A Network Efficiency Measure with Applications to Critical Infrastructure Networks

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4th International Conference on Computational Management Science, April 20-22, 2007, Geneva, Switzerland

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Funding for our research has been provided by:

National Science Foundation

AT&T Foundation

John F. Smith Memorial Fund - University of Massachusetts at Amherst

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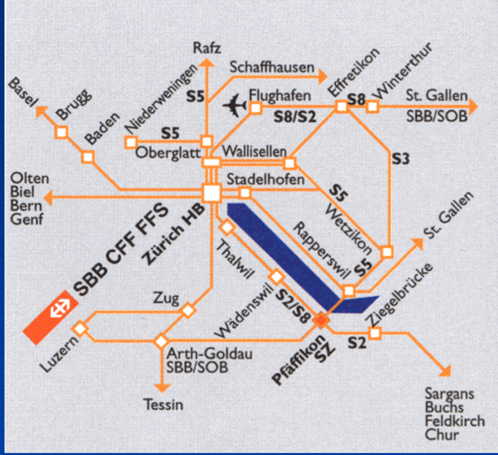
Some Critical Infrastructure Networks

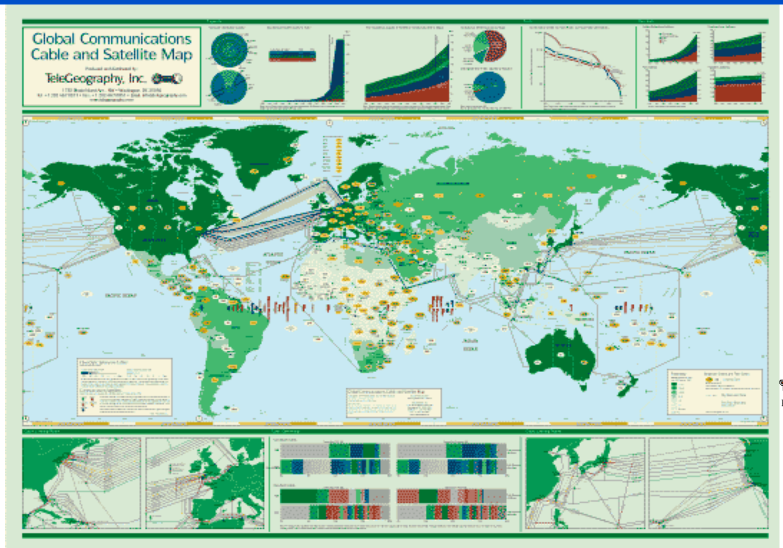
Metro



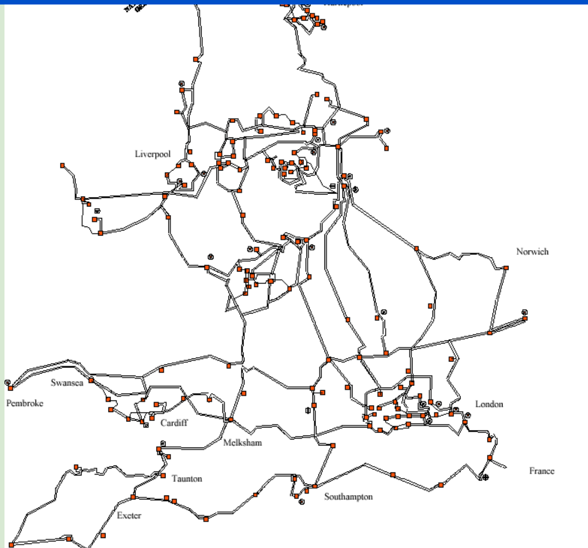
Rail Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks



British Electricity Grid



Network Vulnerability

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.

- For example:
  - Hurricane Katrina, August 23, 2005
  - The biggest blackout in North America, August 14, 2003
  - 9/11 Terrorist Attacks, September 11, 2001

Earthquake Damage
prcs.org.pk

Tsunami
letthesunshinein.wordpress.com

Storm Damage
www.srh.noaa.gov

Infrastructure Collapse
www.10-7.com

An Urgent Need for a Network Efficiency/Performance Measure

In order to be able to assess the performance/efficiency of a network, it is imperative that appropriate measures be devised.

Appropriate network measures can assist in the identification of the importance of network components, that is, nodes and links, and their rankings. Such rankings can be very helpful in the case of the determination of network vulnerabilities as well as when to reinforce/enhance security.

Recent Literature on Network Vulnerability

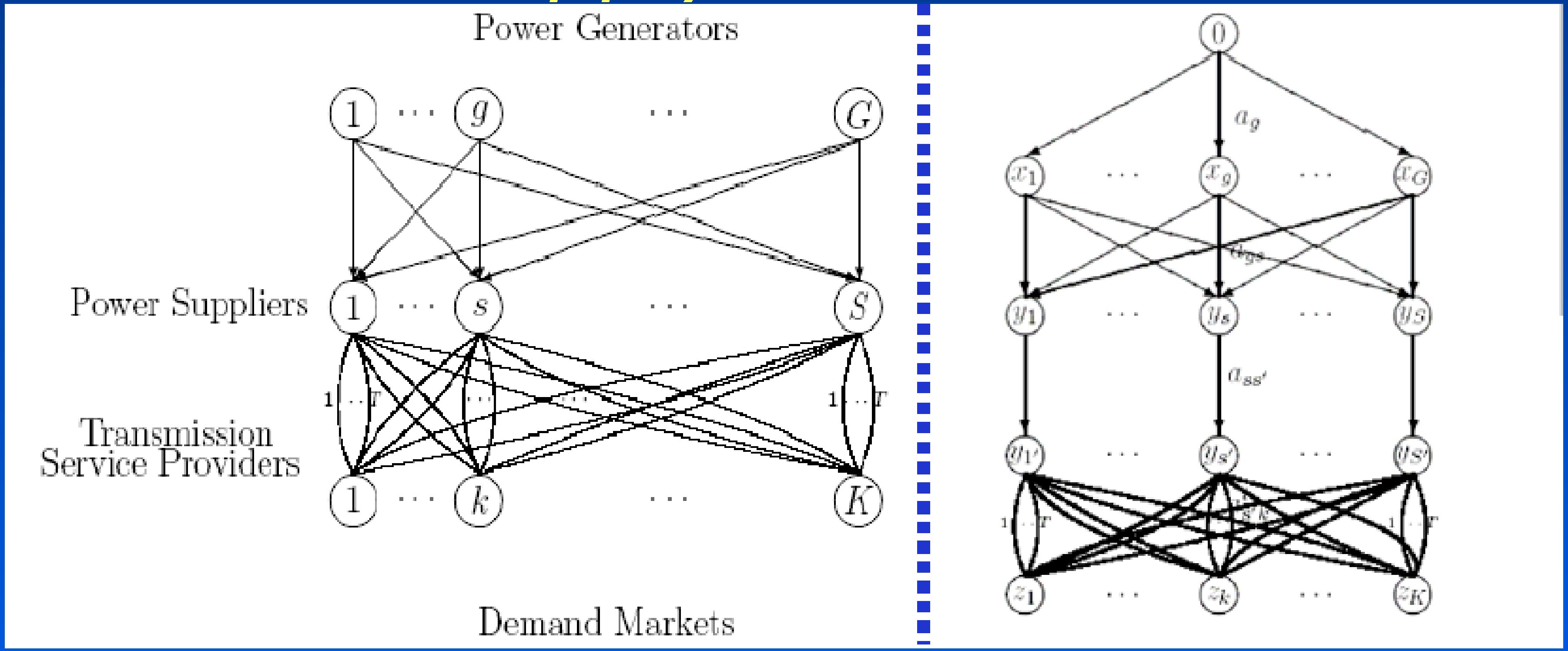
- Latora and Marchiori (2001, 2002, 2004)
- Barrat, Barthélemy and Vespignani (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Chassin and Posse (2005)
- Holme, Kim, Yoon and Han (2002)
- Sheffi (2005)
- Taylor and D'este (2004)
- Jenelius, Petersen and Mattson (2006)
- Murray-Tuite and Mahmassani (2004)

Transportation Network Equilibrium Paradigm

We have recently shown that, as hypothesized over 50 years ago by Beckmann, McGuire, and Winsten (1956), that electric power generation and distribution networks can be reformulated and solved as transportation networks, Wu, Nagurney, Liu, and Stranlund, *Transportation Research D* (2006), Nagurney et al., *Transportation Research D*, in press.

We have demonstrated that financial networks with intermediation can be reformulated and solved as transportation network problems; Liu and Nagurney, *Computational Management Science*, in press.

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks

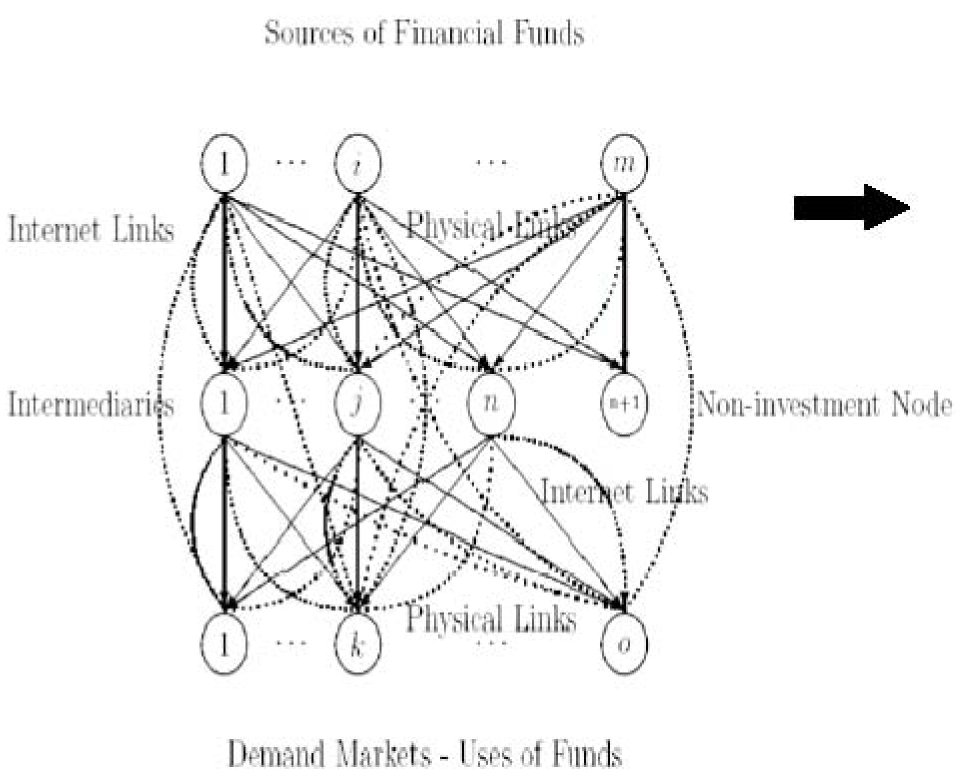


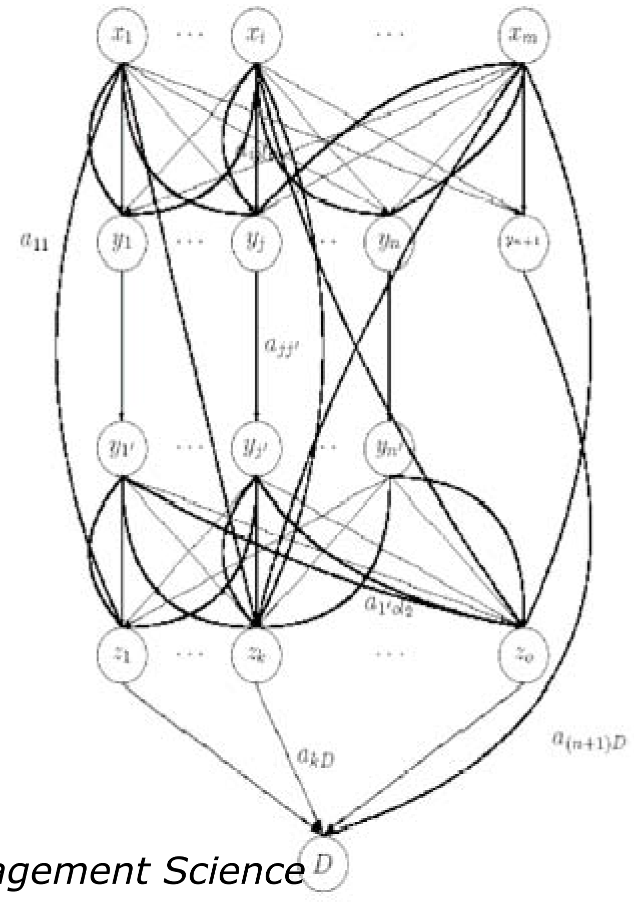
Electric Power Supply Chain Network

Transportation Network

Nagurney et al, to appear in *Transportation Research E*

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation





Liu and Nagurney, to appear in *Computational Management Science*

Transportation Network Equilibrium Problem

Consider a general network $G = [N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. P_w denotes the set of paths, assumed to be acyclic, connecting the O/D pair of nodes w and P the set of all paths.

Let x_p represent the flow on path p and f_a the flow on link a. The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. This expression states that the load on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Transportation science has historically been the discipline that has pushed the frontiers in terms of methodological developments for such problems (which are often large-scale) beginning with the work of Beckmann, McGuire, and Winsten (1956).

Definition: Transportation Network Equilibrium

A route flow pattern $x^* \in K$ is said to be a transportation network equilibrium (according to Wardrop's (1952) first principle) if only the minimum cost routes are used (that is, have positive flow) for each O/D pair. The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every path $p \in P_w$:

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases}$$

VI Formulation of Transportation Network Equilibrium (Dafermos (1980), Smith (1979))

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

The Network Efficiency Measure of Latora and Marchiori (2001)

- Latora and Marchiori (2001) proposed a network efficiency measure (the L-M measure) as follows:

Definition : The L-M Measure

The network performance/efficiency measure, $E(G)$, according to Latora and Marchiori (2001) for a given network topology G, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

Our Research on Network Efficiency and Network Vulnerability

A Network Efficiency Measure with Application to Critical Infrastructure Networks, Nagurney and Qiang (2007a), to appear in *Journal of Global Optimization.*

A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability, Nagurney and Qiang (2007b), to appear in *Proceedings of the POMS 18th Annual Conference*, May 4 to May 7, 2007.

A Unified Network Performance Measure with Importance Identification and the Ranking of Network Components (2007), *Optimization Letters,* in press.

The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007a) (the N-Q Measure) proposed a network efficiency measure for networks with fixed demand, which captures the demand and flow information under the network equilibrium.

Definition : The N-Q Measure

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2007), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w

Importance of a Network Component

Definition : Importance of a Network Component According to the L-M Measure

The importance of a network component $g \in G$, $\bar{I}(g)$, is measured by the network efficiency drop, determined by the L-M measure, after g is removed from the network:

$$\bar{I}(g) = \frac{\triangle E}{E(G)} = \frac{E(G) - E(G-g)}{E(G)}.$$

where $G - g$ is the resulting network after component g is removed from network G.

Definition : Importance of a Network Component According to the N-Q Measure

The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop, determined by the N-Q measure, after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where $G - g$ is the resulting network after component g is removed from network G.

The Approach to Study the Importance of Network Components

The elimination of a link is treated in the N-Q measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Hence, our measure is well-defined even in the case of disconnected networks.

The L-M Measure vs. the N-Q Measure

Theorem

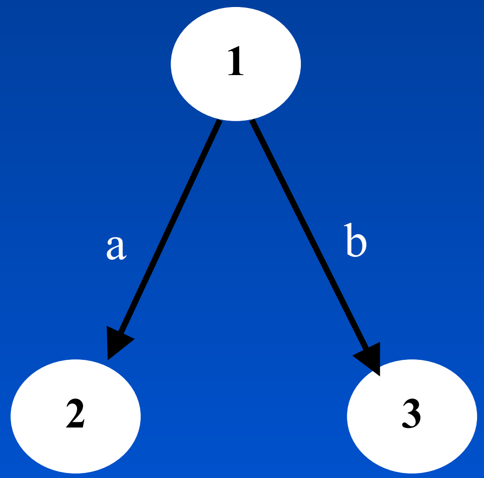
If positive demands exist for all pairs of nodes in the network G*, and each of these demands is equal to* 1 *and if* d_{ij} *is set equal to* λ_w*, where* $w = (i, j)$*, for all* $w \in W$ *then the proposed network efficiency measure and the L-M measure are one and the same.*

Example 1

Assume a network with two O/D pairs: w_1=(1,2) and w_2=(1,3) with demands given, respectively, by d_{w1}=100 and d_{w2}=20. The path for each O/D pair is: for w_1, $p_1=a$; for w_2, $p_2=b$.

The equilibrium path flows are $x_{p_1}^*$= 100, $x_{p_2}^*$=20.

The equilibrium path travel cost is $C_{p_1}=C_{p_2}$=20.



$c_a(f_a)=0.01f_a+19$

$c_b(f_b)=0.05f_b+19$

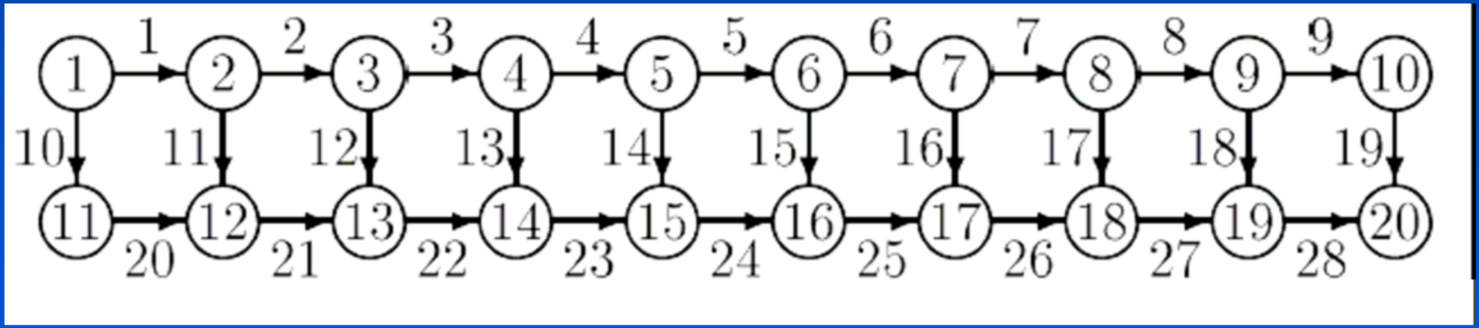
Importance and Ranking of Links and Nodes

Link	Importance Value from the N-Q Measure	Importance Value from the N-Q Measure
a	0.8333	1
b	0.1667	2

Node	Importance Value from the N-Q Measure	Importance Ranking from the N-Q Measure
1	1	1
2	0.8333	2
3	0.1667	3

Example 2

The network topology is the following:



$$w_1 = (1,19),\ w_2 = (1,20)$$

$$d_{w_1} = d_{w_2} = 100$$

Link Cost Functions

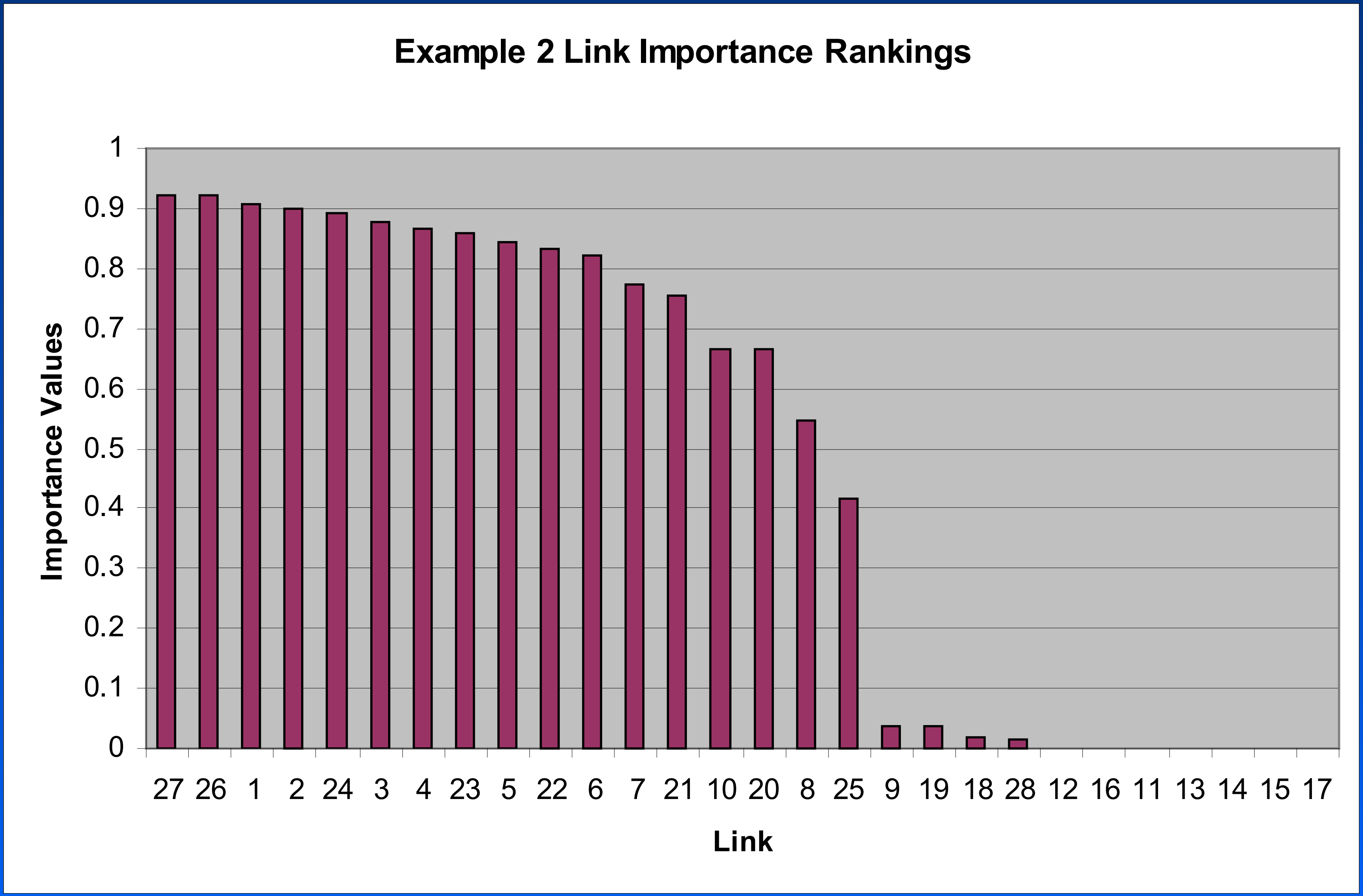
Link a	Link Cost Function $c_a(f_a)$
1	$.00005f_1^4 + 5f_1 + 500$
2	$.00003f_2^4 + 4f_2 + 200$
3	$.00005f_3^4 + 3f_3 + 350$
4	$.00003f_4^4 + 6f_4 + 400$
5	$.00006f_5^4 + 6f_5 + 600$
6	$7f_6 + 500$
7	$.00008f_7^4 + 8f_7 + 400$
8	$.00004f_8^4 + 5f_8 + 650$
9	$.00001f_9^4 + 6f_9 + 700$
10	$4f_{10} + 800$
11	$.00007f_{11}^4 + 7f_{11} + 650$
12	$8f_{12} + 700$
13	$.00001f_{13}^4 + 7f_{13} + 600$
14	$8f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003f_{15}^4 + 9f_{15} + 200$
16	$8f_{16} + 300$
17	$.00003f_{17}^4 + 7f_{17} + 450$
18	$5f_{18} + 300$
19	$8f_{19} + 600$
20	$.00003f_{20}^4 + 6f_{20} + 300$
21	$.00004f_{21}^4 + 4f_{21} + 400$
22	$.00002f_{22}^4 + 6f_{22} + 500$
23	$.00003f_{23}^4 + 9f_{23} + 350$
24	$.00002f_{24}^4 + 8f_{24} + 400$
25	$.00003f_{25}^4 + 9f_{25} + 450$
26	$.00006f_{26}^4 + 7f_{26} + 300$
27	$.00003f_{27}^4 + 8f_{27} + 500$
28	$.00003f_{28}^4 + 7f_{28} + 650$

Importance and Ranking of Links

Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19


Example 2 Link Importance Rankings
Importance Values
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
27 26 1 2 24 3 4 23 5 22 6 7 21 10 20 8 25 9 19 18 28 12 16 11 13 14 15 17
Link

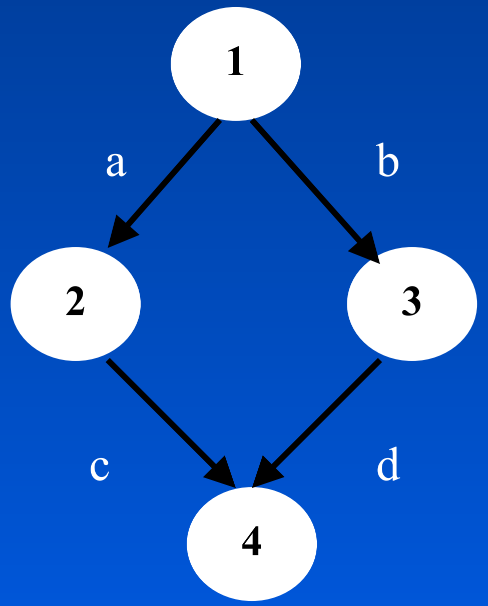
Example 3: the Braess (1968) Network

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of 6, the equilibrium path flows are $\mathbf{x_{p_1}^{*}= x_{p_2}^{*}=3}$.

The equilibrium path travel cost is

$\mathbf{C_{p_1}=C_{p_2}=83}$.



$c_a(f_a)=10f_a$ $\quad$ $c_b(f_b)= f_b+50$

$c_c(f_c)=f_c+50$ $\quad$ $c_d(f_d)= 10f_d$

Adding a Link Increases Travel Cost for All!

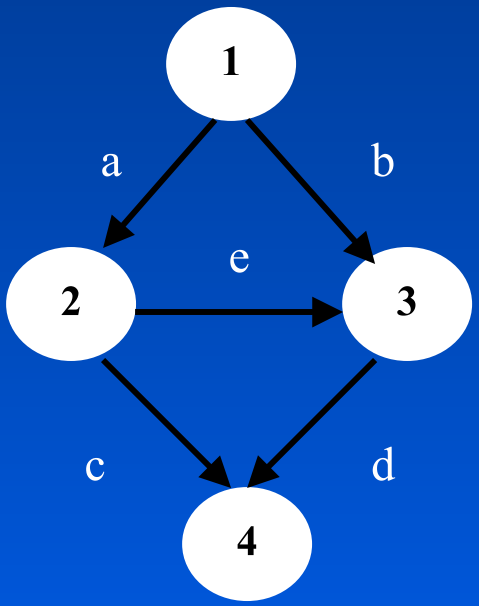
Adding a new link creates a new path **p_3=(a,e,d).**

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow, the cost on path **p_3, C_{p3}=70.**

The new equilibrium flow pattern network is **$x_{p_1}{}^* = x_{p_2}{}^* = x_{p_3}{}^* = 2$.**

The equilibrium path travel cost is

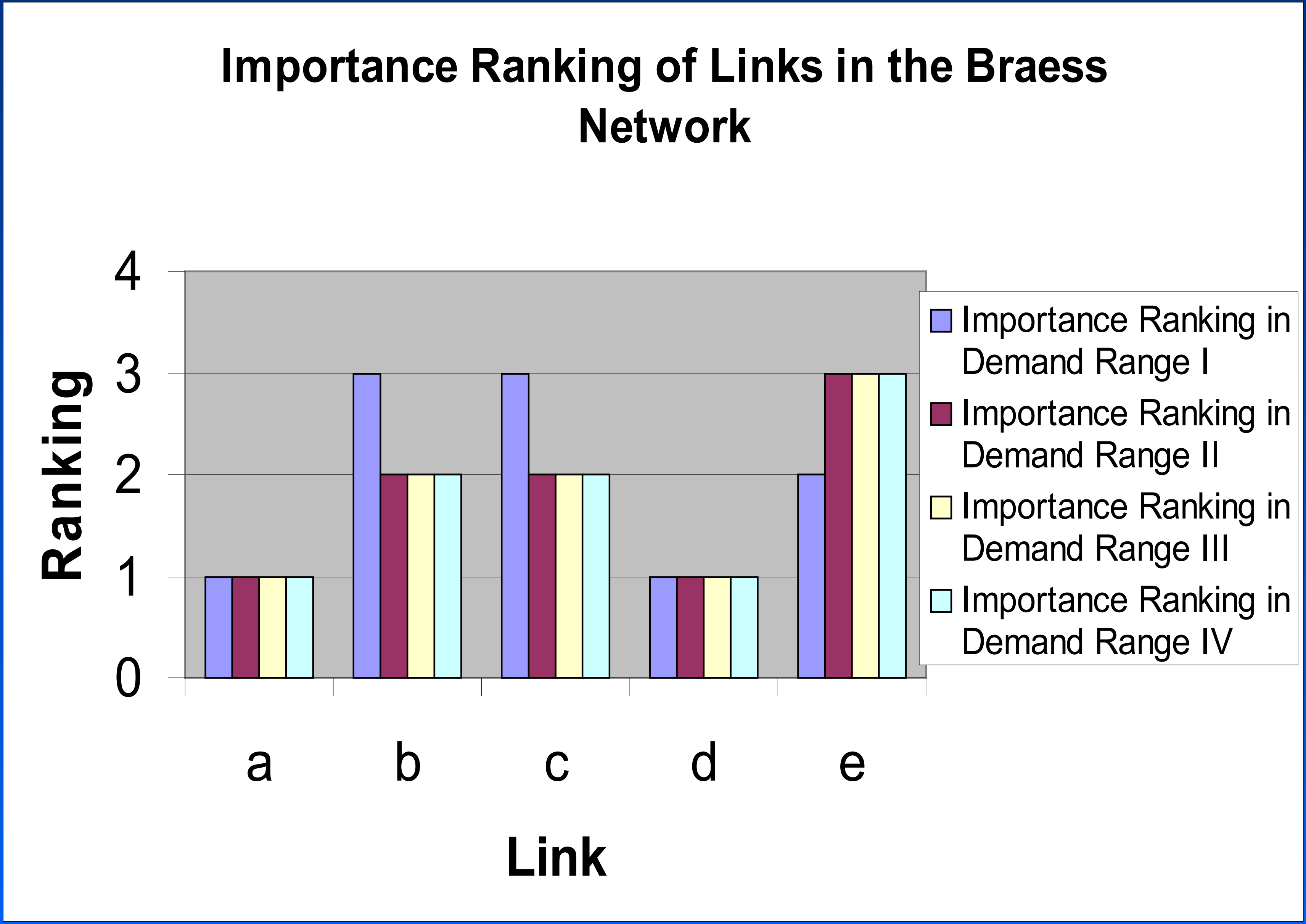
$C_{p1}=C_{p2}= C_{p3}= 92$.

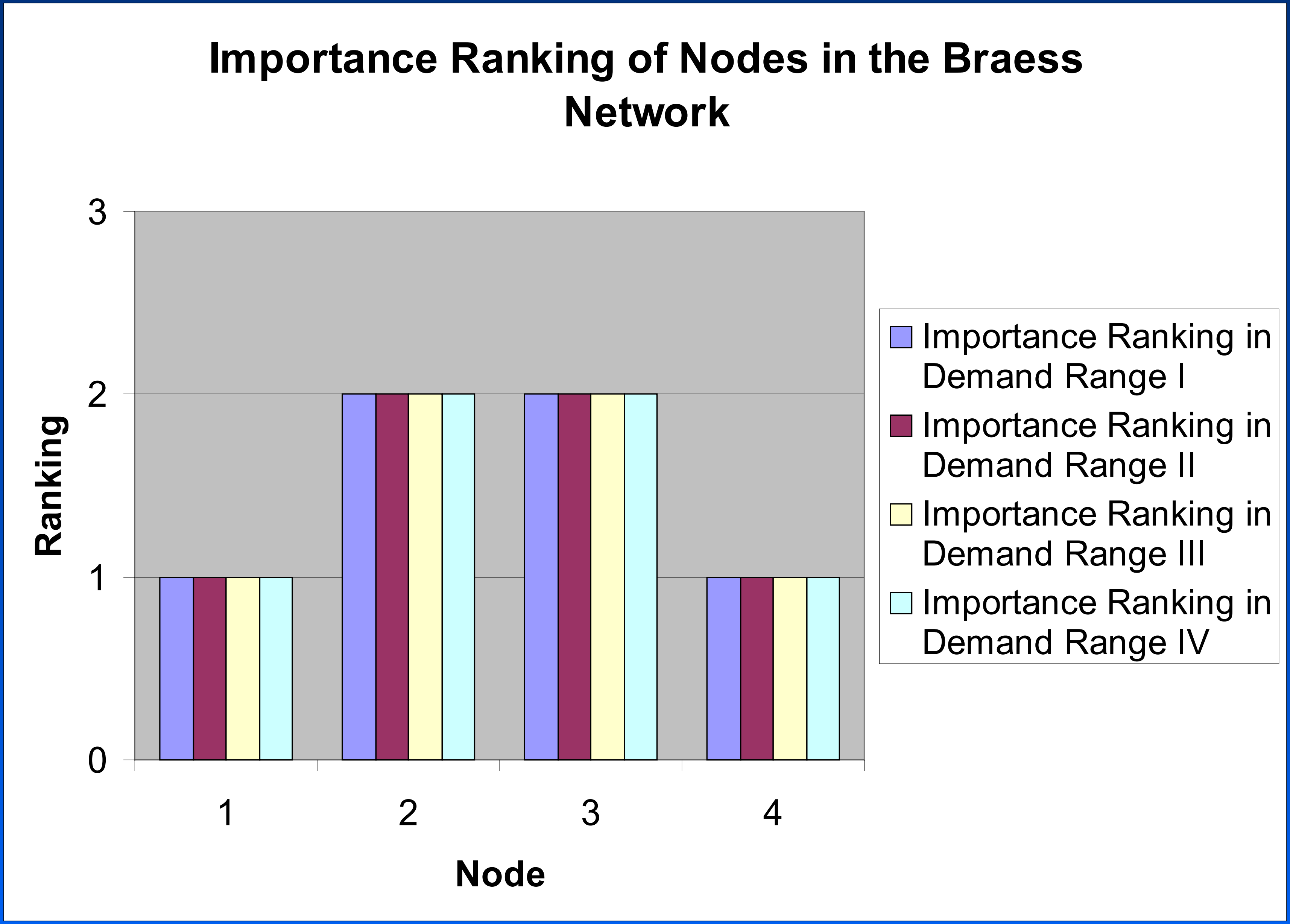


$c_e(f_e)= f_e+10$

Four Demand Ranges

- Demand Range I: $d_w \in [0, 80/31)$
  - Only $\mathbf{p_1}$ and $\mathbf{p_2}$ are used and the Braess Paradox does not occur
- Demand Range II: $d_w \in [80/31, 40/11]$
  - Only $\mathbf{p_1}$ and $\mathbf{p_2}$ are used and the Braess Paradox occurs
- Demand Range III: $d_w \in (40/11, 80/9]$
  - All paths are used and the Braess Paradox still occurs
- Demand Range IV: $d_w \in (80/9, \infty)$
  - Only $\mathbf{p_1}$ and $\mathbf{p_2}$ are used and the Braess Paradox vanishes


Importance Ranking of Links in the Braess Network
Ranking
0
1
2
3
4
a
b
c
d
e
Link
Importance Ranking in Demand Range I
Importance Ranking in Demand Range II
Importance Ranking in Demand Range III
Importance Ranking in Demand Range IV


Importance Ranking of Nodes in the Braess Network
3
2
1
0
Ranking
1
2
3
4
Node
Importance Ranking in Demand Range I
Importance Ranking in Demand Range II
Importance Ranking in Demand Range III
Importance Ranking in Demand Range IV

Discussion

Links *b* and *c* are less important in Demand Range I than Demand Range II, III and IV because they carry zero flow in Demand Range I

Example 4: An Electric Power Supply Chain Network Supernetwork Transformation

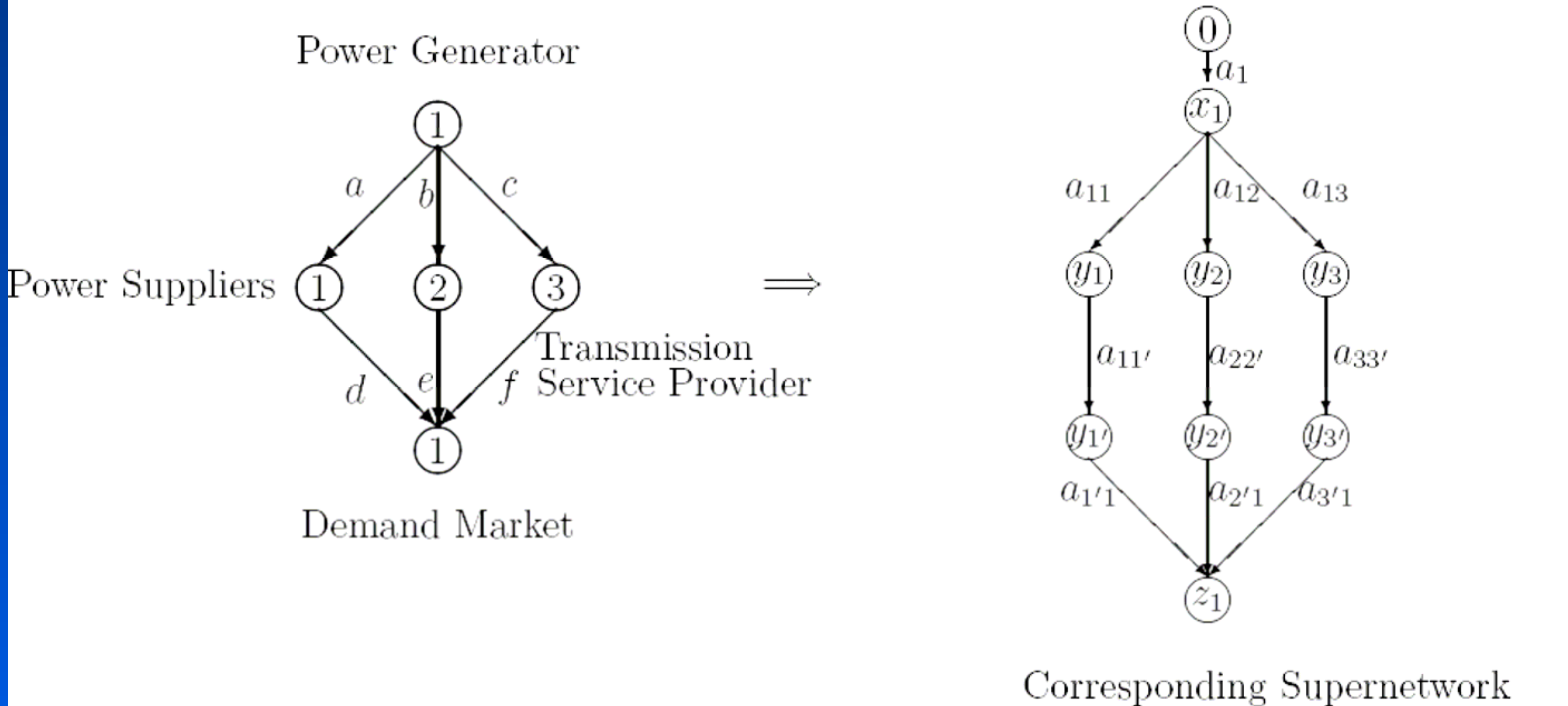


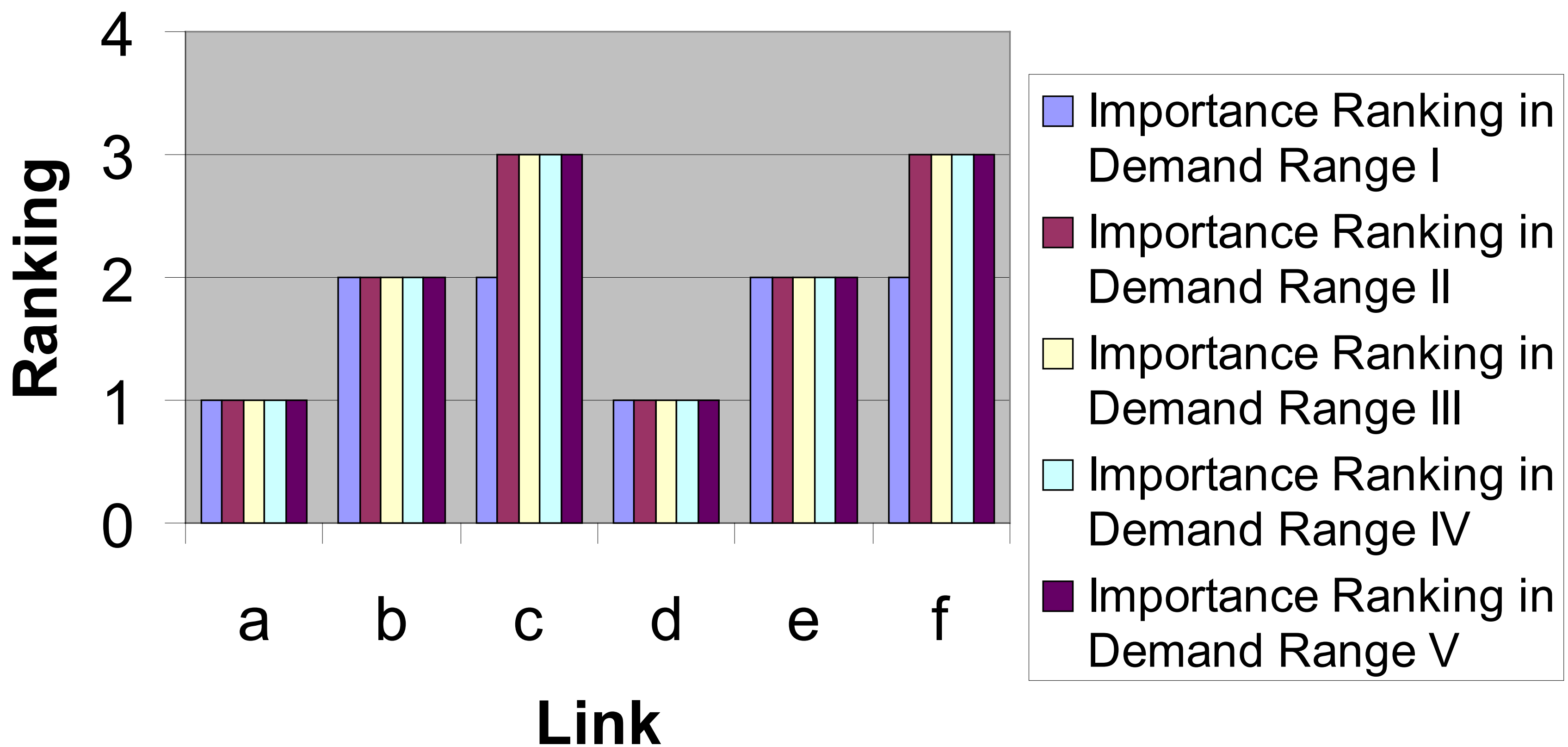
Figure 3: Electric Power Supply Chain Network and the Corresponding Supernetwork

Example 1 from Nagurney, Liu, Cojocaru and Daniele, *TRE* (2005)

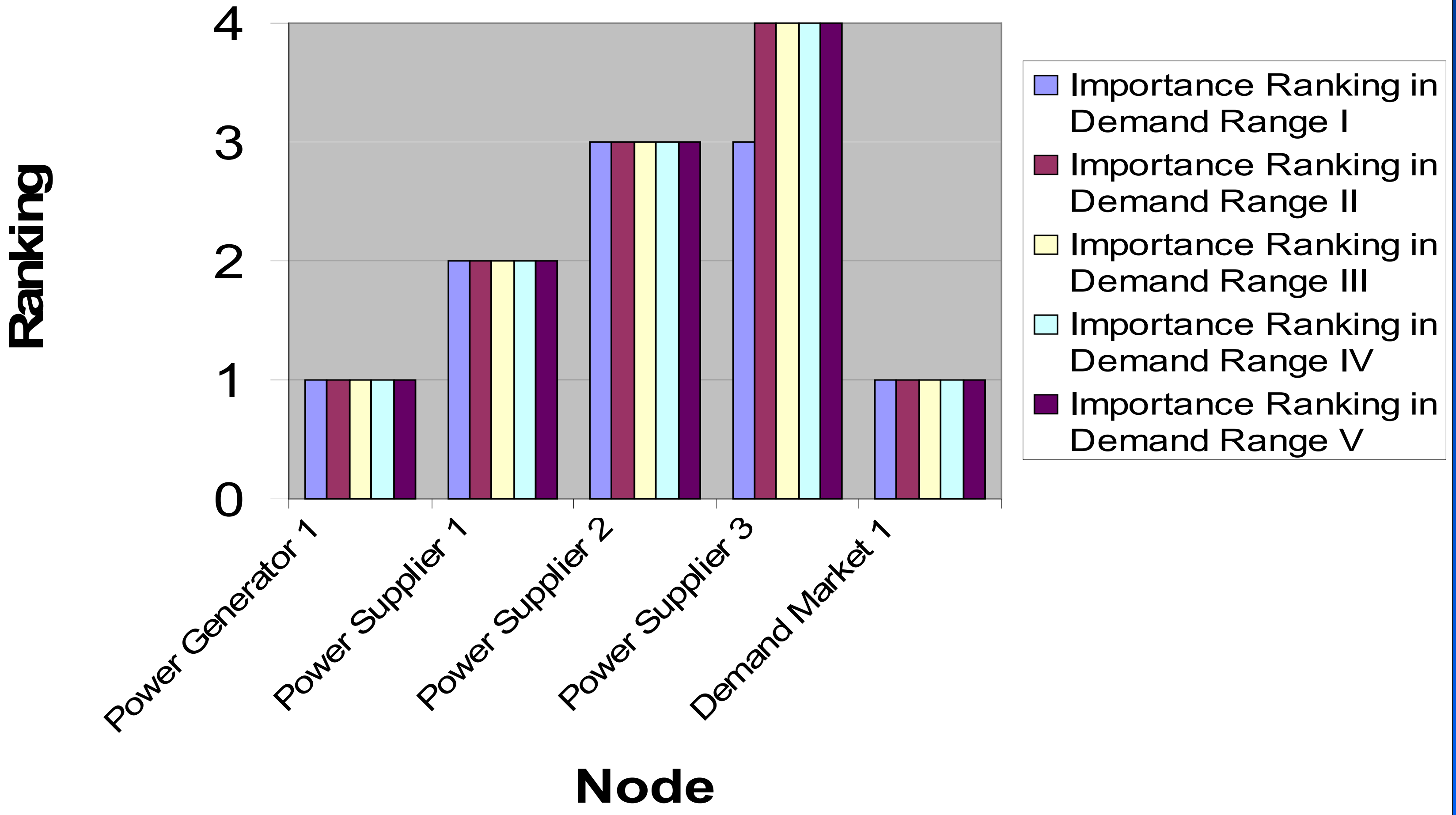
Five Demand Ranges

- Demand Range I: $d_w \in [0, 1]$
- Demand Range II: $d_w \in (1, 4/3]$
- Demand Range III: $d_w \in (4/3, 7/3]$
- Demand Range IV: $d_w \in (7/3, 11/3]$
- Demand Range V: $d_w \in (11/3, \infty)$

Importance Ranking of Links in the Electric Power Supply Chain Network



Importance Ranking of Nodes in the Electric Power Supply Chain Network



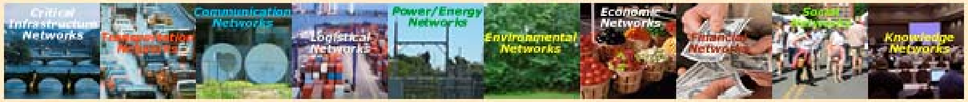
Discussion

Links ***a*** and ***d*** are the most important links and power supplier 1 is ranked the second due to the fact that path $\boldsymbol{p_1}$, which consists of links ***a*** and ***d*** and power supplier 1 carry the largest amount of flow.

The Advantages of the Nagurney and Qiang Network Efficiency Measure

- It captures flows, costs, and behavior of travelers, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
- It can be applied to assess the efficiency/performance of a wide range of critical infrastructure networks.
- It is the unified measure that can be used to assess the network efficiency with either fixed or elastic demands.

The Virtual Center for Supernetworks



Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

Updated: April 12, 2007

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