A Spatial Price Network Equilibrium Paradox

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Outline of Presentation

- **Background and Motivation**
- **The Spatial Price Network Equilibrium Paradox**
- **Additional Examples and Sensitivity Analysis**
- **Insights and Summary**

Background and Motivation

Braess Paradox

Über ein Paradoxon aus der Verkehrsplanung

Von D. BRAESS, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

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On a Paradox of Traffic Planning

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For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Key words: traffic network planning; paradox; equilibrium; critical flows; optimal flows; existence theorem
History: Received: April 2005; revision received: June 2005; accepted: July 2005.

Translated from the original German: Braess, Dietrich. 1968. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* **12** 258–268.

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Preface to "On a Paradox of Traffic Planning"

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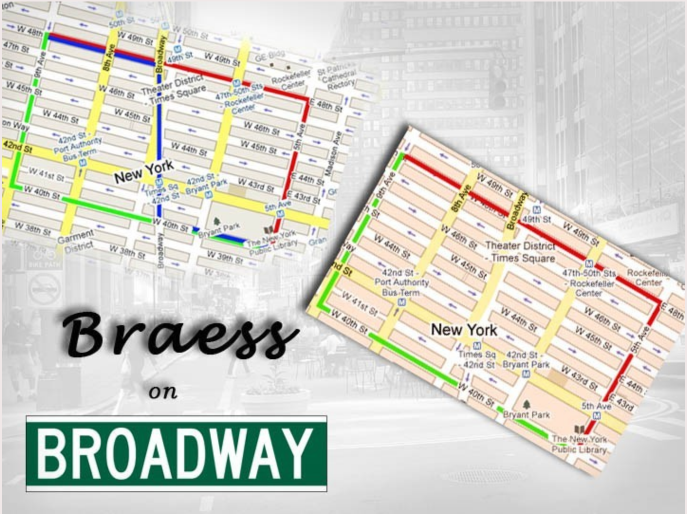
This article is a preface to the translation by Braess, Nagurney, and Wakolbinger of the 1968 paper by Braess, "Über ein Paradoxon aus der Verkehrsplanung" (*Unternehmensforschung* **12** 258–268).

Key words: Braess paradox; user optimization; system optimization
History: Received: May 2005; revision received: July 2005; accepted: July 2005.

Professor Braess Visits the Isenberg School

- **Professor Braess visited Professor Nagurney on April 5-8, 2006 to celebrate the publication of the translation of his 1968 article and the Preface in Transportation Science.**

The Closing of Broadway in NYC to Traffic from 42nd to 47th Streets in 2009 Until Now



Literature Review

This paradox has fascinated researchers and practitioners in transportation and related fields, in which decentralized behavior in congested networks is relevant, such as:

• **computer science,** in the modeling of telecommunication networks and the Internet (Korilis, Lazar, and Orda (1999), Roughgarden and Tardos (2002), Roughgarden (2005), Nagurney, Parkes, and Daniele (2007))
• **electrical engineering,** in the study of power systems (Cohen and Horowitz (1991), Blumsack etal. (2007)) and electronic circuits (cf. Nagurney and Nagurney (2016))
• **physics,** in mechanical (Cohen and Horowitz (1991)) and fluid systems (Calvert and Keady (1993))
• **biology,** in metabolic networks (see Motter (2010)), ecosystems (Sahasrabudhe and Motter (2011)), and targeted cancer therapy (Kippenberger etal. (2016)), and, surprisingly, in
• **sports analytics** in the study of sports teams, where the Braess paradox analogue corresponds to the removal of a player resulting in better team performance (cf. Skinner (2010), Gudmunsson and Horton (2017)).

The Spatial Price Network Equilibrium Paradox

SPE Problems Have Many Applications

Spatial Price Equilibrium (SPE) problems have been widely applied to agriculture, energy, and mineral markets.

Literature Review

• **The models were originated by Samuelson (1952) and Takayama and Judge (1964, 1971). Variational inequalities have been, in particular, successful in the formulation, analysis, and solution of such problems.**

• **Agriculture** (see, e.g., Thompson, 1989; Bishop et al., 1994; Grant et al., 2009; Nagurney et al., 2019) and **mineral and energy markets** (Labys and Yang, 1991, 1997; Birge et al., 2022) have all been the focus of applications of spatial price equilibrium models, which often also have a **network foundation** (see, e.g., Samuelson, 1952; Florian & Los, 1982; Friesz et al., 1984; Nagurney, 1999; Nagurney & Besik, 2022; Nagurney et al., 2023, 2024; and the references therein).

• SPE problems are examples of **network equilibrium problems**, which also include traffic network equilibrium problems (cf. Beckmann et al., 1956; Dafermos & Sparrow, 1969; Dafermos, 1980; Sheffi, 1985; Patriksson, 1993; Florian & Hearn, 1995). Interestingly, Dafermos & Nagurney (1985) and Dafermos (1986) demonstrated **an isomorphism between SPE problems and traffic network equilibrium problems** with elastic demands for single commodity and multiple commodity problems, respectively.

Literature Review

• **Advances in model and algorithmic development**, often conducted by operational researchers (see, e.g., Deckro & Morris, 1980; Florian & Los, 1982; Dafermos & Nagurney, 1984; Friesz et al., 1984; Harker, 1985; Guder, 1988; Nagurney & Aronson, 1989; Nagurney et al., 1996; Nagurney & Zhang, 1996; Daniele, 2004; Önal and Chen, 2021) have included more **general underlying transportation networks** connecting supply markets with demand markets as well as the use of **methodological tools such as variational inequality theory** to allow for more realistic modeling of multicommodity problems.

• The paper by Halljeford et al. (1994) was the first to construct **an elastic demand version of the Braess paradox**, using the classical Braess paradox network data and with a specific elastic demand function. The elastic demand function was such that, when the new route was added, **the travel disutility increased (albeit not by much) and the demand decreased.**

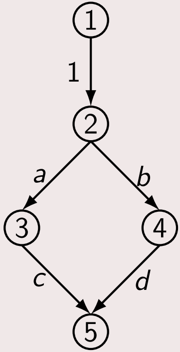
• Yang (1997), in turn, developed a **sensitivity analysis approach** for traffic network equilibrium problems with elastic demand with relevance to Braess Paradox identification in this setting. See also Tu et al. (2019) for additional Braess Paradox examples under **elastic demands**.

The Spatial Price Network Equilibrium Paradox

- The SPE conditions, in the context of a single commodity, state that, for each pair of supply and demand markets, there is **a positive flow of the commodity** from the supply market to the demand market on a path/route joining the two, if the supply price plus the unit transportation cost on the path **is equal to** the demand price.
- On the other hand, if the supply price plus the unit transportation cost **exceeds** the demand price then there will be **zero commodity flow** on the path at the equilibrium.
- Investigation of whether the following paradox can occur in a spatial price network equilibrium problem: **Can the addition of a new route result in a higher demand price for the commodity (and a lower associated volume), a lower supply price, and a higher transportation cost?**

Baseline Example 1: Two Paths with No Shared Links

- **In the baseline SPE network, there is a single supply market at node 1 and a single demand market at node 5. There are two transportation routes: path p_1 consisting of links a and c and path p_2 consisting of links b and d.**



Solution

The supply price is denoted by π_1 and the demand price by ρ_5. The supply is denoted by s_1 and the demand by d_5. The unit transportation costs on the links are denoted by: c_a, c_b, c_c, and c_d. The path cost on path p_1, C_{p_1}, is, hence, $C_{p_1} = c_a + c_c$, whereas the path cost on path p_2, C_{p_2}, is: $C_{p_2} = c_b + c_d$.

The flow on path p_1 is denoted by Q_{p_1} and that on path p_2 by Q_{p_2}. The conservation of flow equations are:

$$s_1 = Q_{p_1} + Q_{p_2}; \tag{1}$$

that is, the supply of the commodity produced at the supply market must be equal to the sum of the commodity shipments out. Also, the demand at the demand market, d_5, must be equal to the sum of the commodity shipments to the demand market; that is:

$$d_5 = Q_{p_1} + Q_{p_2}, \tag{2}$$

with the path flows being nonnegative.

Solution

The transportation link flows are denoted by: f_a, f_b, f_c, and f_d and note that $f_1 = s_1$ is the flow on link 1.

The following conservation of flow equations also must hold:

$$f_a = f_c = Q_{p_1}, \quad f_b = f_d = Q_{p_2}. \tag{3}$$

The unit transportation link cost functions are flow-dependent and are given by:

$$c_a = 10f_a, \quad c_b = f_b + 40, \quad c_c = f_c + 40, \quad c_d = 10f_d. \tag{4}$$

The supply price function is:

$$\pi_1 = s_1 + 4 \tag{5}$$

and the demand price function is:

$$\rho_5 = -d_5 + 89. \tag{6}$$

Solution

The equilibrium commodity shipment pattern and the equilibrium supplies and demands are denoted with a superscript *.

Noting that c_a has the same form as c_d and that c_b has the same form as c_c, it is clear that, in equilibrium, the commodity path flows will be equal; therefore, $Q^*_{p_1} = Q^*_{p_2}$.

For each path p, $p = p_1$ and $p = p_2$, hence, the following SPE condition will hold assuming, of course, a positive equilibrium path flow on each path:

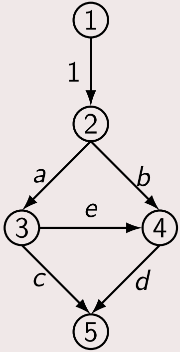
$$\pi_1 + C_p = \rho_5. \tag{7}$$

The spatial price equilibrium solution is: $Q^*_{p_1} = 3$ and $Q^*_{p_2} = 3$. The equilibrium supply $s^*_1 = 6$ and the equilibrium demand $d^*_5 = 6$.

Under this equilibrium commodity flow pattern, it is found that: $\pi_1 = 10$, $C_{p_1} = C_{p_2} = 73$, and $\rho_5 = 83$, so, clearly, the SPE conditions hold.

Examples with New Link e Added and c_e Varied

- **Examples are now considered with the addition of a new link e, joining node 3 to node 4. Such a link could correspond to a road or, in the case of intermodal transportation, a maritime route via a ship or barge, etc.**



Solution

The transportation cost on link e is:

$$c_e = f_e + \alpha \tag{9}$$

with α being a nonnegative term. The impact of varying α from 0.0000 through 13.0000 on the equilibrium commodity flow pattern and on the supply and demand market prices will be investigated. The addition of link e results in a new path $p_3 = (a, e, d)$. The conservation of flow equations now become:

$$s_1 = Q_{p_1} + Q_{p_2} + Q_{p_3},$$

$$d_5 = Q_{p_1} + Q_{p_2} + Q_{p_3},$$

and

$$f_a = Q_{p_1} + Q_{p_3}, \quad f_b = Q_{p_2}, \quad f_c = Q_{p_1}, \quad f_d = Q_{p_2} + Q_{p_3}, \quad f_e = Q_{p_3},$$

with all path flows being nonnegative.

Solution

The following system of equations is set up with $c_e = f_e + \alpha$:

$$\pi_1 + C_{p_1} = \rho_5$$

$$\pi_1 + C_{p_2} = \rho_5$$

$$\pi_1 + C_{p_3} = \rho_5,$$

which, after use of the conservation of flow equations and algebraic simplification, becomes:

$$13Q^*_{p_1} + 2Q^*_{p_2} + 12Q^*_{p_3} = 45$$

$$2Q^*_{p_1} + 13Q^*_{p_2} + 12Q^*_{p_3} = 45$$

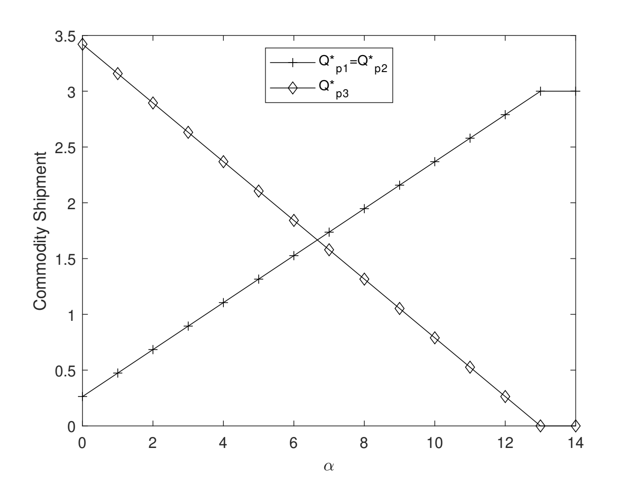
$$12Q^*_{p_1} + 12Q^*_{p_2} + 23Q^*_{p_3} = 85 - \alpha.$$

The Spatial Price Equilibrium Solutions for Different Values of α

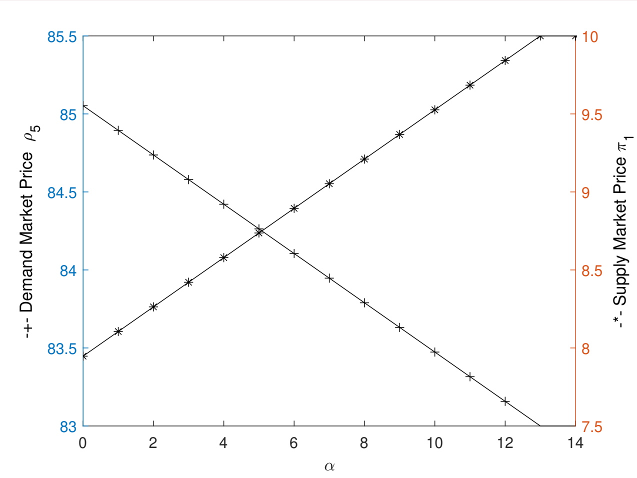
Equilibrium Commodity Path Flows, Supplies and Demands, Prices, and Transportation Path Costs Under Different αs for Baseline Example 1 with New Link e Added and c_e Varied

α	$Q^*_{p_1}$	$Q^*_{p_2}$	$Q^*_{p_3}$	$s^*_1 = d^*_5$	π_1	ρ_5	$C_{p_1} = C_{p_2} = C_{p_3}$
0.0000	0.2632	0.2632	3.4211	3.9474	7.9474	85.0526	77.1052
1.0000	0.4737	0.4737	3.1579	4.1053	8.1053	84.8947	76.7894
2.0000	0.6842	0.6842	2.8947	4.2632	8.2632	84.7368	76.4736
3.0000	0.8947	0.8947	2.6316	4.4211	8.4211	84.5789	76.1578
4.0000	1.1053	1.1053	2.3684	4.5789	8.5789	84.4211	75.8422
5.0000	1.3158	1.3158	2.1053	4.7368	8.7368	84.2632	75.5264
6.0000	1.5263	1.5263	1.8421	4.8947	8.8947	84.1053	75.2106
7.0000	1.7368	1.7368	1.5789	5.0526	9.0526	83.9474	74.8948
8.0000	1.9474	1.9474	1.3158	5.2105	9.2105	83.7895	74.5790
9.0000	2.1579	2.1579	1.0526	5.3684	9.3684	83.6316	74.2632
10.0000	2.3684	2.3684	0.7895	5.5263	9.5263	83.4737	74.9474
11.0000	2.5789	2.5789	0.5263	5.6842	9.6842	83.3158	73.6316
12.0000	2.7895	2.7895	0.2632	5.8421	9.8421	83.1579	73.3158
13.0000	3.0000	3.0000	0.0000	6.0000	10.0000	83.0000	73.0000

Commodity Shipments at the Equilibrium for Different Values of α



Demand and Supply Market Prices at the Equilibrium for Different Values of α



Discussion of Results

- The demand price, ρ_5, at Example 1 was 83 and the demand, d_5^*, was 6. The supply price, π_1, was 10, with the transportation costs on both paths p_1 and p_2 being equal to 73.
- Now, with the addition of the new link e, with α in the range of [0.0000,13.0000), the demand price ρ_5 is always higher than 83; the supply price π_1 is always less than 10, and the transportation path costs C_{p_1}, C_{p_2}, and C_{p_3} are always greater than 73!
- The spatial price network equilibrium paradox occurs in a range of values for α. And, if α is greater than or equal to 13.0000, then the new path p_3 is never used. It is clear that the addition of a link that results in a new path can make both consumers and producers worse-off. This paradox, with multiple instances, expands the literature on the Braess paradox.

Additional Examples and Sensitivity Analysis

Example 2: Same Data as that in Example 1 Except for Different c_b and c_c

- **In this example, the impacts of changes to the transportation link costs c_b and c_c are explored, with the cost functions now being:**

$$c_b = f_b + 30, \quad c_c = f_c + 30.$$

The spatial price equilibrium solution is now: $Q^*_{p_1} = Q^*_{p_2} = 3\frac{2}{3}$ and, hence, $s^*_1 = d^*_5 = 7\frac{1}{3}$. The demand price ρ_5 is now $81\frac{2}{3}$, which is lower than in the analogous example above where $\rho_5 = 83$.

With the reduction in the transportation link costs on links b and c, the commodity flows in equilibrium on both paths p_1 and p_2 increase.

The demand price decreases as compared to the results for the Example 1 in which the fixed terms in both c_b and c_c being equal to 40.

The supply price π_5, on the other hand, now increases to $11\frac{1}{3}$.

Discussion of Results

- **The addition of a new link e, with $c_e = f_e + \alpha$ (as previously), is now considered.**
- **It is observed that for any nonnegative α, the new path p_3 is never used, and therefore, the spatial price network equilibrium paradox does not occur.**
- **The equilibrium solution for Example 2 for paths p_1 and p_2 remains the same for all nonnegative values of α with $Q^*_{p_3} = 0.0000$.**

Example 3: Same Data as that in Example 1 Except for Different c_b and c_c

- **An increase in the fixed terms in the transportation cost functions for links b and c, as compared to their values in both Examples 1 and 2, is now considered so that:**

$$c_b = f_b + 50, \quad c_c = f_c + 50$$

The equilibrium commodity shipment pattern is now: $Q^*_{p_1} = Q^*_{p_2} = 2\frac{1}{3}$, resulting in: $s^*_1 = d^*_5 = 4\frac{2}{3}$.

The transportation cost on both paths is $75\frac{2}{3}$.

The demand price $\rho_5 = 84\frac{1}{3}$, which is higher than either the demand price in Example 1 or that in Example 2. This is reasonable because, in the former, the fixed transportation link cost term is 40 and, in the latter, it is 30.

The supply price $\pi_1 = 8\frac{2}{3}$.

Examples with New Link e Added and c_e Varied

- **We now proceed to conduct a similar experiment as previously. We add a new link e, as before, and we investigate the impact of varying the α in the cost for link e, c_e.**

The system of equations that was set up:

$$13Q^*_{p_1} + 2Q^*_{p_2} + 12Q^*_{p_3} = 35$$

$$2Q^*_{p_1} + 13Q^*_{p_2} + 12Q^*_{p_3} = 35$$

$$12Q^*_{p_1} + 12Q^*_{p_2} + 23Q^*_{p_3} = 85 - \alpha.$$

By solving the above system, the following is obtained:

$$\begin{bmatrix} Q^*_{p_1} \\ Q^*_{p_2} \\ Q^*_{p_3} \end{bmatrix} = \begin{bmatrix} -3.77197 \\ -3.77197 \\ 7.6316 \end{bmatrix} - \alpha \begin{bmatrix} -.2105 \\ -.2105 \\ .2632 \end{bmatrix}.$$

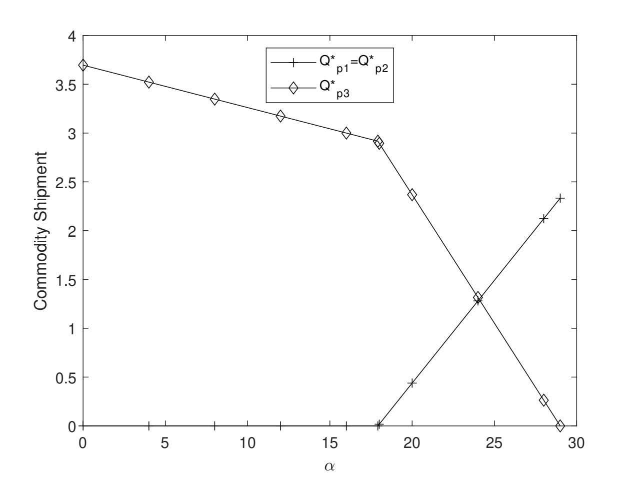
It can be seen that both $Q^*_{p_1}$ and $Q^*_{p_2}$ become 0.0000 for $\alpha = 17.9167$, whereas $Q^*_{p_3}$ becomes 0.0000 for $\alpha = 29.0000$.

The Spatial Price Equilibrium Solutions for Different Values of α

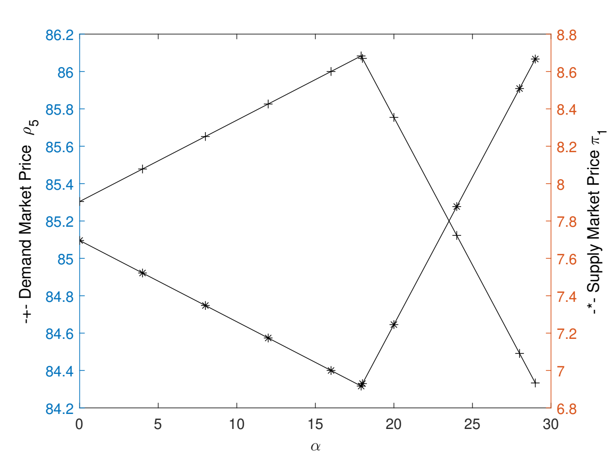
Equilibrium Commodity Path Flows, Supplies and Demands, Prices, and Transportation Path Costs Under Different αs for Example 3 with New Link e Added and c_e Varied

α	$Q^*_{p_1}$	$Q^*_{p_2}$	$Q^*_{p_3}$	$s^*_1 = d^*_5$	π_1	ρ_5	C_{p_3}
0.0000	0.0000	0.0000	3.6957	3.6957	7.6957	85.3043	77.6086
4.0000	0.0000	0.0000	3.5217	3.5217	7.5217	85.4783	77.9566
8.0000	0.0000	0.0000	3.3478	3.3478	7.3478	85.6522	78.3044
12.0000	0.0000	0.0000	3.1739	3.1739	7.1739	85.8261	78.6522
16.0000	0.0000	0.0000	3.0000	3.0000	7.0000	86.0000	79.0000
17.9167	0.0000	0.0000	2.9167	2.9167	6.9167	86.0833	79.1666
18.0000	0.01750	0.01750	2.8947	2.9298	6.9298	86.0702	79.1404
20.0000	0.4386	0.4386	2.3684	3.2456	7.2456	85.7544	78.5088
24.0000	1.2807	1.2807	1.3158	3.8770	7.8770	85.1228	77.2456
28.0000	2.1228	2.1228	0.2632	4.5088	8.5088	84.4912	75.9824
29.0000	2.3333	2.3333	0.0000	4.6667	8.6667	84.3333	75.6666

Commodity Shipments at the Equilibrium for Different Values of α



Demand Market Prices and Supply Market Prices at the Equilibrium for Different Values of α



Discussion of Results

- **For values of α greater than** 29.0000 **the same equilibrium solution holds as in Example 3, since path p_3 is no longer used. Hence, for the range of α until 29.0000, the demand market price ρ_5 at the equilibrium is higher and the supply price is lower than that in Example 3.**
- **Note that the total equilibrium commodity shipments, which equal the equilibrium supply and the equilibrium demand, are also lower.**
- **In agricultural commodities, this paradox has negative implications for food security, since consumers have to pay a higher price and they receive a lower volume of the commodity with a new transportation link. Furthermore, the transportation path costs are all higher than in Example 3, except when the same equilibrium solution is attained at $\alpha =$** 29.0000 **and path p_3 is no longer used.**

Insights and Summary

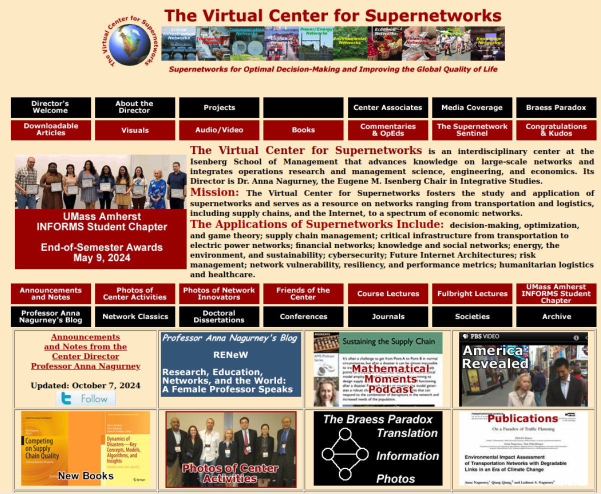
Insights

- **Through multiple examples, it is demonstrated that, indeed, a spatial price network equilibrium paradox can occur.**
- **After the addition of a link to an existing spatial chain network in the form of a transportation link, the demand price for the commodity can increase; the supply price can decrease, and all transportation routes connecting a supply market to a demand market can become more costly.**
- **Such a result also has implications for food security, if the commodity is an agricultural one.**
- **The results herein demonstrate the importance of the quantification of the potential impacts of network redesign in terms of transportation routes.**

Summary

- **Many of SPE models are network-based and they have also been formulated and solved using various methodologies. There has been a resurgence of interest in such models due to global issues such as wars and other crises and climate change.**
- **Alternative transportation routes have become increasingly important and, hence, SPE models in which there are alternative transportation routes from supply markets to demand markets are garnering renewed attention.**
- **This paper investigates whether a Braess type of paradox can occur in spatial price equilibrium problems and why such a paradox is meaningful.**
- **This paper adds to the literature on the Braess paradox, in the case of a network economic problem with elastic demands.**

Thank You Very Much!



More related information can be found on the Supernetwork Center site: https://supernet.isenberg.umass.edu/