A Dynamic Network Oligopoly Model with Transportation Costs, Product Differentiation, and Quality Competition

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where a full list of references can be found.

Outline

- Motivation
- The Dynamic Network Oligopoly Model
- Stability Analysis
- The Algorithm
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Motivation

Oligopolies constitute fundamental industrial organization market structures of numerous industries world-wide.

In classical oligopoly problems, the product is assumed to be homogeneous. However, in many cases, consumers may consider the products to be differentiated according to the producer.

Motivation

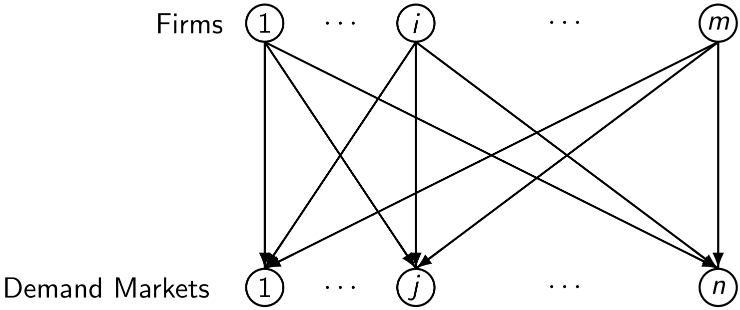
Quality is emerging as an important feature in numerous products, and *it is implicit in product differentiation*.

Motivation

Cabral (2012) recently articulated the need for new dynamic oligopoly models, combined with network features, as well as quality.

In this paper, we develop a network oligopoly model with differentiated products and quality levels. We present both the static version, in an equilibrium context, which we formulate as a finite-dimensional variational inequality problem, and then we develop its dynamic counterpart, using projected dynamical systems theory.

The Network Structure of the Dynamic Network Oligopoly Problem with Product Differentiation



The Dynamic Network Oligopoly Model

Conservation of flow equations

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m, \tag{1}$$

$$d_{ij} = Q_{ij}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{2}$$

$$Q_{ij} \geq 0, \quad i = 1, \ldots, m; j = 1, \ldots, n. \tag{3}$$

We group the production outputs into the vector $s \in R^m_+$, the demands into the vector $d \in R^{mn}_+$, and the product shipments into the vector $Q \in R^{mn}_+$.

Production cost function for firm i

$$\hat{f}_i = \hat{f}_i(s, q_i), \quad i = 1, \ldots, m. \tag{4}$$

We assume, hence, that the functions in (5) also capture the total quality cost, since, as a special case, the above functions can take on the form

$$\hat{f}_i(s, q_i) = f_i(s, q_i) + g_i(q_i), \quad i = 1, \ldots, m. \tag{5}$$

The production cost functions (4) (and (5)) are assumed to be convex and continuously differentiable. We group the quality levels of all firms into the vector $q \in R^m_+$.

Interestingly, the second term in (5) can also be interpreted as the R&D cost (cf. Matsubara 2010), which is the cost that occurs in the processes of the development and introduction of new products to market as well as the improvement of existing products. Evidence indicates that the R&D cost depends on the quality level of its products (see, Klette and Griliches 2000; Hoppe and Lehmann-Grube 2001; Symeonidis 2003).

Nonnegative quality level for firm i's product

$$q_i \geq 0, \quad i = 1, \ldots, m. \tag{6}$$

Demand price function for firm i's product at demand market j

$$p_{ij} = p_{ij}(d, q), \quad i = 1, \ldots, m; j = 1, \ldots, n. \tag{7}$$

We allow the demand price for a product at a demand market to depend, in general, upon the entire consumption pattern, as well as on all the levels of quality of all the products. The generality of the expression in (6) allows for modeling and application flexibility. The demand price functions are, typically, assumed to be monotonically decreasing in product quantity but increasing in terms of product quality.

The Dynamic Network Oligopoly Model

Transportation cost function

$$\hat{c}_{ij} = \hat{c}_{ij}(Q_{ij}), \quad i = 1, \ldots, m; j = 1, \ldots, n. \tag{8}$$

The demand price functions (7) and the total transportation cost functions (8) are assumed to be continuous and continuously differentiable.

The Dynamic Network Oligopoly Model

The strategic variables of firm i are its product shipments $\{Q_i\}$ where $Q_i = (Q_{i1}, \ldots, Q_{in})$ and its quality level q_i.

Utility function

$$U_i = \sum_{j=1}^{n} p_{ij} d_{ij} - \hat{f}_i - \hat{g}_i - \sum_{j=1}^{n} \hat{c}_{ij}. \tag{9}$$

In view of (1) - (9), one may write the profit as a function solely of the shipment pattern and quality levels, that is,

$$U = U(Q, q), \tag{10}$$

where U is the m-dimensional vector with components: $\{U_1, \ldots, U_m\}$.

Definition: A Network Cournot-Nash Equilibrium

Let K^i denote the feasible set corresponding to firm i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{m} K^i$.

Definition 1

A product shipment and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a Cournot-Nash equilibrium if for each firm $i; i = 1, \ldots, m$,

$$U_i(Q_i^*, q_i^*, \hat{Q}_i^*, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q}_i^*, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{11}$$

where

$$\hat{Q}_i^* \equiv (Q^*{}_1, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*); \quad \text{and}$$

$$\hat{q}_i^* \equiv (q^*{}_1, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*). \tag{12}$$

Theorem: Variational Inequality Formulation

Theorem 1

Assume that for each firm i the profit function $U_i(Q,q)$ is concave with respect to the variables $\{Q_{i1},\dots,Q_{in}\}$, and q_i, and is continuous and continuously differentiable. Then $(Q^*,q^*)\in K$ is a network Cournot-Nash equilibrium according to the above Definition if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ij}}\times(Q_{ij}-Q^*_{ij})-\sum_{i=1}^{m}\frac{\partial U_i(Q^*,q^*)}{\partial q_i}\times(q_i-q^*_i)\geq 0,$$

$$\forall (Q,q)\in K, \tag{13}$$

Theorem: Variational Inequality Formulation

$(s^*, Q^*, d^*, q^*) \in K^1$ is an equilibrium production, shipment, consumption, and quality level pattern if and only if it satisfies

$$\sum_{i=1}^{m} \frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial s_i} \times (s_i - s_i^*)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial \hat{c}_{ij}(Q_{ij}^*)}{\partial Q_{ij}} - \sum_{k=1}^{n} \frac{\partial p_{ik}(d^*, q^*)}{\partial d_{ij}} \times d_{ik}^*\right] \times (Q_{ij} - Q_{ij}^*)$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}(d^*, q^*) \times (d_{ij} - d_{ij}^*)$$

$$+\sum_{i=1}^{m}\left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial q_i} - \sum_{k=1}^{n} \frac{\partial p_{ik}(d^*, q^*)}{\partial q_i} \times d_{ik}^*\right] \times (q_i - q_i^*) \geq 0,$$

$$(s, Q, d, q) \in K^1, \tag{14}$$

where $K^1 \equiv \{(s, Q, d, q) |\, Q \geq 0,\ q \geq 0, \text{and (1) and (2) hold}\}$.

The Projected Dynamical System Model

A dynamic adjustment process for quantity and quality levels

$$\dot{Q}_{ij} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ij}}, & \text{if} \quad Q_{ij} > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ij}}\}, & \text{if} \quad Q_{ij} = 0. \end{cases} \tag{15}$$

$$\dot{q}_i = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_i}, & \text{if} \quad q_i > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\}, & \text{if} \quad q_i = 0. \end{cases} \tag{16}$$

The Projected Dynamical System Model

The pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments and quality levels, in vector form, is:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{17}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{18}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{x \in \mathcal{K}} \|Q - x\|, \tag{19}$$

and where $\|\cdot\| = \langle x^T, x \rangle$. Hence, $F(X) = -\nabla U(Q, q)$.

Theorem: Equilibrium Condition

Theorem 2

X^* solves the variational inequality problem (13) if and only if it is a stationary point of the ODE (17), that is,

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{20}$$

Stability Under Monotonicity

Definition 2
An equilibrium shipment and quality level pattern X^* is stable, if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for all initial shipments and quality levels $X \in B(X^*, \delta)$ and all $t \geq 0$

$$X(t) \in B(X^*, \epsilon). \tag{21}$$

The equilibrium point X^* is unstable, if it is not stable.

Definition 3
An equilibrium shipment and quality level X^* is asymptotically stable, if it is stable and there exists a $\delta > 0$ such that for all initial shipments and quality levels $X \in B(X^*, \delta)$

$$\lim_{t \to \infty} X(t) \longrightarrow X^*. \tag{22}$$

Definition 4
An equilibrium shipment and quality level X^* is exponentially stable, if there exists a neighborhood $N(X^*)$ of X^* and constants $b > 0$ and $\mu > 0$ such that

$$\|X^0(t) - X^*\| \leq b\|X^0 - X^*\| e^{-\mu t}, \quad \forall t \geq 0, \ \forall X^0 \in N(X^*); \quad (23)$$

X^* is globally exponentially stable, if (23) holds true for all $X^0 \in \mathcal{K}$.

Definition 5
An equilibrium shipment and quality level pattern X^* is a monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$, the Euclidean distance between $X(t)$ and X^*, $\|X(t) - X^*\|$, is a nonincreasing function of t; X^* is a global monotone attractor, if $\|X(t) - X^*\|$ is nonincreasing in t for all $X \in \mathcal{K}$.

Definition 6

An equilibrium X^* is a strictly monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$, $\|X(t) - X^*\|$ is monotonically decreasing to zero in t; X^* is a strictly global monotone attractor, if $\|X(t) - X^*\|$ is monotonically decreasing to zero in t for all $X \in \mathcal{K}$.

Stability Under Monotonicity

Recall (cf. Nagurney (1999)) that $F(X)$ is locally monotone at X^*, if there is a neighborhood $N(X^*)$ of X^*, such that

$$\langle (F(X) - F(X^*))^T, X - X^* \rangle \geq 0, \quad \forall X \in N(X^*); \tag{24}$$

$F(X)$ is monotone at X^*, if (24) holds for all $X \in \mathcal{K}$; F is *monotone over* $\mathcal{K}$, if (24) holds for all X and X^* in $\mathcal{K}$.

$F(X)$ is locally strictly monotone at X^*, if there exists a neighborhood $N(X^*)$ of X^*, such that

$$\langle (F(X) - F(X^*))^T, X - X^* \rangle > 0, \quad \forall X \in N(X^*),\ X \neq X^*; \tag{25}$$

$F(X)$ is strictly monotone at X^*, if (25) holds for all $X \in \mathcal{K}$; F is *strictly monotone over* $\mathcal{K}$, if (25) holds true for all X and X^* in $\mathcal{K}$, with $X \neq X^*$.

Stability Under Monotonicity

$F(X)$ is locally strongly monotone at X^*, if there is a neighborhood $N(X^*)$ of X^* and $\eta > 0$, such that

$$\langle (F(X) - F(X^*))^T, X - X^* \rangle \geq \eta \|X - X^*\|^2, \quad \forall X \in N(X^*); \quad (26)$$

$F(X)$ is strongly monotone at X^*, if (26) holds for all $X \in \mathcal{K}$; F is *strongly monotone over* $\mathcal{K}$, if (26) holds true for all X and X^* in $\mathcal{K}$.

Stability Under Monotonicity

The monotonicity of a function F is closely related to the positive-definiteness of its Jacobian ∇F (cf. Nagurney (1999)). Particularly, if ∇F is positive-semidefinite, F is monotone; if ∇F is positive-definite, F is strictly monotone; and, if ∇F is strongly positive definite, in the sense that the symmetric part of $\nabla F, (\nabla F^T + \nabla F)/2$, has only positive eigenvalues, then F is strongly monotone.

Existence and Uniqueness Results of the Equilibrium Pattern

Assumption 1

Suppose that in a network oligopoly model there exists a sufficiently large M, such that for any (i, j),

$$\frac{\partial U_i(Q, q)}{\partial Q_{ij}} < 0, \tag{27}$$

for all shipment patterns Q with $Q_{ij} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any i,

$$\frac{\partial U_i(Q, q)}{\partial q_i} < 0, \tag{28}$$

for all quality level patterns q with $q_i \geq \bar{M}$.

Proposition 1

Any network oligopoly problem that satisfies Assumption 1 possesses at least one equilibrium shipment and quality level pattern.

Existence and Uniqueness Results of the Equilibrium Pattern

Proposition 2 Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in (13). Then it has at most one equilibrium point.

Theorem 4 (Under Local Monotonicity)
Let X^* be a network Cournot-Nash equilibrium by Definition 1. We have the following stability results under various local monotonicity conditions:
(i). If $-\nabla U(Q, q)$ is monotone (locally monotone) at (Q^*, q^*), then (Q^*, q^*) is a global monotone attractor (monotone attractor) for the utility gradient process.
(ii). If $-\nabla U(Q, q)$ is strictly monotone (locally strictly monotone) at (Q^*, q^*), then (Q^*, q^*) is a strictly global monotone attractor (strictly monotone attractor) for the utility gradient process.
(iii). If $-\nabla U(Q.q)$ is strongly monotone (locally strongly monotone) at (Q^*, q^*), then (Q^*, q^*) is globally exponentially stable (exponentially stable) for the utility gradient process.

Existence and Uniqueness Results of the Equilibrium Pattern

Theorem 4 (Under Global Monotonicity)
(i). If $-\nabla U(Q,q)$ is monotone, then every network Cournot-Nash equilibrium, provided its existence, is a global monotone attractor for the utility gradient process.
(ii). If $-\nabla U(Q,q)$ is strictly monotone, then there exists at most one network Cournot-Nash equilibrium. Furthermore, provided existence, the unique network Cournot-Nash equilibrium is a strictly global monotone attractor for the utility gradient process.
(iii). If $-\nabla U(Q,q)$ is strongly monotone, then there exists a unique network Cournot-Nash equilibrium, which is globally exponentially stable for the utility gradient process.

Stability Under Monotonicity: Example 1

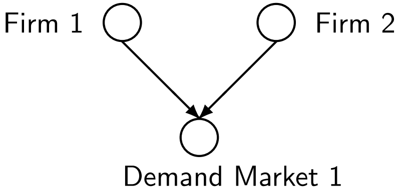


Figure: Example 1

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10.$$

Stability Under Monotonicity: Example 1

The demand price functions are:

$$p_{11}(d,q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2,$$

$$p_{21}(d,q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2.$$

The utility function of firm 1 is, hence:

$$U_1(Q,q) = p_{11}d_{11} - \hat{f}_1 - \hat{c}_{11},$$

whereas the utility function of firm 2 is:

$$U_2(Q,q) = p_{21}d_{21} - \hat{f}_2 - \hat{c}_{21}.$$

Stability Under Monotonicity: Example 1

The Jacobian matrix of -$\nabla U(Q,q)$, denoted by $J(Q_{11},Q_{21},q_1,q_2)$, is

$$J(Q_{11},Q_{21},q_1,q_2)=\begin{pmatrix} 6 & 1.4 & -0.3 & -0.5 \\ 2.6 & 21 & -0.1 & -0.5 \\ -0.3 & 0 & 4 & 0 \\ 0 & -0.5 & 0 & 2 \end{pmatrix}.$$

Stability Under Monotonicity: Example 1

The equilibrium solution, which is:
$Q^*_{11}= 16.08$, $Q^*_{21}= 2.79$, $q^*_1= 1.21$, and $q^*_2= 0.70$ is globally exponentially stable. In addition, the utility gradient process has the following convergence rate:

$$\|X(t) - X^0\| \ \leq \ \|X^* - X^0\|e^{-t}, \quad \forall t \geq 0, \ \forall X^0 \in R_+^{mn+m}.$$

Stability Under Monotonicity: Example 2

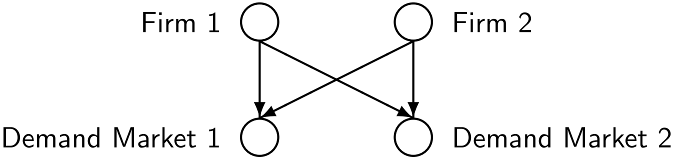


Figure: Example 2

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10,$$

$$\hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5.$$

Stability Under Monotonicity: Example 2

The demand price functions are:

$$p_{11}(d,q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2,$$

$$p_{12}(d,q) = 100 - 2d_{12} - d_{22} + 0.4q_1 + 0.2q_2,$$

$$p_{21}(d,q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2,$$

$$p_{22}(d,q) = 100 - 0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2.$$

The utility function of firm 1 is:

$$U_1(Q,q) = p_{11}d_{11} + p_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12})$$

with the utility function of firm 2 being:

$$U_2(Q,q) = p_{21}d_{21} + p_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}).$$

The Jacobian of $-\nabla U(Q,q)$, denoted by $J(Q_{11},Q_{12},Q_{21},Q_{22},q_1,q_2)$, is

$$J(Q_{11},Q_{12},Q_{21},Q_{22},q_1,q_2)$$

$$=\begin{pmatrix} 6 & 2 & 1.4 & 1 & -0.3 & -0.05 \\ 2 & 16 & 1 & 2 & -0.4 & -0.2 \\ 2.6 & 2 & 21 & 4 & -0.1 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & -0.01 & -0.6 \\ -0.3 & -0.4 & 0 & 0 & 4 & 0 \\ 0 & 0 & -0.5 & -0.6 & 0 & 2 \end{pmatrix}.$$

Moreover, the equilibrium solution (stationary point) is:
$Q_{11}^*= 14.27$, $Q_{12}^*= 3.81$, $Q_{21}^*= 1.76$, $Q_{22}^*= 4.85$, $q_1^*= 1.45$, $q_2^*= 1.89$ and it is globally exponentially stable. In addition, as was also the case for Example 1 above, the utility gradient process has the following convergence rate:

$$\|X(t) - X^0\| \ \leq \ \|X^* - X^0\| e^{-t}, \quad \forall t \geq 0, \, \forall X^0 \in R_+^{mn+m}.$$

The Algorithm-The Euler Method

Iteration τ of the Euler method (see also Nagurney and Zhang (1996)) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{29}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the function that enters the variational inequality problem (19).
The sequence $\{a_{\tau}\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to 0$, as $\tau \to \infty$.

Explicit Formulae for the Euler Method Applied to the Network Oligopoly

$$Q_{ij}^{\tau+1} = \max\{0, Q_{ij}^{\tau} + a_\tau (p_{ij}(d^\tau, q^\tau) + \sum_{k=1}^{n} \frac{\partial p_{ik}(d^\tau, q^\tau)}{\partial d_{ij}} d_{ik}^{\tau}$$

$$- \frac{\partial \hat{f}_i(s^\tau, q_i^\tau)}{\partial s_i} - \frac{\partial \hat{c}_{ij}(Q_{ij}^\tau)}{\partial Q_{ij}})\}, \tag{30}$$

$$q_i^{\tau+1} = \max\{0, q_i^\tau + a_\tau (\sum_{k=1}^{n} \frac{\partial p_{ik}(d^\tau, q^\tau)}{\partial q_i} d_{ik}^\tau - \frac{\partial \hat{f}_i(s^\tau, q_i^\tau)}{\partial q_i})\}. \tag{31}$$

$$d_{ij}^{\tau+1} = Q_{ij}^{\tau+1}; \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{32}$$

$$s_i^{\tau+1} = \sum_{j=1}^{n} Q_{ij}^{\tau+1}, \quad s = 1, \ldots, m. \tag{33}$$

Theorem 5

In the network oligopoly problem with product differentiation and quality levels let $F(X) = -\nabla U(Q, q)$ *be strictly monotone at any equilibrium pattern and assume that Assumption 1 is satisfied. Also, assume that* F *is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern* $(Q^*, q^*) \in K$ *and any sequence generated by the Euler method as given by (29) above, where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* (Q^*, q^*).

Numerical Examples

We implemented the Euler method, as described in Section 3, using Matlab on a LenovoE46A. The convergence criterion was $\epsilon = 10^{-6}$; that is, the Euler method was considered to have converged if, at a given iteration, the absolute value of the difference of each product shipment and each quality level differed from its respective value at the preceding iteration by no more than ϵ.

The sequence $\{a_\tau\}$ was: $.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)$. We initialized the algorithm by setting each product shipment $Q_{ij} = 2.5$, $\forall i, j$, and by setting the quality level of each firm $q_i = 0.00$, $\forall i$.

Example 1 Revisited

The Euler method required 39 iterations for convergence to the equilibrium pattern for Example 1 described in Section 3. The utility/profit of firm 1 was 723.89 and that of firm 2 was 34.44.

The Trajectory for the Product Shipments for Example 1

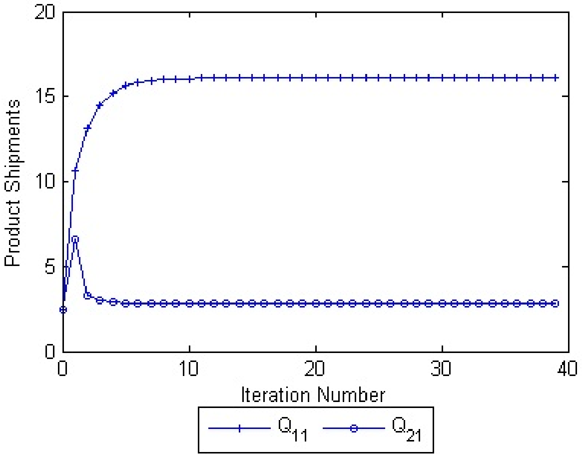


Figure: Product shipments for Example 1

The Trajectory for the Quality Levels for Example 1

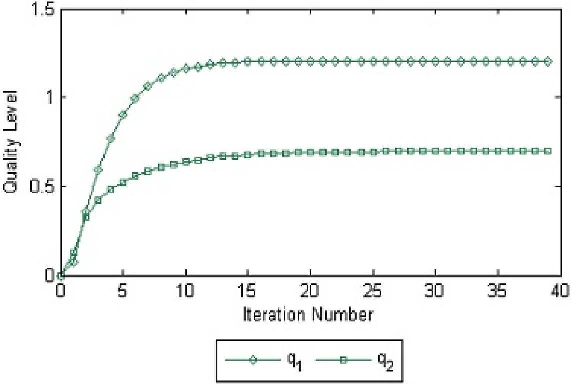


Figure: Quality levels for Example 1

Example 2 Revisited

For Example 2, described in Section 3, the Euler method required 45 iterations for convergence. The profit of firm 1 was 775.19, whereas that of firm 2 was 145.20.

The Trajectory for the Product Shipments for Example 2

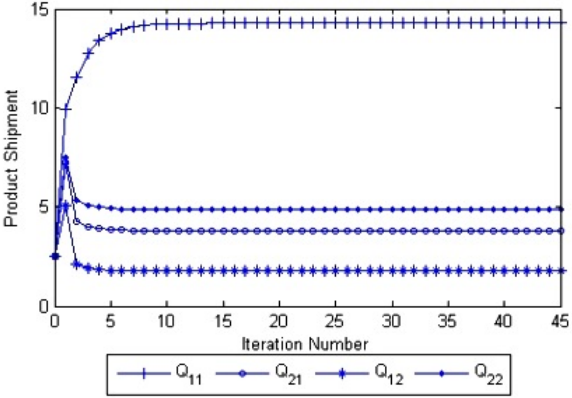


Figure: Product shipments for Example 2

The Trajectory for the Quality Levels for Example 2

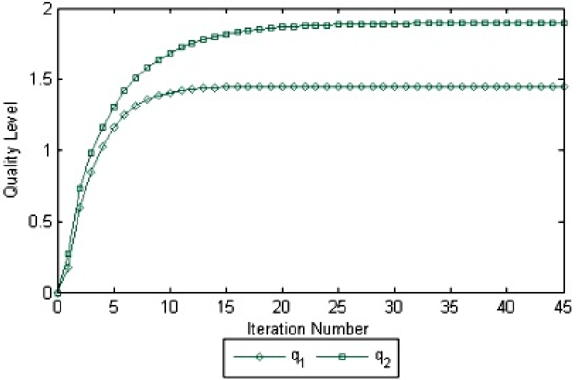


Figure: Quality levels for Example 2

Example 3

We assume, in this example, that there is another firm, firm 3, entering the oligopoly and its quality cost is much higher than those of firms 1 and 2.

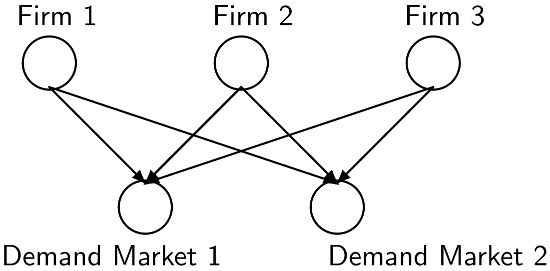


Figure: Example 3

Example 3

The production cost functions were:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + s_1 s_3 + 2q_1^2 + 39,$$

$$\hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + 2s_3 s_2 + q_2^2 + 37,$$

$$\hat{f}_3(s, q_3) = s_3^2 + s_1 s_3 + s_3 s_2 + 8q_3^2 + 60.$$

The total transportation cost functions were:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7,$$

$$\hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$

$$\hat{c}_{31}(Q_{31}) = 2Q_{31}^2 + 9, \quad \hat{c}_{32}(Q_{32}) = 3Q_{32}^2 + 8,$$

Example 3

The demand price functions were:

$$p_{11}(d,q) = 100 - d_{11} - 0.4d_{21} - 0.1d_{31} + 0.3q_1 + 0.05q_2 + 0.05q_3,$$

$$p_{12}(d,q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.4q_1 + 0.2q_2 + 0.2q_3,$$

$$p_{21}(d,q) = 100 - 0.6d_{11} - 1.5d_{21} - 0.1d_{31} + 0.1q_1 + 0.5q_2 + 0.1q_3,$$

$$p_{22}(d,q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.6q_2 + 0.01q_3,$$

$$p_{31}(d,q) = 100 - 0.2d_{11} - 0.4d_{21} - 1.8d_{31} + 0.2q_1 + 0.2q_2 + 0.7q_3,$$

$$p_{32}(d,q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 0.4q_3.$$

The utility function expressions of firm 1, firm 2, and firm 3 were, respectively:

$$U_1(Q,q) = p_{11}d_{11} + p_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12}),$$

$$U_2(Q,q) = p_{21}d_{21} + p_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}),$$

$$U_3(Q,q) = p_{31}d_{31} + p_{32}d_{32} - \hat{f}_3 - (\hat{c}_{31} + \hat{c}_{32}).$$

Example 3

The Jacobian of $-\nabla U(Q, q)$ was

$$J_{(}Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\ 2 & 16 & 1 & 2 & 1 & 1.1 & -0.4 & -0.2 & -0.2 \\ 2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.6 & -0.01 \\ 1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\ 1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -0.4 \\ -0.3 & -0.4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.5 & -0.6 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & -0.4 & 0 & 0 & 16 \end{pmatrix}.$$

Example 3

The Euler method converged to the equilibrium solution:
$Q^*_{11}= 12.63$, $Q^*_{12}= 3.45$, $Q^*_{21}= 1.09$, $Q^*_{22}= 3.21$, $Q^*_{31}= 6.94$, $Q^*_{32}= 5.42$, $q^*_1= 1.29$, $q^*_2= 1.23$, $q^*_3= 0.44$ in 42 iterations.
The profits of the firms were: $U_1 = 601.67$, $U_2 = 31.48$, and $U_3 = 403.97$.

The Trajectory for the Product Shipments for Example 3

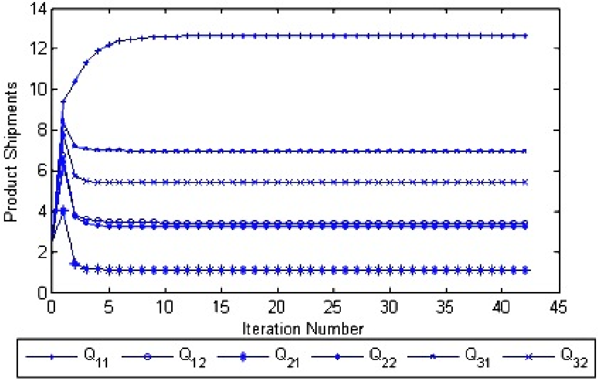


Figure: Product shipments for Example 3

The Trajectory for the Quality Levels for Example 3

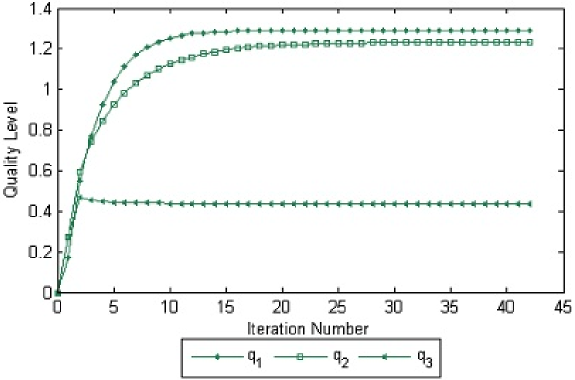


Figure: Quality levels for Example 3

Example 4

The new demand price functions associated with demand market 2 were now:

$$p_{12}(d,q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.49q_1 + 0.2q_2 + 0.2q_2,$$

$$p_{22}(d,q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.87q_2 + 0.01q_3,$$

and

$$p_{32}(d,q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 1.2q_3.$$

Example 4

The Jacobian of $-\nabla U(Q,q)$ was now:

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\ 2 & 16 & 1 & 2 & 1 & 1.1 & -0.49 & -0.2 & -0.2 \\ 2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.87 & -0.01 \\ 1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\ 1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -1.2 \\ -0.3 & -0.49 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.5 & -0.87 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & -1.2 & 0 & 0 & 16 \end{pmatrix}.$$

Example 4

The computed equilibrium solution was now: $Q_{11}^* = 13.41$, $Q_{12}^* = 3.63$, $Q_{21}^* = 1.41$, $Q_{22}^* = 4.08$, $Q_{31}^* = 3.55$, $Q_{32}^* = 2.86$, $q_1^* = 1.45$, $q_2^* = 2.12$, $q_3^* = 0.37$. The profits of the firms were now: $U_1 = 682.44$, $U_2 = 82.10$, and $U_3 = 93.19$.

The Trajectory for the Product Shipments for Example 4

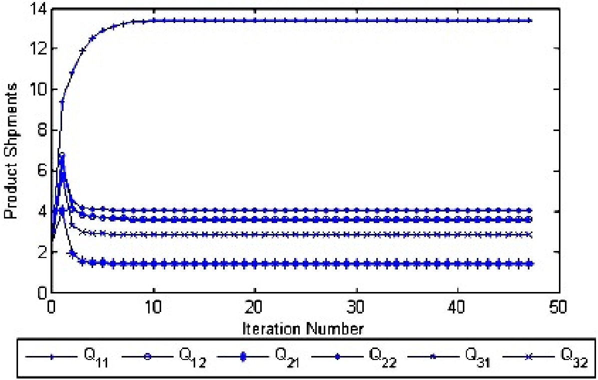


Figure: Product shipments for Example 4

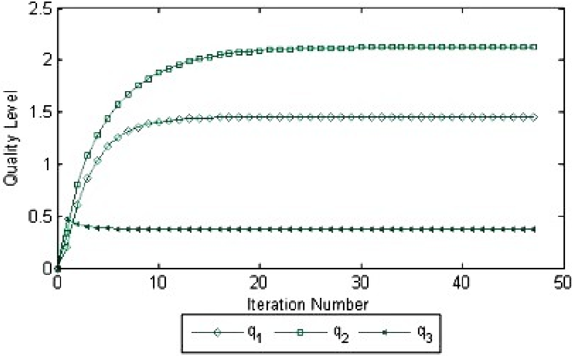


Figure: Quality levels for Example 4

Example 4

The equilibrium quality levels of the three firms changed, with those of firm 1 and firm 2, increasing, relative to their values in Example 3.

Since it costs much more for firm 3 to achieve higher quality levels than it does for firm 1 and firm 2, the profit of firm 3 decreased by 76.9%, while the profits of the firms 1 and 2 increased 13.4% and 160.8%, respectively.

Example 5

The data were as in Example 4 except for the production cost functions, which were now:

$$\hat{f}_1(s, q_1) = 2s_1^2 + 0.005s_1q_1 + 2q_1^2 + 30,$$

$$\hat{f}_2(s, q_2) = 4s_2^2 + 0.005s_2q_2 + q_2^2 + 30,$$

$$\hat{f}_3(s, q_3) = 4s_3^2 + 0.005s_3q_3 + 8q_3^2 + 50.$$

Example 5

The Jacobian of $-\nabla U(Q, q)$, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$, was

$$J_(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 8 & 4 & 0.4 & 0 & 0.1 & 0 & -0.295 & -0.05 & -0.05 \\ 4 & 18 & 0 & 1 & 0 & 0.1 & -0.395 & -0.2 & -0.2 \\ 0.6 & 0 & 25 & 8 & 0.1 & 0 & -0.1 & -0.495 & -0.1 \\ 0 & 0.7 & 8 & 15.4 & 0 & 0.1 & -0.01 & -0.595 & -0.01 \\ 0.2 & 0 & 0.4 & 0 & 9.6 & 2 & -0.2 & -0.2 & -0.695 \\ 0 & 0.1 & 0 & 0.3 & 2 & 12 & -0.2 & -0.1 & -0.395 \\ -0.295 & -0.395 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.495 & -0.595 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.695 & -0.395 & 0 & 0 & 16 \end{pmatrix}.$$

Example 5

The Euler method converged to the equilibrium solution: $Q_{11}^*= 10.95$, $Q_{12}^*= 2.84$, $Q_{21}^*= 2.04$, $Q_{22}^*= 5.34$, $Q_{31}^*= 4.47$, $Q_{32}^*= 3.49$, $q_1^*= 1.09$, $q_2^*= 2.10$, $q_3^*= 0.28$ in 46 iterations. The profits of the firms were: $U_1 = 1222.89$, $U_2 = 668.03$, and $U_3 = 722.03$.

The Trajectory for the Product Shipments for Example 5

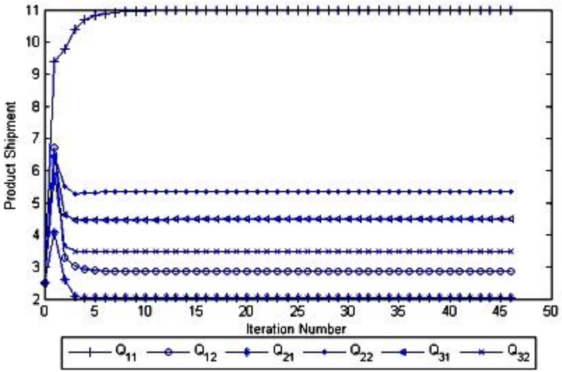


Figure: Product shipments for Example 5

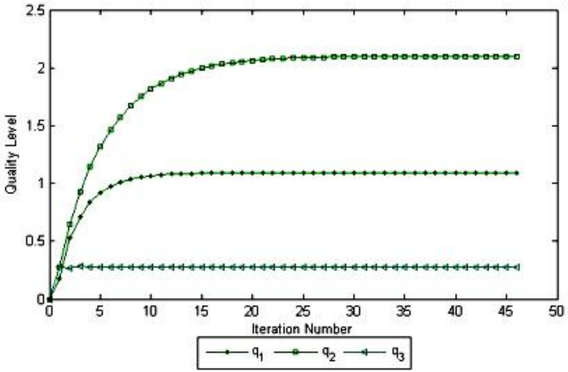


Figure: Quality levels for Example 5

Summary and Conclusions

- We developed a new network oligopoly model with product differentiation and quality levels, in a network framework.
- We derived the governing equilibrium conditions and provided alternative variational inequality formulations.
- We then proposed a continuous-time adjustment process and showed how our projected dynamical systems model of the network oligopoly problem under consideration here guarantees that the product shipments and quality levels remain nonnegative.

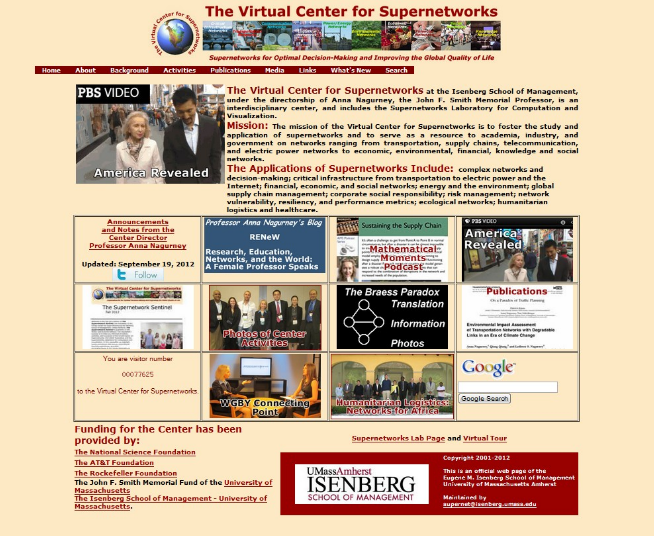
Summary and Conclusions

- We provided qualitative properties of existence and uniqueness of the dynamic trajectories and also gave conditions, using a monotonicity approach, for stability analysis and associated results.
- We, subsequently, described an algorithm, which yields closed form expressions for the product shipment and quality levels at each iteration and which provides a discrete-time discretization of the continuous-time product shipment and quality level trajectories.
- We then, through several numerical examples, illustrated the model and theoretical results, in order to demonstrate how the contributions in this paper could be applied in practice.

Summary and Conclusions

- The models are not limited to a preset number of firms (such as two, in the case of duopoly) or to specific functional forms (linear demand functions, for example).
- The models capture quality levels both on the supply side as well as on the demand side, with linkages through the transportation costs, yielding an integrated economic network framework.
- Restrictive assumptions need not be imposed on the underlying dynamics, since we make use of projected dynamical systems.
- Both qualitative results, including stability analysis results, as well as an effective, and easy to implement, computational procedure are provided, along with numerical examples.

Thank you!



For more information, please visit http://supernet.isenberg.umass.edu.

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