# A Cournot-Nash–Bertrand Game Theory Model of a Service-Oriented Internet with Price and Quality Competition Among Network Transport Providers


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# Outline

- Background and Motivation
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- The Underlying Dynamics
- The Algorithm
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# Background and Motivation

# Envisioning a New Kind of Internet

We are one of five teams funded by NSF as part of the Future Internet Architecture (FIA) project.

Our project is: *Network Innovation Through Choice* and the envisioned architecture is *ChoiceNet*.

Team:

University of Kentucky: Jim Griffioen, Ken Calvert
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# ChoiceNet Principles

*Competition Drives Innovation!*

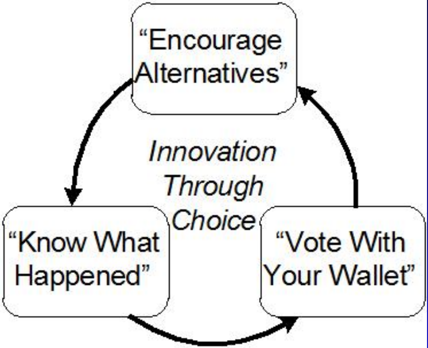
Services are at core of ChoiceNet ("everything is a service")

Services provide a benefit, have a cost
Services are created, composed, sold, verified, etc.

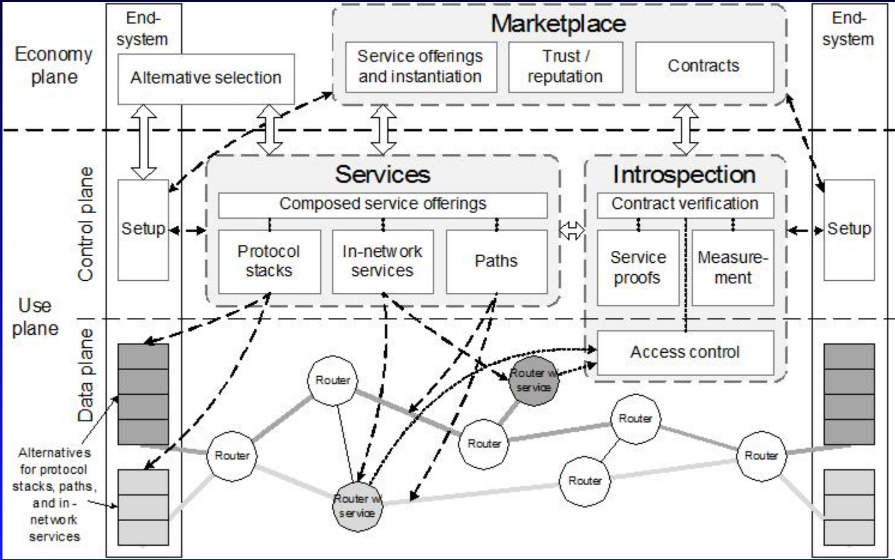
"Encourage alternatives" Provide building blocks for different types of services

"Know what happened" Ability to evaluate services

"Vote with your wallet" Reward good services!



# ChoiceNet Architecture



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• Although the technology associated with the existing Internet is rather well-understood, the economics of the associated services have been less studied.

• Modeling and computational frameworks that capture the competitive behavior of decision-makers ranging from service providers to network providers are still in their infancy. This may be due, in part, to unawareness of appropriate methodological frameworks.

# Some Features of Our Model

We build on the recent work on game theory frameworks for a service-oriented Internet with the goal of expanding the generality of applicable game theory models that are also computable.

By services we mean not only content, such as news, videos, music, etc., but also services associated with, for example, cloud computing.

The game theory model that we present here is inspired by that of Zhang, Nabipay, Odlyzko, and Guerin (2010) who employed Cournot and Bertrand games to model competition among service providers and among network providers, with the former competing in a Cournot manner, and the latter in a Bertrand manner. The two types of competition were then unified in a Stackelberg game.

# Some Features of Our Model

Zhang et al. (2010) focused only on a two service provider, two network provider, and two user network configuration along with a linear demand function to enable closed form analytical solutions. They did not capture the quality of network provision.

Altman et al. (2011) emphasized the need for metrics for quality of service and the Internet and also provided an excellent review of game theory models, and noted that many of the models in the existing literature considered only one or two service providers.

# Some Features of Our Model

A notable feature of our modeling approach is that it allows for *composition*, in that users at demand markets have associated demand price functions that reflect how much they are willing to pay for the service and the network provision combination, as a function of service volumes and quality levels. Such an idea is motivated, in part, to provide consumers with more choices (see Wolf et al. (2012)).

Our framework can be used as the foundation for the further disaggregation of decision-making and the inclusion of additional topological constructs, say, in expanding the paths, which may reflect the transport of services at the more detailed level of expanded sequences of links.

# Some More References

Our contributions fall under *network economics* as well as *computational management science*.

Some of the early papers on network economics and the Internet are the works of: MacKie-Mason and Varian (1995), Varian (1996), Kelly (1997), MacKnight and Bailey (1997), Kausar, Briscoe, and Crowcroft (1999), and Odlyzko (2000).

More recent contributions: Ros and Tuffin (2004), He and Walrand (2005), Shakkottai and Srikant (2006), Shen and Basar (2007), and Neely (2007).

# Some More References

Lv and Rouskas (2010) focused on the modeling of Internet service providers and the pricing of tiered network services. They provided both models and an algorithm, along with computational results, a contribution that is rare in this stream of literature. They assumed that the users are homogeneous, whereas we consider distinct demand price functions associated with the demand markets and the composition of service provider services and network provision.

# The Game Theory Model

# The Model

There are $m$ service providers, with a typical service provider denoted by $i$, $n$ network providers, which provide "transport" of the services to the demand markets, with a typical one denoted by $j$, and $o$ demand markets associated with the users of the services and network provision. A typical demand market is denoted by $k$.

The service providers offer multiple different services such as movies for video streaming, music for downloading, news, etc. Users can select among different service offerings (e.g., movie streaming from service provider 1 vs. movie streaming from service provider 2).

Different network providers can be used for data communication over the Internet (i.e., "transport") between the service providers and the users.

In addition, we explicitly handle the quality level among network providers.

# The Model

We allow for consumers to differentiate among the services provided by the service providers.

It is assumed that the service providers compete under the Cournot-Nash equilibrium concept of non-cooperative behavior and select their service volumes (quantities).

The network providers, in turn, compete with prices a la Bertrand and with quality levels.

The consumers, in turn, signal their preferences for the services and network provision via the demand price functions associated with the demand markets. The demand price functions are, in general, functions of the service/network provision combinations at all the demand markets as well as the quality levels of network provision, since the focus here is on *composition* and having choices.

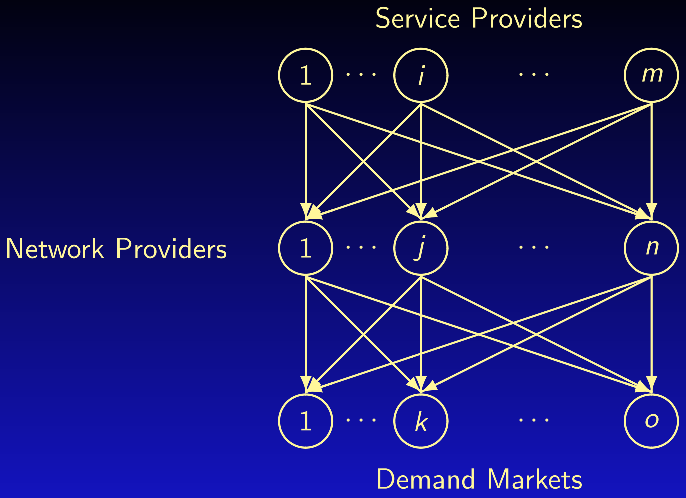


Figure 1: The Network Structure of the Cournot-Nash-Bertrand Model for a Service-Oriented Internet

Table 1: Notation for the Game Theoretic Cournot-Nash-Bertrand Model

| Notation | Definition |
|---|---|
| $Q_{ijk}$ | nonnegative service volume from $i$ to $k$ via $j$. We group the $\{Q_{ijk}\}$ elements into vector $Q \in R^{mno}_+$. |
| $s_i$ | service volume (output) produced by service provider $i$. We group the $\{s_i\}$ elements into vector $s \in R^m_+$. |
| $d_{ijk}$ | demand for service $i$ transported by $j$ to demand market $k$. We group the $\{d_{ijk}\}$ elements into vector $d \in R^{mno}$. |
| $q_{ijk}$ | nonnegative quality level of network provider $j$ transporting service $i$ to $k$. We group $\{q_{ijk}\}$ elements into vector $q \in R^{mno}_+$. |
| $\pi_{ijk}$ | price charged by network provider $j$ for transporting a unit of service provided by $i$ via $j$ to $k$. We group the $\{\pi_{ijk}\}$ elements into vector $\pi \in R^{mno}$. |
| $f_i(s)$ | total production cost of service provider $i$. |
| $\rho_{ijk}(d,q)$ | demand price at $k$ with service $i$ transported via $j$. |
| $c_{ijk}(Q,q)$ | transportation cost with delivering service $i$ via $j$ to $k$. |
| $oc_{ijk}(\pi_{ijk})$ | opportunity cost with pricing by network provider $j$ services from $i$ to $k$. |

# The Behavior of the Service Providers and Their Optimality Conditions

The service providers seek to maximize their individual profits, where the profit function for service provider $i$; $i = 1, \ldots, m$ is given by the expression:

$$\sum_{j=1}^{n} \sum_{k=1}^{o} \rho_{ijk}(d, q^*) Q_{ijk} - f_i(s) - \sum_{j=1}^{n} \sum_{k=1}^{o} \pi^*_{ijk} Q_{ijk} \tag{1}$$

subject to the constraints:

$$s_i = \sum_{j=1}^{n} \sum_{k=1}^{o} Q_{ijk}, \quad i = 1, \ldots, m, \tag{2}$$

$$d_{ijk} = Q_{ijk}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{3}$$

$$Q_{ijk} \geq 0, \quad j = 1, \ldots, n; k = 1, \ldots, o. \tag{4}$$

# The Behavior of the Service Providers and Their Optimality Conditions

In view of constraint (2), we can define the production cost functions $\hat{f}_i(Q)$; $i = 1, \ldots, m$, as follows:

$$\hat{f}_i(Q) \equiv f_i(s), \tag{5}$$

and, in view of constraint (3), we can also define the demand price functions $\hat{\rho}_{ijk}(Q, q)$; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$, such that

$$\hat{\rho}_{ijk}(Q, q) \equiv \rho_{ijk}(d, q). \tag{6}$$

We assume that the production cost and the demand price functions are continuous and continuously differentiable. We also assume that the production cost functions are convex and that the demand price functions are monotonically decreasing in service volumes but increasing in the quality of network provision.

# The Behavior of the Service Providers and Their Optimality Conditions

Therefore, the profit maximization problem for service provider $i$; $i = 1, \dots, m$, with its profit expression denoted by $U_i^1$, which also represents its utility function, with the superscript 1 reflecting the first (top) tier of decision-makers in Figure 1, can be reexpressed as:

$$\text{Maximize } U_i^1(Q, q^*, \pi^*) = \sum_{j=1}^{n}\sum_{k=1}^{o} \hat{\rho}_{ijk}(Q, q^*)Q_{ijk} - \hat{f}_i(Q)$$

$$-\sum_{j=1}^{n}\sum_{k=1}^{o} \pi_{ijk}^* Q_{ijk} \quad (7)$$

subject to: (4).

# The Behavior of the Service Providers and Their Optimality Conditions

For service provider $i$, we group all its $\{Q_{ijk}\}$ elements, which are its strategic variables, into vector $Q_i$. The strategic variables of service provider $i$ are its service transport volumes $\{Q_i\}$. In view of (1) - (7), we may write the profit functions of the service providers as functions of the service provision/transportation pattern, that is,

$$U^1 = U^1(Q, q, \pi), \tag{8}$$

where $U^1$ is the $m$-dimensional vector with components: $\{U_1^1, \ldots, U_m^1\}$. Let $K^{1i}$ denote the feasible set corresponding to service provider $i$, where $K^{1i} \equiv \{Q_i | Q_i \geq 0\}$ and define $K^1 \equiv \prod_{i=1}^m K^{1i}$.

# The Behavior of the Service Providers and Their Optimality Conditions

We consider the oligopolistic market mechanism, in which the $m$ service providers supply their services in a non-cooperative fashion, each one trying to maximize its own profit.

We seek to determine a nonnegative service volume pattern $Q^*$ for which the $m$ service providers will be in a state of equilibrium as defined below. In particular, Nash (1950, 1951) generalized Cournot's concept of an equilibrium among several players, in what has been come to be called a non-cooperative game.

**Definition 1: Cournot-Nash Equilibrium with Service Differentiation and Network Provision Choices.** *A service volume pattern $Q^* \in K^1$ is said to constitute a Cournot-Nash equilibrium if for each service provider $i$; $i = 1, \ldots, m$:*

$$U_i^1(Q_i^*, \hat{Q}_i^*, q^*, \pi^*) \geq U_i^1(Q_i, \hat{Q}_i^*, q^*, \pi^*), \quad \forall Q_i \in K^{1i}, \qquad (9)$$

*where*

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*). \qquad (10)$$

**Theorem 1: Variational Inequality Formulations of Cournot-Nash Equilibrium.** *Assume that for each service provider i the profit function* $U_i^1(Q,q,\pi)$ *is concave with respect to the variables in* $\{Q_i\}$ *and is continuous and continuously differentiable. Then,* $Q^* \in K^1$ *is a Cournot-Nash equilibrium according to Definition 1 if and only if it satisfies*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial U_i^1(Q^*,q^*,\pi^*)}{\partial Q_{ijk}} \times (Q_{ijk}-Q_{ijk}^*) \geq 0, \quad \forall Q \in K^1, \tag{11}$$

*or, equivalently,* $Q^* \in K^1$ *is a Cournot-Nash equilibrium service volume pattern if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}} + \pi_{ijk}^* - \rho_{ijk}(Q^*,q^*) - \sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \rho_{ihl}(Q^*,q^*)}{\partial Q_{ijk}} \times Q_{ihl}^*\right] \times (Q_{ijk}-Q_{ijk}^*) \geq 0, \quad \forall Q \in K^1. \tag{12}$$

**Proof:** (11) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987). In order to obtain (12) from (11), we note that $\forall i, j, k$:

$$-\frac{\partial U_i^1(Q^*, q^*, \pi^*)}{\partial Q_{ijk}}$$

$$= \left[ \frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}} + \pi^*_{ijk} - \rho_{ijk}(Q^*, q^*) - \sum_{h=1}^{n} \sum_{l=1}^{o} \frac{\partial \rho_{ihl}(Q^*, q^*)}{\partial Q_{ijk}} \times Q^*_{ihl} \right] . \quad (13)$$

Multiplying the expression in (13) by $(Q_{ijk} - Q^*_{ijk})$ and summing the resultant over all $i$, $j$, and $k$ yields (12). □

# The Behavior of the Network Providers and Their Optimality Conditions

The network providers also seek to maximize their individual profits. They have as their strategic variables the prices that they charge for the transport of the services and the quality levels. The optimization problem faced by network provider $j$; $j = 1, \ldots, n$ is given by

$$\text{Maximize } U_j^2(Q^*, q, \pi) = \sum_{i=1}^{m}\sum_{k=1}^{o} \pi_{ijk} Q_{ijk}^* - \sum_{i=1}^{m}\sum_{k=1}^{o} c_{ijk}(Q^*, q) - oc_{ijk}(\pi_{ijk}) \tag{14}$$

subject to:

$$\pi_{ijk} \geq 0, \quad i = 1, \ldots, m; k = 1, \ldots, o, \tag{15}$$

$$q_{ijk} \geq 0, \quad i = 1, \ldots, m; k = 1, \ldots, o. \tag{16}$$

# The Behavior of the Network Providers and Their Optimality Conditions

We group network provider $j$'s prices $\{\pi_{ijk}\}$ into the vector $\pi_j$ and its quality levels $\{q_{ijk}\}$ into the vector $q_j$. We also group the network provider utility functions, as given in (14), into the vector $U^2$ as in (17):

$$U^2 = U^2(Q, q, \pi). \tag{17}$$

Let $K^{2j}$ denote the feasible set corresponding to network provider $j$, such that $K^{2j} \equiv \{(q_j, \pi_j) | q_j \geq 0, \pi_j \geq 0\}$ and $K^2 \equiv \prod_{j=1}^{n} K^{2j}$.

**Definition 2: Bertrand Equilibrium in Transport Prices and Quality.** *A quality level pattern and transport price pattern* $(q^*, \pi^*) \in K^2$ *is said to constitute a Bertrand equilibrium if for each network provider* $j$*;* $j = 1, \ldots, n$*:*

$$U_j^2(Q^*, q_j^*, \hat{q}_j^*, \pi_j^*, \hat{\pi}_j^*) \geq U_j^2(Q^*, q_j, \hat{q}_j^*, \pi_j, \hat{\pi}_j^*), \quad \forall (q_j, \pi_j) \in K^{2j}, \tag{18}$$

*where*

$$\hat{q}_j^* \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_n^*), \tag{19}$$

$$\hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_n^*). \tag{20}$$

According to (18), a Bertrand equilibrium is established if no network provider can unilaterally improve upon its profits by selecting an alternative vector of quality levels and transport prices.

**Theorem 2: Variational Inequality Formulations of Bertrand Equilibrium.** *Assume that for each network provider $j$ the profit function $U_j^2(Q,q,\pi)$ is concave with respect to the variables in $\{q_j\}$ and in $\{\pi_j\}$ and is continuous and continuously differentiable. Then, $(q^*,\pi^*) \in K^2$ is a Bertrand equilibrium according to Definition 2 if and only if it satisfies the variational inequality*

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*,q^*,\pi^*)}{\partial q_{ijk}} \times (q_{ijk} - q_{ijk}^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*,q^*,\pi^*)}{\partial \pi_{ijk}} \times (\pi_{ijk} - \pi_{ijk}^*) \geq 0, \quad \forall (q,\pi) \in K^2, \tag{21}$$

*or, equivalently,* $(q^*, \pi^*) \in K^2$ *is a Bertrand price and quality level equilibrium pattern if and only if it satisfies*

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial c_{hjl}(Q^*, q^*)}{\partial q_{ijk}}\right] \times (q_{ijk} - q^*_{ijk})$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[-Q^*_{ijk} + \frac{\partial oc(\pi^*_{ijk})}{\partial \pi_{ijk}}\right] \times (\pi_{ijk} - \pi^*_{ijk}) \geq 0, \quad \forall (q, \pi) \in K^2. \tag{22}$$

**Proof:** Similar to the proof of Theorem 1.

# The Integrated Cournot-Nash-Bertrand Equilibrium Conditions and Variational Inequality Formulations

We are now ready to present the Cournot-Nash-Bertrand equilibrium conditions. We let $K^3 \equiv K^1 \times K^2$ denote the feasible set for the integrated model. We assume the same assumptions on the functions as previously.

**Definition 3: Cournot-Nash-Bertrand Equilibrium in Service Differentiation, Transport Network Prices, and Quality.** *A service volume, quality level, and transport price pattern $(Q^*, q^*, \pi^*) \in K^3$ is a Cournot-Nash-Bertrand equilibrium if it satisfies (9) and (18) simultaneously.*

**Theorem 3: Variational Inequality Formulations of Cournot-Nash-Bertrand Equilibrium.** *Under the same assumptions as given in Theorems 1 and 2, $(Q^*, q^*, \pi^*) \in K^3$ is a Cournot-Nash-Bertrand equilibrium according to Definition 3 if and only if it satisfies the variational inequality:*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial U_i^1(Q^*, q^*, \pi^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial q_{ijk}} \times (q_{ijk} - q_{ijk}^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial \pi_{ijk}} \times (\pi_{ijk} - \pi_{ijk}^*) \geq 0, \quad \forall (Q, q, \pi) \in K^3, \tag{23}$$

*or, equivalently, the variational inequality problem:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}} + \pi^*_{ijk}\right.$$

$$\left. -\rho_{ijk}(Q^*, q^*) - \sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \rho_{ihl}(Q^*, q^*)}{\partial Q_{ijk}} \times Q^*_{ihl}\right] \times (Q_{ijk} - Q^*_{ijk})$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial c_{hjl}(Q^*, q^*)}{\partial q_{ijk}}\right] \times (q_{ijk} - q^*_{ijk})$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[-Q^*_{ijk} + \frac{\partial oc(\pi^*_{ijk})}{\partial \pi_{ijk}}\right] \times (\pi_{ijk} - \pi^*_{ijk}) \geq 0, \quad \forall (Q, q, \pi) \in K^3. \tag{24}$$

We now put variational inequality (24) into standard form: determine $X^* \in \mathcal{K}$ where $X$ is a vector in $R^N$, $F(X)$ is a continuous function such that $F(X) : X \mapsto \mathcal{K} \subset R^N$, and

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{25}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the $N$-dimensional Euclidean space, and $\mathcal{K}$ is closed and convex. We define the vector $X \equiv (Q, q, \pi)$ and $\mathcal{K} \equiv K^3$. Also, here $N = 3mno$. The components of $F$ are then given by: for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$:

$$F^1_{ijk}(X) = \frac{\partial \hat{f}_i(Q)}{\partial Q_{ijk}} + \pi_{ijk} - \rho_{ijk}(Q, q) - \sum_{h=1}^{n} \sum_{l=1}^{o} \frac{\partial \rho_{ihl}(Q, q)}{\partial Q_{ijk}} \times Q_{ihl}, \tag{26}$$

$$F^2_{ijk}(X) = \sum_{h=1}^{m} \sum_{l=1}^{o} \frac{\partial c_{hjl}(Q, q)}{\partial q_{ijk}}, \tag{27}$$

$$F^3_{ijk}(X) = -Q_{ijk} + \frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}}. \tag{28}$$

## The Underlying Dynamics

# The Underlying Dynamics

In our framework, the rate of change of the service volume between a service provider $i$ and demand market $k$ via network provider $j$ is in proportion to $-F^1_{ijk}(X)$, as long as the service volume $Q_{ijk}$ is positive. Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U^1_i(Q, q, \pi)}{\partial Q_{ijk}}, \tag{29}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of $Q_{ijk}$. However, when $Q_{ijk} = 0$, the nonnegativity condition (4) forces the service volume $Q_{ijk}$ to remain zero when $\frac{\partial U^1_i(Q,q,\pi)}{\partial Q_{ijk}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the service volume. Namely, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\{0, \frac{\partial U^1_i(Q, q, \pi)}{\partial Q_{ijk}}\}. \tag{30}$$

# The Underlying Dynamics

We may write (29) and (30) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U_i^1(Q,q,\pi)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\{0, \frac{\partial U_i^1 Q,q,\pi)}{\partial Q_{ijk}}\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \qquad (31)$$

# The Underlying Dynamics

As for the quality levels (cf. (16)), when $q_{ijk} > 0$, then

$$\dot{q}_{ijk} = \frac{\partial U_j^2(Q, q, \pi)}{\partial q_{ijk}}, \tag{32}$$

where $\dot{q}_{ijk}$ denotes the rate of change of $q_{ijk}$; otherwise:

$$\dot{q}_{ijk} = \max\{0, \frac{\partial U_j^2(Q, q, \pi)}{\partial q_{ijk}}\}, \tag{33}$$

since $q_{ijk}$ must be nonnegative.
Combining (32) and (33), we may write:

$$\dot{q}_{ijk} = \begin{cases} \frac{\partial U_j^2(Q,q)}{\partial q_{ijk}}, & \text{if} \quad q_{ijk} > 0 \\ \max\{0, \frac{\partial U_j^2(Q,q)}{\partial q_{ijk}}\}, & \text{if} \quad q_{ijk} = 0. \end{cases} \tag{34}$$

# The Underlying Dynamics and Stability Analysis

Using similar arguments as above, we can conclude that for the network transport prices, when $\pi_{ijk} > 0$, then

$$\dot{\pi}_{ijk} = \frac{\partial U_j^2(Q, q, \pi)}{\partial \pi_{ijk}}, \tag{35}$$

where $\dot{\pi}_{ijk}$ denotes the rate of change of $\pi_{ijk}$; otherwise (cf. (15)):

$$\dot{\pi}_{ijk} = \max\{0, \frac{\partial U_j^2(Q, q, \pi)}{\partial \pi_{ijk}}\}, \tag{36}$$

since $\pi_{ijk}$ must be nonnegative. Hence, we have that:

$$\dot{\pi}_{ijk} = \begin{cases} \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}, & \text{if} \quad \pi_{ijk} > 0 \\ \max\{0, \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}\}, & \text{if} \quad \pi_{ijk} = 0. \end{cases} \tag{37}$$

# The Underlying Dynamics

Applying (31), (34), and (37) to all $i = 1, \ldots, m$; $j = 1, \ldots, n$, and $k = 1, \ldots, o$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the service volumes, quality levels, and transport network prices, in vector form, as:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{38}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at $X$ defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{39}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathcal{K}} \|X - z\|, \tag{40}$$

and where $\| \cdot \| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q, \pi)$, where $\nabla U(Q, q, \pi)$ is the vector of marginal utilities (profits).

We cite the following theorem from Dupuis and Nagurney (1993). See also the book by Nagurney and Zhang (1996).

**Theorem 4**
*$X^*$ solves the variational inequality problem (25) if and only if it is a stationary point of the ODE (38), that is,*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{41}$$

This theorem demonstrates that the necessary and sufficient condition for a pattern $X^* = (Q^*, q^*, \pi^*)$ to be a Cournot-Nash-Bertrand equilibrium, according to Definition 3, is that $X^* = (Q^*, q^*, \pi^*)$ is a stationary point of the adjustment process defined by ODE (38), that is, $X^*$ is the point at which $\dot{X} = 0$.

# The Algorithm

The projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). Specifically, iteration $\tau$ of the Euler method (see also Nagurney and Zhang (1996)) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau}F(X^{\tau})), \tag{42}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and $F$ is the function that enters the variational inequality problem (19).

# The Algorithm

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$.

Specific conditions for convergence of this scheme as well as various applications to the solutions of other game theory models can be found in Nagurney, Dupuis, and Zhang (1994), Nagurney, Takayama, and Zhang (1995), Cruz (2008), Nagurney (2010), and Nagurney and Li (2012).

# The Algorithm

The elegance of this procedure for the computation of solutions to our model (in both the dynamic and static, that is, equilibrium, versions) can be seen in the following explicit formulae.

**Explicit Formulae for the Euler Method Applied to the Cournot-Nash-Bertrand Game Theory Model**

We have the following closed form expression for the service volumes:

$$Q_{ijk}^{\tau+1} = \max\{0, Q_{ijk}^{\tau} + a_{\tau}(\rho_{ijk}(Q^{\tau}, q^{\tau}) + \sum_{h=1}^{n}\sum_{l=1}^{o} \frac{\partial \rho_{ihl}(Q^{\tau}, q^{\tau})}{\partial Q_{ijk}} \times Q_{ihl}^{\tau}$$

$$-\pi_{ijk}^{\tau} - \frac{\partial \hat{f}_i(Q^{\tau})}{\partial Q_{ijk}})\}, \forall i, j, k,$$

and the following closed form expression for the quality levels:

$$q_{ijk}^{\tau+1} = \max\{0, q_{ijk}^{\tau} + a_{\tau}(-\sum_{h=1}^{m}\sum_{l=1}^{o} \frac{\partial c_{hjl}(Q^{\tau}, q^{\tau})}{\partial q_{ijk}})\}, \forall i, j, k,$$

# The Algorithm

with the explicit formulae for the network transport prices being:

$$\pi_{ijk}^{\tau+1} = \max\{0, \pi_{ijk}^{\tau} + a_\tau (Q_{ijk}^{\tau} - \frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}})\}, \forall i, j, k.$$

# The Algorithm

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

**Theorem 5**

*In the Cournot-Nash-Bertrand model for a service-oriented Internet, let* $F(X) = -\nabla U(Q, q, \pi)$ *be strongly monotone. Also, assume that* $F$ *is uniformly Lipschitz continuous. Then there exists a unique equilibrium service volume, quality leel, and price pattern* $(Q^*, q^*, \pi^*) \in \mathcal{K}$ *and any sequence generated by the Euler method as given by (42) above, where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* $(Q^*, q^*, \pi^*)$.

# Numerical Examples

# Numerical Examples

We applied the Euler method to compute the Cournot - Nash - Bertrand equilibrium for several examples. We set $\{a_\tau\} = 1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$. The convergence criterion was that the absolute value of the difference of the iterates at two successive iterations was less than or equal to $10^{-4}$. There were 3 service providers, 2 network providers, and 2 demand markets.

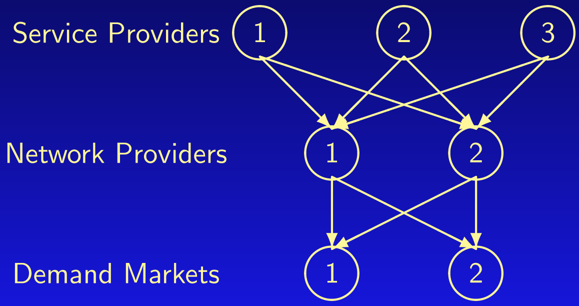


Figure 2: Network Topology for the Numerical Examples

# Baseline Example 1

The data for this numerical example, from which we then construct subsequent variants, were as follows.

The production cost functions were:

$$\hat{f}_1 = 2(Q_{111} + Q_{112} + Q_{121} + Q_{122})^2 + (Q_{111} + Q_{112} + Q_{121} + Q_{122}),$$
$$\hat{f}_2 = (Q_{211} + Q_{212} + Q_{221} + Q_{222})^2 + (Q_{211} + Q_{212} + Q_{221} + Q_{222}),$$
$$\hat{f}_3 = 3(Q_{311} + Q_{312} + Q_{321} + Q_{322})^2 + (Q_{311} + Q_{312} + Q_{321} + Q_{322}).$$

The demand price functions were:

$$\hat{\rho}_{111} = -Q_{111} - .5Q_{112} + q_{111} + 100, \quad \hat{\rho}_{112} = -2Q_{112} - 1Q_{111} + q_{112} + 200,$$
$$\hat{\rho}_{121} = -2Q_{121} - .5Q_{111} + .5q_{121} + 100, \quad \hat{\rho}_{122} = -3Q_{122} - Q_{112} + .5q_{122} + 150,$$
$$\hat{\rho}_{211} = -1Q_{211} - .5Q_{212} + .3q_{211} + 100, \quad \hat{\rho}_{212} = -3Q_{212} + .8q_{212} + 200,$$
$$\hat{\rho}_{221} = -2Q_{221} - 1Q_{222} + q_{221} + 140, \quad \hat{\rho}_{222} = -3Q_{222} - Q_{121} + q_{221} + 300,$$
$$\hat{\rho}_{311} = -4Q_{311} + .5q_{311} + 230, \quad \hat{\rho}_{312} = -2Q_{312} - Q_{321} + .3q_{312} + 150,$$
$$\hat{\rho}_{321} = -3Q_{321} - Q_{311} + .2q_{321} + 200, \quad \rho_{322} = -4Q_{322} + .7q_{322} + 300.$$

# Baseline Example 1

The transportation cost functions were:

$$c_{111} = q_{111}^2 - .5q_{111}, \; c_{112} = .5q_{112}^2 - q_{112}, \; c_{121} = .1q_{121}^2 - q_{121}, \; c_{122} = q_{122}^2,$$

$$c_{211} = .1q_{211}^2 - q_{211}, \; c_{212} = q_{212}^2 - .5q_{212}, \; c_{221} = 2q_{221}^2, \; c_{222} = .5q_{222}^2 - q_{222},$$

$$c_{311} = q_{311}^2 - q_{311}, \; c_{312} = .5q_{312}^2 - q_{312}, \; c_{321} = q_{321}^2 - q_{321}, \; c_{322} = 2q_{322}^2 - 2q_{322}.$$

The opportunity cost functions were:

$$oc_{111} = 2\pi_{111}^2, \; oc_{112} = 2\pi_{112}^2, \; oc_{121} = \pi_{121}^2, \; oc_{122} = .5\pi_{122}^2,$$

$$oc_{211} = \pi_{211}^2, \; oc_{212} = .5\pi_{212}^2, \; oc_{221} = 2\pi_{221}^2, \; oc_{222} = 1.5\pi_{222}^2,$$

$$oc_{311} = \pi_{311}^2, \; oc_{312} = 2.5\pi_{312}^2, \; oc_{321} = 1.5\pi_{321}^2, \; oc_{322} = \pi_{322}^2.$$

The Euler method converged in 432 iterations and yielded the approximation to the equilibrium solution reported in Table 2.

Table 2: Equilibrium Solution for the Baseline Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.00 | 0.25 | 0.00 |
| 1 | 1 | 2 | 22.67 | 1.00 | 5.67 |
| 1 | 2 | 1 | 0.00 | 5.00 | 0.00 |
| 1 | 2 | 2 | 3.24 | 0.00 | 3.24 |
| 2 | 1 | 1 | 0.00 | 5.00 | 0.00 |
| 2 | 1 | 2 | 14.53 | 0.25 | 14.53 |
| 2 | 2 | 1 | 2.24 | 0.00 | 0.56 |
| 2 | 2 | 2 | 31.97 | 1.00 | 10.66 |
| 3 | 1 | 1 | 7.55 | 0.50 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.18 | 0.50 | 1.39 |
| 3 | 2 | 2 | 15.80 | 0.50 | 7.90 |

# Baseline Example 1

The profit of service provider 1 was: 2402.31 and that of service provider 2: 6086.77 and service provider 3: 3549.49. The profit of network provider 1 was: 184.04 and that of network provider 2 was: 241.54.

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Demand market 2, however, obtains services from all three service providers. Network provider 1 handles positive volumes of services from all service providers as does network provider 2. It is also interesting to see that two of the quality levels are equal to zero.

Noting that $Q^*_{111} = 0.00$ we then constructed Variant 1 as follows.

# Example 2: Variant 1 of Example 1

We explored the effects of a change in the price function $\rho_{111}$ since, in Example 1, $Q^*_{111} = 0.00$. Such a change in a price function could occur, for example, through enhanced marketing.

Specifically, we sought to determine the change in the equilibrium pattern if the consumers at demand market 1 are willing to pay more for the services of the service provider 1 and network provider 1 combination.

The new demand price function was:

$$\hat{\rho}_{111}(Q, q) = -Q_{111} - .5Q_{112} + q_{111} + 200,$$

with the remainder of the data as in Example 1. The new computed solution is reported in Table 3. The algorithm converged in 431 iterations.

Table 3: Equilibrium Solution for Example 2: Variant 1 of Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 25.40 | 0.25 | 6.35 |
| 1 | 1 | 2 | 8.67 | 1.00 | 2.17 |
| 1 | 2 | 1 | 0.00 | 4.45 | 0.00 |
| 1 | 2 | 2 | 0.37 | 0.00 | 0.37 |
| 2 | 1 | 1 | 0.00 | 4.45 | 0.00 |
| 2 | 1 | 2 | 14.52 | 0.25 | 14.53 |
| 2 | 2 | 1 | 2.24 | 0.00 | 0.56 |
| 2 | 2 | 2 | 31.97 | 1.00 | 10.66 |
| 3 | 1 | 1 | 7.55 | 0.50 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.18 | 0.50 | 1.39 |
| 3 | 2 | 2 | 15.80 | 0.50 | 7.90 |

# Example 2: Variant 1 of Example 1

The profit of service provider 1 was now: 3168.18. The profits of the other two service providers remained as in Example 1. The profit of network provider 1 was now: 209.85 and that of network provider 2 was: 236.35.

# Example 2: Variant 1 of Example 1

The profit of service provider 1 was now: 3168.18. The profits of the other two service providers remained as in Example 1. The profit of network provider 1 was now: 209.85 and that of network provider 2 was: 236.35.

Hence, both service provider 1 and network provider 1 had higher profits than in Example 1 and the service volume $Q^*_{111}$ increased from 0.00 to 25.40. There was a reduction in service volume $Q^*_{112}$ and in $Q^*_{122}$.

In the final example, we returned to Example 1 and modified all of the transportation cost functions to include an additional term: $Q_{ijk}q_{ijk}$ to reflect that cost could depend on both congestion level and on quality of transport.

The solution obtained via the Euler method for this example is given in Table 4. The Euler method required 705 iterations for convergence.

Table 4: Equilibrium Solution for Example 3: Variant 2 of Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.00 | 0.25 | 0.00 |
| 1 | 1 | 2 | 22.52 | 0.00 | 5.63 |
| 1 | 2 | 1 | 0.00 | 4.98 | 0.00 |
| 1 | 2 | 2 | 3.31 | 0.00 | 3.31 |
| 2 | 1 | 1 | 0.00 | 4.99 | 0.00 |
| 2 | 1 | 2 | 14.52 | 0.00 | 14.52 |
| 2 | 2 | 1 | 2.31 | 0.00 | 0.58 |
| 2 | 2 | 2 | 31.84 | 0.00 | 10.61 |
| 3 | 1 | 1 | 7.53 | 0.00 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.19 | 0.00 | 1.40 |
| 3 | 2 | 2 | 15.77 | 0.00 | 7.89 |

# Example 3: Variant 2 of Example 1

The profit of service provider 1 was now: 2380.87. The profit of service provider 2 was: 6053.76 and that of service provider 3 was: 3541.93. The profit of network provider 1 was now: 181.89 and that of network provider 2 was: 237.21.

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Observe that, in this, as in the previous two examples, if $Q^*_{ijk} = 0$, then the price $\pi^*_{ijk} = 0$, which is reasonable.

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Observe that, in this, as in the previous two examples, if $Q^*_{ijk} = 0$, then the price $\pi^*_{ijk} = 0$, which is reasonable.

It is interesting to note that, in this example, the inclusion of an additional term $Q_{ijk}q_{ijk}$ to each transportation cost function $c_{ijk}$, with the remainder of the data as in Example 1, results in a decrease in the quality levels in eight out of the twelve computed equilibrium variable values, with the other quality values remaining unchanged.

Having an effective modeling and computational framework allows one to explore the effects of changes in the underlying functions on the equilibrium pattern to gain insights that may not be apparent from smaller scale, analytical solutions.

# Summary and Conclusions

- We developed a game theory model for a service-oriented Internet. The motivation for the research stems, in part, from a need to understand the underlying economics of a service-oriented Internet with more choices as well as to demonstrate the integration of complex competitive behaviors on multitiered networks.

# Summary and Conclusions

• We developed a game theory model for a service-oriented Internet. The motivation for the research stems, in part, from a need to understand the underlying economics of a service-oriented Internet with more choices as well as to demonstrate the integration of complex competitive behaviors on multitiered networks.

• We developed both static and dynamic versions of the Cournot-Nash-Bertrand game theory model in which the service providers offer differentiated, but substitutable, services and the network providers transport the services to consumers at the demand markets. Consumers respond to the composition of service and network provision choices and to the quality levels and service volumes, through the prices. The service providers compete in a Cournot-Nash manner, whereas the network providers compete a la Bertrand in prices charged for the transport of the services, as well as with the quality levels associated with the transport.

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• The paper on which this presentation is based has additional results in the form of stability analysis along with illustrative examples.

# THANK YOU!


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