352** 1991 699.

Pala M.G., S. Baltazar, P. Liu, H. Sellier, B. Hackens, F. Martins, V. Bayot, X. Wallart, L. Desplanque, and S. Huant, *Physical Review Letters* **108** 2012 076802.

Pala, M., H. Sellier, B. Hackens, F. Martins, V. Bayot, and S. Huant, *Nanoscale Research Letters* **7** 2012 472.

Witthaut, D. and M. Timme, *New Journal of Physics* **14** 2012 083036.

References - IV

Penchina, C. M. and L. J. Penchina, *American Journal of Physics* **71** 2003 479.

Bjørndal M. and Jørnsten K., In **Pareto Optimality, Game Theory and Equilibria**. A. Chinchuluun, O. M. Pardalos and A. Migdalas, editors, Springer, Berlin, Germany 2008 593.

Bloy, L. A. K., **An Investigation into Braess' Paradox** Master's Thesis, University of South Africa 2007