A Relative Total Cost Index for the Evaluation of Transportation Network Robustness in the Presence of Degradable Links and Alternative Travel Behavior

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The U.S. Ailing Infrastructure

- The U.S. is experiencing a freight capacity crisis that threatens the strength and productivity of the U.S. economy. According to the American Road & Transportation Builders Association (see Jeanneret (2006)), nearly 75% of US freight is carried in the US on highways and bottlenecks are causing truckers 243 million hours of delay annually with an estimated associated cost of $8 billion.

- It is noted that the US government is facing a $1.6 trillion deficit over the five years in terms of infrastructure repairing and reconstruction according to a recent estimate (Environment News Service (2008)).
The U.S. Ailing Infrastructure

- **Over one-quarter** of the nation’s 590,750 bridges were rated structurally deficient or functionally obsolete. The degradation of transportation networks due to poor maintenance, natural disasters, deterioration over time, as well as unforeseen attacks now lead to estimates of **$94 billion** in the U.S. in terms of needed repairs for roads alone. Poor road conditions in the U.S. cost motorists **$54 billion** in repairs and operating costs annually (ASCE Survey (2005)).
The U.S. Ailing Infrastructure

- Due to the constant breakdowns of the U.S. transportation networks and the increasing number of vehicles, American commuters now spend 3.5 billion hours a year stuck in traffic, which translates to a cost of $63.2 billion a year to the economy (ASCE (2005)).

- The number of motor vehicles in the U.S. has risen by 157 million (or 212.16%) since 1960 while the population of licensed drivers grew by 109 million (or 125.28%) (U.S. Department of Transportation (2004)).
Figure: Road Damaged after Flooding, Source: AccuWeather.com
Figure: Congestion Caused by One Lane Closed, Source: Flicker.com
Robustness Measure

- The focus of the robustness of networks (and complex networks) has been on the impact of different network measures when facing the removal of nodes on networks.

- We focus on the degradation of links through reductions in their capacities and the effects on the induced travel costs in the presence of known travel demands and different functional forms for the links.
“Robustness” in Engineering and Computer Science

- IEEE (1990) defined robustness as “the degree to which a system of component can function correctly in the presence of invalid inputs or stressful environmental conditions.”

- Gribble (2001) defined system robustness as “the ability of a system to continue to operate correctly across a wide range of operational conditions, and to fail gracefully outside of that range.”

- Schilllo et al. (2001) argued that robustness has to be studied “in relation to some definition of the performance measure.”
“Robustness” in Transportation

- Sakakibara et al. (2004) proposed a topological index. The authors considered a transportation network to be robust if it is dispersed in terms of the number of links connected to each node.

- Scott et al. (2005) examined transportation network robustness by analyzing the increase in the total network cost after removal of certain network components.
Related Literature in Transportation Network Vulnerability and Robustness

- Berdica (2002)
- Sakakibara, Kajitani, and Okada (2004)
- Scott et al. (2006)
- Taylor, Sekhar, and D’Este (2006)
- Murray and Grubesic (2007)
- Jenelius (2007)
Our Research on Network Efficiency, Vulnerability, and Robustness I


Our Research on Network Efficiency, Vulnerability, and Robustness II


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Notation for the Transportation Network Models

- Network $G$ with the set of links $L$ with $n_L$ elements, the set of paths $P$ with $n_P$ elements, and the set of O/D pairs $W$ with $n_W$ elements.

- The set of (acyclic) paths connecting O/D pair $w$ are denoted by $P_w$, the links by $a, b,$ etc., the paths by $p, q,$ etc., and the O/D pairs by $w_1, w_2,$ etc.

- The flow on path $p$ is denoted by $x_p$ and the flow on link $a$ by $f_a$. $x$ is the column vector of all path flows and $f$ is the column vector of all link flows.

- Travel cost on a path $p$ is denoted by $C_p$ and the travel cost on a link $a$ by $c_a$. $C$ is the column vector of all path costs and $c$ is the column vector of all link costs. Assume that the user link cost functions are continuous.

- Denote the travel demand between O/D pair $w$ by $d_w$ and the travel disutility by $\lambda_w$, where $d_w$ is fixed and known for all $w$. 
Conservation of Flow Between Demands and Path Flows

\[ d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W, \quad (1) \]

Conservation of Flow Between Link Flows and Path Flows

\[ f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \quad (2) \]

where \( \delta_{ap} = 1 \), if path \( p \) contains link \( a \), and \( \delta_{ap} = 0 \), otherwise.

Link Costs and Path Costs

\[ C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \quad (3) \]

The user cost on link \( a \) is denoted by \( c_a \) where

\[ c_a = c_a(f_a), \quad \forall a \in L. \quad (4) \]
Motivation

Relative Total Cost Indices

Bureau of Public Roads (BPR) (1964) Link Cost Function

\[ c_a(f_a) = t_a^0[1 + k\left(\frac{f_a}{u_a}\right)^\beta], \quad \forall a \in L, \quad (5) \]

\(u_a\) is the “practical” capacity on link \(a\), which also has the interpretation of the level-of-service flow rate; \(t_a^0\) is the free-flow travel time or cost on link \(a\); \(k\) and \(\beta\) are the model parameters and both take on positive values. Typically, in applications, \(k = .15\) and \(\beta = 4\).
The Total Cost on Link $a$, denoted by $\hat{c}_a$

$$\hat{c}_a = \hat{c}_a(f_a) = c_a(f_a) \times f_a = t_a^0[1 + k\left(\frac{f_a}{u_a}\right)^\beta] \times f_a, \quad \forall a \in L, \quad (6)$$

The Total Cost on a Network, denoted by $TC$

$$TC = \sum_{a \in L} \hat{c}_a(f_a), \quad (7)$$

where the link flows $f$ must lie in the feasible set $\mathcal{K}$:

$$\mathcal{K} \equiv \{f \in R_{+}^{nl} | \exists x \in R_{+}^{np} \text{ satisfying (1), (2)}\}.$$
The U-O and S-O models that we utilize are classical (see Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969)).
The User-Optimized (U-O) Problem

A path flow pattern \( x^* \in K^1 \), where
\[ K^1 \equiv \{ x | x \in R^{np_+} and d_w = \sum_{p \in P_w} x_p, \forall w \in W \text{ holds} \} \], is said to be a transportation network equilibrium, in the case of fixed demands, if the following condition holds for each O/D pair \( w \in W \) and every path \( p \in P_w \):

\[
C_p(x^*) - \lambda^*_w \begin{cases} 
= 0, & \text{if } x^*_p > 0, \\
\geq 0, & \text{if } x^*_p = 0.
\end{cases}
\] (8)

Equivalent Optimization Problem

The equilibrium link flow (and path flow pattern) can be obtained via the solution of the following optimization problem:

\[
\text{Minimize}_{f \in K} \sum_{a \in L} \int_0^{f_a} c_a(y)dy.
\] (9)
The System-Optimized (S-O) Problem

Assume that there is a central controller of the traffic who routes the traffic in an optimal manner so as to minimize the total cost in the network. That is,

$$\text{Minimize}_{f \in K} \sum_{a \in L} \hat{c}_a(f_a)$$  \hspace{1cm} (10)

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \ \forall p \in P,$$  \hspace{1cm} (11)

where the user cost on a path, $C_p$, is given by (3).

In view of (2), (3), and (4), one may express the cost on a path $p$ as a function of the path flow variables and, hence, an alternative version of the above S-O problem with objective function (10) can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize}_{x \in K^1} \sum_{p \in P} C_p(x) x_p$$  \hspace{1cm} (12)
System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function (10) in the S-O problem is convex, and the feasible set $\mathcal{K}^1$ is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: for each O/D pair $w \in W$ and each path $p \in P_w$, the flow pattern $x \in \mathcal{K}^1$, must satisfy:

$$\hat{C}'_p(x) \begin{cases} = \mu_w, & \text{if } x_p > 0, \\ \geq \mu_w, & \text{if } x_p = 0, \end{cases}$$

(13)

where $\hat{C}'_p(x)$ denotes the marginal of the total cost on path $p$, given by:

$$\hat{C}'_p(x) = \sum_{a \in L} \frac{\partial \hat{c}(f_a)}{\partial f_a} \delta_{ap}$$

(14)

evaluated in (13) at the solution and $\mu_w$ is the Lagrange multiplier associated with constraint (1) for that O/D pair $w$. 
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Relative Total Cost Index for Assessing the Robustness of a Transportation Network Under the U-O Flow Pattern

The relative total cost index under the U-O flow pattern for a transportation network $G$ with the vector of demands $d$, the vector of user link cost functions $c$, and the vector of link capacities $u$ is defined as the relative total cost increase under a given uniform capacity retention ratio $\gamma$ ($\gamma \in (0, 1]$) so that the new capacities (cf. 5) are given by $\gamma u$. Let $c$ denote the vector of BPR user link cost functions and let $d$ denote the vector of O/D pair travel demands.

$$I_{U-O}^\gamma = I_{U-O}(G, c, d, \gamma, u) = \frac{TC_{U-O}^\gamma - TC_{U-O}}{TC_{U-O}} \times 100\% \quad (15)$$

where $TC_{U-O}$ and $TC_{U-O}^\gamma$ are the total network costs (7) evaluated under the user-optimizing flow pattern with the original capacities and the remaining capacities (i.e., $\gamma u$), respectively.
Relative Total Cost Index for Assessing the Robustness of a Transportation Network Under the S-O Flow Pattern

The relative total cost index under the S-O flow pattern for a transportation network $G$ with the vector of demands $d$, the vector of user link cost functions $c$, and the vector of link capacities $u$ is defined as the relative total cost increase under a given uniform capacity retention ratio $\gamma$ ($\gamma \in (0, 1]$) so that the new capacities (cf. 5) are given by $\gamma u$. Let $c$ denote the vector of BPR user link cost functions and let $d$ denote the vector of O/D pair travel demands.

$$I_{S-O}^\gamma = I_{S-O}(G, c, d, \gamma, u) = \frac{TC_{S-O}^\gamma - TC_{S-O}}{TC_{S-O}} \times 100\% \quad (16)$$

where $TC_{S-O}$ and $TC_{S-O}^\gamma$ are the total network costs (7) evaluated under the system-optimizing flow pattern with the original capacities and the remaining capacities (i.e., $\gamma u$), respectively.
Remark

From the above definition(s), a transportation network, under a given capacity retention/deterioration ratio $\gamma$ (and either the S-O or the U-O travel behavior) is considered to be robust if the index $\mathcal{I}^{\gamma}$ is low. This means that the relative total cost does not change much and, hence, the transportation network may be viewed as being more robust than if the relative total cost is small.
Theoretical Results I

Theorem

Consider a network consisting of two nodes 1 and 2 as in the figure below, which are connected by \( n \) parallel links. If the free-flow term, \( t_0^a, \forall a \in L \), is the same for all links \( a \in L \) in the BPR link cost function, the S-O flow pattern coincides with the U-O flow pattern and, therefore, \( I_{U-O} = I_{S-O} \).

Theorem

The upper bound for \( I_{S-O}^\gamma \) for a transportation network with BPR link cost functions is \( \frac{1-\gamma^\beta}{\gamma^\beta} \times 100\% \).
Theoretical Results II

Theorem

Consider a network consisting of two nodes 1 and 2 as in the previous figure, which are connected by \( n \) parallel links. Assume that the associated BPR link cost functions have \( \beta = 1 \). Furthermore, assume that there are positive flows on all the links at both the original and the partially degraded capacity levels given, respectively, by \( u \) and \( \gamma u \). Then the relative total cost index under the U-O flow pattern is given by the explicit formula:

\[
I_{U-O}^\gamma = \left( \frac{\gamma U + kd_w}{\gamma U + k\gamma d_w} - 1 \right) \times 100\%,
\]

where \( d_w \) is the given demand for O/D pair \( w = (1,2) \) and \( U \equiv u_a + u_b + \cdots + u_n \). Moreover, the network robustness \( I_{U-O}^\gamma \) is bounded from above by \( \frac{1-\gamma}{\gamma} \times 100\% \).
Relationship with Price of Anarchy

- The price of anarchy (PoA) (cf. Roughgarden (2005)), denoted by $\rho$, is defined as:

$$\rho = \frac{TC_{U-O}}{TC_{S-O}}.$$  \hspace{1cm} (18)

- Observe that $\rho$ captures the relationship between total costs across distinct behavioral principles whereas the proposed indices are focused on the degradation of network performance within the U-O or the S-O behavior.

- We have the following relationship between the ratio of the two indices and the PoA:

$$\frac{I_{S-O}^\gamma}{I_{U-O}^\gamma} = \frac{[TC_{S-O}^\gamma - TC_{S-O}]}{[TC_{U-O}^\gamma - TC_{U-O}]} \times \rho.$$  \hspace{1cm} (19)
Example 1

Consider the following simple transportation network as shown in the figure on the right. There are two O/D pairs, namely, \( w_1=(1,3) \) and \( w_2=(1,4) \). The demands for the two O/D pairs are: \( d_{w_1} = 10 \) and \( d_{w_2} = 20 \). The paths connecting O/D pair \( w_1 \) are: \( p_1 = (a, b) \) and \( p_2 = d \). The paths connecting O/D pair \( w_2 \) are: \( p_3 = (a, c) \) and \( p_4 = e \). The capacities for links \( a, b, c, d, \) and \( e \) are: 100, 50, 60, 10, and 20, respectively. Let \( t_0 \) and \( k \) be identical for all the links and equal to 10 and 1, respectively. The BPR link cost functions for the links are, hence, given by:

\[
\begin{align*}
    c_a(f_a) &= 10[1 + (\frac{f_a}{100})^\beta], \\
    c_b(f_b) &= 10[1 + (\frac{f_b}{50})^\beta], \\
    c_c(f_c) &= 10[1 + (\frac{f_c}{60})^\beta], \\
    c_d(f_d) &= 10[1 + (\frac{f_d}{10})^\beta], \\
    c_e(f_e) &= 10[1 + (\frac{f_e}{20})^\beta].
\end{align*}
\]
Relative Total Cost Index Under the U-O Flow Pattern

The diagram illustrates the relative total cost index ($\pi_{U-O}$) as a function of the capacity retention ratio ($\gamma$) for different values of $\beta$. Each line represents a different value of $\beta$: $\beta=1$, $\beta=2$, $\beta=3$, and $\beta=4$. The index increases significantly as the capacity retention ratio decreases, indicating higher costs for lower capacity retention. The $x$-axis represents the capacity retention ratio, while the $y$-axis represents the relative total cost index.
This transportation network under the U-O solution concept is more robust than under the S-O solution concept.
Assume that $k = 1$. Let $t_{a}^0 = t_{d}^0 = 1$; $t_{b}^0 = t_{c}^0 = 50$, and $t_{e}^0 = 10$. Furthermore, let $u_{a} = u_{d} = 20$; $u_{b} = u_{c} = 50$, and $u_{e} = 100$. The user link cost functions are, thus, given by: $c_{a}(f_{a}) = 1 + \left( \frac{f_{a}}{20} \right)^{\beta}$, $c_{b}(f_{b}) = 50\left[1 + \left( \frac{f_{b}}{50} \right)^{\beta}\right]$, $c_{c}(f_{c}) = 50\left[1 + \left( \frac{f_{b}}{50} \right)^{\beta}\right]$, $c_{d}(f_{d}) = 1 + \left( \frac{f_{d}}{20} \right)^{\beta}$, $c_{e}(f_{e}) = 10\left[1 + \left( \frac{f_{e}}{100} \right)^{\beta}\right]$. Recall $w_{1} = (1, 4)$. The demand is given by $d_{w_{1}} = 110$. 

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Relative Total Cost Index Under the U-O Flow Pattern for the Braess Network Example

\[ I_{U-O} \]
Relative Total Cost Index Under the S-O Flow Pattern for the Braess Network Example

- $\beta = 1$
- $\beta = 2$
- $\beta = 3$
- $\beta = 4$
The Sioux-Falls Network

There are 528 O/D pairs, 24 nodes, and 76 links in the Sioux-Falls network.
Algorithms

- The projection method (cf. Dafermos (1980) and Nagurney (1999)) with the embedded Dafermos and Sparrow (1969) equilibration algorithm (see also, e.g., Nagurney (1984)) and the column generation algorithm (cf. Leventhal, Nemhauser, and Trotter (1973)) were utilized to compute the equilibrium solutions.

- Based on the equilibrium solutions, the network efficiency was determined and the importance values and the importance rankings of the links were computed.
From the above figure, we can see that the Sioux-Falls network is always more robust under the U-O behavior except when $\beta$ is equal to 2 and the capacity retention ratio is between 0.5 and 0.9.
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We proposed two relative total cost indices to analyze transportation network robustness associated with different user behaviors.

We examined the impacts of the degradation of the links in terms of capacity reductions on the relative total cost in the network at the original capacities and at the degraded capacities by assuming that the user link cost functions for the transportation network are of BPR form.

We established new theoretical results and also provided numerical examples.

For future research, we plan on assessing the robustness of transportation networks with asymmetric BPR-type link cost functions and with multiple modes of transportation. We would also like to study the robustness of transportation networks in the case of stochastic costs (and demands).
The Virtual Center for Supernetworks

Motivation

The Virtual Center for Supernetworks at the Isenberg School of Management at the University of Massachusetts Amherst was established under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, to serve as a resource to academia, industry, and government. The mission of the Virtual Center for Supernetworks is to foster the development of the theory and application of supernetworks and to serve as a resource to academia, industry, and government. Supernetworks are multi-element connected network systems that are composed of multiple networks to economic, environmental, financial, knowledge, and social networks. The applications of supernetworks include multimodal transportation, critical infrastructure, energy, and the environment, the Internet and electronic commerce, supply chain management, international financial networks, web-based advertising, social networks, and decision-making, and integrated social and economic networks.

Relative Total Cost Index

Numerical Examples

Summary

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Relative Total Cost Index
Thank You!

For more information,
see http://supernet.som.umass.edu