



A VARIATIONAL APPROACH FOR
INTERNATIONAL HUMAN MIGRATION
NETWORKS WITH AND WITHOUT
REGULATIONS

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EUROPT 2021 - TOULOUSE, JULY 7-9, 2021

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Definition (Human Migration)

*It is the **movement** that people do from one place to another with the intention of settling temporarily or permanently in the new location*

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Definition (Human Migration)

*It is the **movement** that people do from one place to another with the intention of settling temporarily or permanently in the new location*

Main Causes

Many social and economical factors affect the dynamics of human populations, such as **poverty, violence, war, dictatorships, persecutions, oppression, genocide, ethnic cleansing, climate change, tsunamis, floods, earthquakes, famines, family reunification as well as economic and educational possibilities or a job.**

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Figure: World's congested human migration routes

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Today there are **258 million** people living in a country different from that of birth, with an **increase of 49%** since 2000, which means that 3.4% of the world's inhabitants are international migrants (*International Migration Report, 2017*).

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Today there are **258 million** people living in a country different from that of birth, with an **increase of 49%** since 2000, which means that 3.4% of the world's inhabitants are international migrants (*International Migration Report, 2017*).

Between 2000 and 2015, migration contributed **42%** of the population growth in **Northern America** and **31%** in **Oceania**. In Europe, the size of the total population would have declined during the period 2000-2015 in the absence of migration.

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During 2018 Mediterranean arrivals were **141,475**, with more than 2,000 dead and missing people.

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During 2018 Mediterranean arrivals were **141,475**, with more than 2,000 dead and missing people.

From 2018 until January 2019, **17% of arrivals** by sea were registered in **Italy**, compared to 69% in 2017 (UNHCR).

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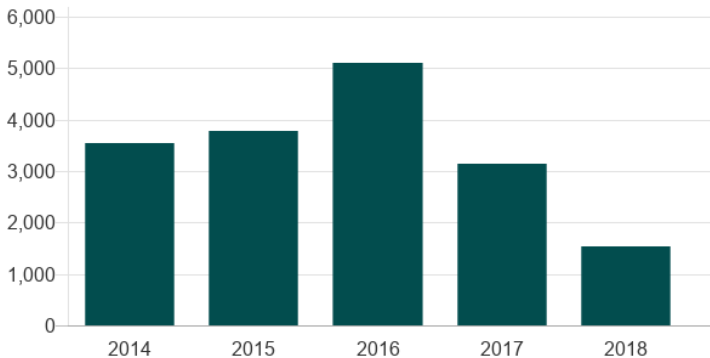
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Deaths in the Mediterranean



Source: UNHCR, figs to 11 Sep 2018

BBC

Figure: World's congested human migration routes

State-of-the-art

- Nagurney, 1989: a multiclass migration equilibrium model, which did not include migration/movement costs, isomorphic to a traffic network equilibrium.

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- Nagurney, 1989: a multiclass migration equilibrium model, which did not include migration/movement costs, isomorphic to a traffic network equilibrium.
- Nagurney, 1990: network equilibrium model and reformulation of the equilibrium conditions as the solution to an equivalent quadratic programming problem.

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- Cojocaru, 2007: application of the double-layer dynamics theory for modelling dynamics of human migration problems reformulated as transportation network problems.

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- Cui and Bai, 2014: evolution of population density and spread of epidemics in population systems.

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- Cui and Bai, 2014: evolution of population density and spread of epidemics in population systems.
- Volpert, Petrovskii, Zencenok, 2017: interaction of human migration and wealth distribution.

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- Causa, Jadamba, Raciti, 2017: inclusion of uncertainty in the utility functions, the migration cost functions, and the populations.

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- Causa, Jadamba, Raciti, 2017: inclusion of uncertainty in the utility functions, the migration cost functions, and the populations.
- Cappello, Daniele, 2020: a network based model where the aim of each migration class is to maximize the attractiveness of the origin country and the optimization model is formulated in terms of a Nash equilibrium problem and a variational inequality.

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- Cappello, Daniele, 2020: a network based model where the aim of each migration class is to maximize the attractiveness of the origin country and the optimization model is formulated in terms of a Nash equilibrium problem and a variational inequality.
- Nagurney, Daniele 2021: development of a network model with regulations.

The mathematical model

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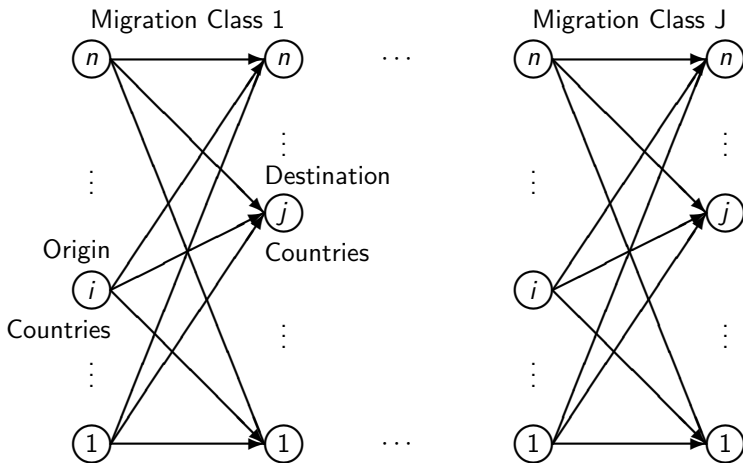


Figure: The Network Structure of International Human Migration

Common Notation

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Notation	Definition
f_{ij}^k	the flow of migrants of class k from country i to country j . The $\{f_{ij}^k\}$ elements for all i and j and fixed k are grouped into the vector $f^k \in R_+^{nn}$. We then further group the f^k vectors; $k = 1, \dots, J$, into the vector $f \in R_+^{Jnn}$.
p_i^k	the nonnegative population of migrant class k in country i . We group the populations of class k ; $k = 1, \dots, J$, into the vector $p^k \in R_+^n$. We then further group all such vectors into the vector $p \in R_+^{Jn}$.
\bar{p}_i^k	the initial fixed population of class k in country i ; $i = 1, \dots, n$; $k = 1, \dots, J$.
$u_i^k(p)$	the utility perceived by class k in country i ; $i = 1, \dots, n$; $k = 1, \dots, J$.
$c_{ij}^k(f)$	the cost of international migration , which includes economic, psychological, and social costs encumbeblue by class k in migrating from country i to country j ; $i = 1, \dots, n$; $j = 1, \dots, n$; $k = 1, \dots, J$.

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Conservation of flow equations

$$\bar{p}_i^k = \sum_l f_{il}^k, \quad (a) \quad \text{and} \quad p_i^k = \sum_l f_{li}^k, \quad (b) \quad \forall i, \forall k$$



$$p_i^k - \bar{p}_i^k = \sum_l f_{li}^k - \sum_l f_{il}^k, \quad \forall i, \forall k$$

Equilibrium Conditions

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Definition (International Human Migration Equilibrium without Regulations)

A vector of populations and international migration flows $(p^*, f^*) \in K^1$ is in equilibrium if it satisfies the equilibrium conditions: For each class k ; $k = 1, \dots, J$ and each pair of countries i, j ; $i = 1, \dots, n$; $j = 1, \dots, n$:

$$u_i^k(p^*) + c_{ij}^k(f^*) \begin{cases} = u_j^k(p^*) - \lambda_i^{k*}, & \text{if } f_{ij}^{k*} > 0 \\ \geq u_j^k(p^*) - \lambda_i^{k*}, & \text{if } f_{ij}^{k*} = 0 \end{cases}$$

and

$$\lambda_i^{k*} \begin{cases} \geq 0, & \text{if } \sum_{l \neq i} f_{il}^{k*} = \bar{p}_i^k, \\ = 0, & \text{if } \sum_{l \neq i} f_{il}^{k*} < \bar{p}_i^k \end{cases}$$

Variational Formulation of the International Human Migration Model without Regulations

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Theorem

A population and migration flow pattern $(p^, f^*) \in K^1$ is an international human migration equilibrium without regulations according to Definition 1, if and only if it satisfies the variational inequality problem*

$$-\langle u(p^*), p - p^* \rangle + \langle c(f^*), f - f^* \rangle \geq 0,$$

$$\forall (p, f) \in K^1 \equiv \{(p, f) | f \geq 0, \text{ and (a) and (b) hold}\}$$

Variational Formulation of the International Human Migration Model with Regulations

We now consider **regulations** imposed by a single country \bar{j} :

$$\sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^k \leq U_{\bar{j}} \quad (c)$$

Different types of regulations

- restriction of the migratory flow from a specific country \bar{i} and specific class of migrant \bar{k} :

$$f_{\bar{i}\bar{j}}^{\bar{k}} \leq U_{\bar{j}}$$

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$$\sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^k \leq U_{\bar{j}} \quad (c)$$

Different types of regulations

- restriction of the migratory flow from a specific country \bar{i} and specific class of migrant \bar{k} :

$$f_{\bar{i}\bar{j}}^{\bar{k}} \leq U_{\bar{j}}$$

- upper bounds on all incoming migrants from a specific country \bar{i} , irrespective of class:

$$\sum_k f_{\bar{i}\bar{j}}^k \leq U_{\bar{j}}$$

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$$\sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^k \leq U_{\bar{j}} \quad (c)$$

Different types of regulations

- restriction of the migratory flow from a specific country \bar{i} and specific class of migrant \bar{k} :
$$f_{ij}^{\bar{k}} \leq U_{\bar{j}}$$
- upper bounds on all incoming migrants from a specific country \bar{i} , irrespective of class:
$$\sum_k f_{ij}^k \leq U_{\bar{j}}$$
- regulations restricting the number of all incoming migrants of class \bar{k} from a group of countries:
$$\sum_{i \in C^1} f_{ij}^{\bar{k}} \leq U_{\bar{j}}$$

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New Feasible Set

$$K^2 \equiv K^1 \cap \{f | (c) \text{ is satisfied}\}$$

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New Feasible Set

$$K^2 \equiv K^1 \cap \{f \mid (c) \text{ is satisfied}\}$$

Theorem

A population and migration flow pattern $(p^, f^*) \in K^2$ is an international human migration equilibrium with regulations, if and only if it satisfies the variational inequality problem*

$$-\langle u(p^*), p - p^* \rangle + \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall (p, f) \in K^2$$

Equivalent Variational Inequality Formulation

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VI in flows

Determine $f^* \in K^3 \equiv \{f \mid f \in R_+^{Jnn} \text{ and } (a) \text{ and } (c) \text{ hold}\}$ such that

$$\sum_i \sum_j \sum_k (-\hat{u}_j^k(f^*) + c_{ij}^k(f^*)) \times (f_{ij}^k - f_{ij}^{k*}) \geq 0, \quad \forall f \in K^3$$

Lagrange Function

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K^3 can be rewritten as follows:

$$K^3 = \left\{ f : -f \leq 0; \sum_j f_{ij}^k - \bar{p}_i^k = 0, \forall i, \forall k; \sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^k - U_j \leq 0 \right\}$$

and the last variational inequality can be rewritten as a **minimization problem**, since if we set:

$$V(f) = \sum_i \sum_j \sum_k (-\hat{u}_j^k(f^*) + c_{ij}^k(f^*)) \times (f_{ij}^k - f_{ij}^{k*}),$$

then we have:

$$V(f) \geq 0 \text{ for } f \in K^3 \text{ and } \min_{f \in K^3} V(f) = V(f^*) = 0.$$

Existence of Lagrange Multipliers and Strong Duality

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Theorem

If $f^* \in K^3$ is a solution to variational inequality, then the Lagrange multipliers $\bar{\gamma} \in R_+^{Jnn}$, $\bar{\delta} \in R^{Jn}$, and $\bar{\mu}_{\bar{j}} \in R_+$ do exist, and for all i, j, k , and \bar{j} , the following conditions hold true:

$$\bar{\gamma}_{ij}^k (-f_{ij}^{k*}) = 0, \quad \bar{\delta}_{ik} \left(\sum_j f_{ij}^{k*} - \bar{p}_i^k \right) = 0, \quad (d)$$

$$\bar{\mu}_{\bar{j}} \left(\sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^{k*} - U_{\bar{j}} \right) = 0$$

$$-\hat{u}_j^k(f^*) + c_{ij}^k(f^*) - \bar{\gamma}_{ij}^k + \bar{\delta}_{ik} = 0, \quad \text{if } j \neq \bar{j} \quad (e)$$

$$-\hat{u}_{\bar{j}}^k(f^*) + c_{i\bar{j}}^k(f^*) - \bar{\gamma}_{i\bar{j}}^k + \bar{\delta}_{ik} + \bar{\mu}_{\bar{j}} = 0, \quad \text{if } j = \bar{j} \quad (f)$$

Moreover, the **strong duality** also holds true; namely:

$$V(f^*) = \min_{f \in K^3} V(f) = \max_{\gamma \in R_+^{Jnn}, \delta \in R^{Jn}, \mu_{\bar{j}} \in R_+} \min_{f \in R^{Jnn}} \mathcal{L}(f, \gamma, \delta, \mu_{\bar{j}})$$

Interpretation of the Lagrange Analysis

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- If $f_{ij}^{k*} > 0$ for some $j \neq \bar{j}$; from (d) we know that then $\bar{\gamma}_{ij}^k = 0$.
It then follows from (e) that:
$$\bar{\delta}_{ik} + c_{ij}^k(f^*) = \hat{u}_j^k(f^*)$$

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- If $f_{ij}^{k*} > 0$ for some $j \neq \bar{j}$; from (d) we know that then $\bar{\gamma}_{ij}^k = 0$. It then follows from (e) that: $\bar{\delta}_{ik} + c_{ij}^k(f^*) = \hat{u}_j^k(f^*)$
- If $f_{ij}^{k*} = 0$, for some $j \neq \bar{j}$, then $\bar{\gamma}_{ij}^k \geq 0$, and, from (e), we can infer that:

$$\bar{\delta}_{ik} + c_{ij}^k(f^*) = \hat{u}_j^k(f^*) + \bar{\gamma}_{ij}^k \quad \text{equivalently:} \quad \bar{\delta}_{ik} + c_{ij}^k(f^*) \geq \hat{u}_j^k(f^*)$$

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- If $f_{ij}^{k*} = 0$, for some $j \neq \bar{j}$, then $\bar{\gamma}_{ij}^k \geq 0$, and, from (e), we can infer that:

$$\bar{\delta}_{ik} + c_{ij}^k(f^*) = \hat{u}_j^k(f^*) + \bar{\gamma}_{ij}^k \quad \text{equivalently:} \quad \bar{\delta}_{ik} + c_{ij}^k(f^*) \geq \hat{u}_j^k(f^*)$$

- If $f_{ii}^{k*} > 0$, then from (d) it follows, since $c_{ii} = 0$, that

$$0 + \bar{\delta}_{ik} = \hat{u}_i^k(f^*) \quad \text{and, hence,} \quad \bar{\delta}_{ik} = \hat{u}_i^k(f^*).$$

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If we consider a destination node \bar{j} , under the regulation, and, if $f_{ij}^{k*} > 0$, then (d) applies and we obtain:

$$\bar{\delta}_{ik} + c_{ij}^k(f^*) = \hat{u}_j^k(f^*) - \bar{\mu}_{\bar{j}}$$

If the upper bound holds tightly, then the migrants incur a higher utility at destination node \bar{j} than just the sum of the origin node utility and the migration cost.

Alternative Variational Inequality Formulation

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Variational Inequality

determine $(f^*, \delta^*, \mu_{\bar{j}}^*) \in K^4$ such that

$$\begin{aligned} & \sum_{i; i \notin C^1} \sum_{j \neq \bar{j}} \sum_{k; k \notin C^1} (-\hat{u}_j^k(f^*) + c_{ij}^k(f^*) + \delta_{ik}^*) \times (f_{ij}^k - f_{ij}^{k*}) \\ & + \sum_{i \in C^1} \sum_{k \in C^1} (-\hat{u}_j^k(f^*) + c_{ij}^k(f^*) + \delta_{ik}^* + \mu_{\bar{j}}^*) \times (f_{ij}^k - f_{ij}^{k*}) \\ & + \sum_i \sum_k (\bar{p}_i^k - \sum_j f_{ij}^{k*}) \times (\delta_{ik} - \delta_{ik}^*) \\ & + (U_{\bar{j}} - \sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^{k*}) \times (\mu_{\bar{j}} - \mu_{\bar{j}}^*) \geq 0, \end{aligned}$$

$$\forall (f, \delta, \mu_{\bar{j}}) \in K^4 \equiv \{(f, \delta, \mu_{\bar{j}}) | f \in R_+^{Jnn}, \delta \in R^{Jn}, \mu_{\bar{j}} \in R_+\}$$

Standard Form

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Positions

- $\mathcal{K} \equiv K^4$
- $X \equiv (f, \delta, \mu_{\bar{j}})$
- $N = Jnn + Jn + 1$
- $F \equiv (F_1, F_2, F_3, F_4)$: the components of F_1 consist of the elements: $-\hat{u}_j^k(f) + c_{ij}^k(f) + \delta_{ik}$, for $i; i \notin C^1$ and $j \neq \bar{j}$, and $k; k \notin C^1$; the components of F_2 consist of the elements: $-\hat{u}_j^k(f) + c_{ij}^k(f) + \delta_{ik} + \mu_{\bar{j}}$, for $i \in C^1$ and $k \in C^1$; F_3 consists of the elements: $\bar{p}_i^k - \sum_j f_{ij}^k, \forall i, k$, and, finally, F_4 consists of the single element: $U_{\bar{j}} - \sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^k$

Standard Variational Inequality

Determine $X^* \in \mathcal{K}$ such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}$$

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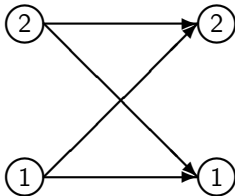
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Origin Countries



Destination Countries

Figure: International Migration Network for Illustrative Examples

Case without Regulations

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Data

- $\bar{p}_1 = 50$ and $\bar{p}_2 = 0$
- $u_1(p) = -p_1 + 100$ and $u_2(p) = -p_2 + 120$
- $c_{11}(f) = c_{22}(f) = 0$, $c_{12}(f) = .1f_{12} + 7$, $c_{21}(f) = f_{21} + 10$

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Data

- $\bar{p}_1 = 50$ and $\bar{p}_2 = 0$
- $u_1(p) = -p_1 + 100$ and $u_2(p) = -p_2 + 120$
- $c_{11}(f) = c_{22}(f) = 0$, $c_{12}(f) = .1f_{12} + 7$, $c_{21}(f) = f_{21} + 10$

Equilibrium Solution

- $f_{12}^* = 30$, $f_{11}^* = 20$, $f_{21}^* = 0$, $f_{22}^* = 0$
- $p_1^* = 20$, $p_2^* = 30$
- $\hat{u}_1(f^*) = u_1(p^*) = 80$, $\hat{u}_2(f^*) = u_2(p^*) = 90$
- $c_{11}(f^*) = c_{22}(f^*) = 0$, $c_{12}(f^*) = 10$, and $c_{21}(f^*) = 10$
- $\bar{\delta}_{11} = 80$, $\bar{\delta}_{21} = 90$, and $\bar{\gamma}_{11} = \bar{\gamma}_{21} = \bar{\gamma}_{12} = \bar{\gamma}_{22} = 0$

and the equilibrium conditions hold true.

Case with regulations

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$$f_{12} \leq U_2 = 20$$

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Regulation

$$f_{1\bar{2}} \leq U_2 = 20$$

New Equilibrium Solution

- $f_{11}^* = 30$, $f_{1\bar{2}}^* = 20$, $f_{21}^* = 0$, $f_{2\bar{2}}^* = 0$
- $p_1^* = 30$ and $p_2^* = 20$
- $\hat{u}_1(f^*) = u_1(p^*) = 70$, $\hat{u}_2(f^*) = u_2(p^*) = 100$
- $c_{11}(f^*) = c_{2\bar{2}}(f^*) = 0$; $c_{1\bar{2}}(f^*) = 9$, $c_{21}(f^*) = 10$
- $\bar{\mu}_2 = 21$
- $\bar{\delta}_{11} = 70$, $\bar{\delta}_{21} = 100$, and $\bar{\gamma}_{11} = \bar{\gamma}_{21} = \bar{\gamma}_{12} = \bar{\gamma}_{22} = 0$

and the equilibrium conditions hold true.

The Modified Projection Method

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- **Step 0: Initialization** Initialize with $X^0 \in \mathcal{K}$. Set $t := 1$ and let β be a scalar such that $0 < \beta \leq \frac{1}{L}$, where L is the Lipschitz constant
- **Step 1: Computation** Compute \bar{X}^t by solving the variational inequality subproblem:

$$\langle \bar{X}^t + \beta F(X^{t-1}) - X^{t-1}, X - \bar{X}^t \rangle \geq 0, \quad \forall X \in \mathcal{K}$$

- **Step 2: Adaptation** Compute X^t by solving the variational inequality subproblem:

$$\langle X^t + \beta F(\bar{X}^t) - X^{t-1}, X - X^t \rangle \geq 0, \quad \forall X \in \mathcal{K}$$

- **Step 3: Convergence Verification**
If $|X^t - X^{t-1}| \leq \epsilon$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $t := t + 1$ and go to Step 1

Explicit Formulae

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$$\bar{f}_{ij}^{kt} = \max\{0, f_{ij}^{k(t-1)} + \beta(\hat{u}_j^k(f^{t-1}) - c_{ij}^k(f^{t-1}) - \delta_{ik}^{t-1})\}, \quad i \notin C^1; j \neq \bar{j}; k \notin C^1$$

$$\bar{f}_{ij}^{kt} = \max\{0, f_{ij}^{k(t-1)} + \beta(\hat{u}_j^k(f^{t-1}) - c_{ij}^k(f^{t-1}) - \delta_{ik}^{t-1} - \mu_j^{t-1})\}, \quad i \in C^1; k \in C^1$$

$$\bar{\delta}_{ik}^t = \delta_{ik}^{t-1} + \beta\left(\sum_j f_{ij}^{k(t-1)} - \bar{p}_i^k\right), \quad \forall i, \forall k$$

$$\bar{\mu}_j^t = \max\{0, \mu_j^{t-1} + \beta\left(\sum_{i \in C^1} \sum_{k \in C^1} f_{ij}^{k(t-1)} - U_j\right)\}$$

Single Class Example without and with a Regulation

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Example 1

3 countries, no regulations and a single class of migrants

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Example 1

3 countries, no regulations and a single class of migrants

Data

- $\bar{p}_1 = 10,000$, $\bar{p}_2 = 5,000$, and $\bar{p}_3 = 1,000$
- $u_1(p) = -p_1 - .5p_2 + 30,000$, $u_2(p) = -2p_2 - p_1 + 20,000$,
 $u_3(p) = -3p_3 + .5p_2 + 10,000$
- $c_{ij} = 0$, $i = 1, 2, 3$
- $c_{12}(f) = 2f_{12} + 20$, $c_{13}(f) = f_{13} + 30$
- $c_{21}(f) = 5f_{21} + 40$, $c_{23}(f) = 4f_{23} + 20$
- $c_{31}(f) = 6f_{31} + 80$, $c_{32}(f) = 4f_{32} + 60$
- $\beta = .1$

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Equilibrium migration flow pattern

- $f_{11}^* = 10,000.00$, $f_{12}^* = 0.00$, $f_{13}^* = 0.00$
 $f_{21}^* = 2,447.90$, $f_{22}^* = 1,519.66$, $f_{23}^* = 1,032.44$
 $f_{31}^* = 1,000.00$, $f_{32}^* = 0.00$, $f_{33}^* = 0.00$
- $c_{11}(f^*) = 0.00$, $c_{12}(f^*) = 20.00$, $c_{13}(f^*) = 30.00$
 $c_{21}(f^*) = 12,279.50$, $c_{22}(f^*) = 0.00$, $c_{23}(f^*) = 4,149.76$
 $c_{31}(f^*) = 6,080.09$, $c_{32}(f^*) = 60.00$, $c_{33}(f^*) = 0.00$
- $p_1^* = 13,447.92$, $p_2^* = 1,519.66$, $p_3^* = 1,032.44$
- $u_1(p^*) = 15,792.25$, $u_2(p^*) = 3,512.75$, $u_3(p^*) = 7,662.51$
- $\delta_1^* = 15,792.25$, $\delta_2^* = 3,512.75$, $\delta_3^* = 9,712.17$

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Regulation

$$f_{2\bar{1}} + f_{3\bar{1}} \leq 2,000$$

Equilibrium migration pattern

- $f_{11}^* = 10,000.00$, $f_{12}^* = 0.00$, $f_{13}^* = 0.00$,
 $f_{2\bar{1}}^* = 1,458.43$, $f_{22}^* = 2,545.92$, $f_{23}^* = 995.65$
 $f_{3\bar{1}}^* = 541.59$, $f_{32}^* = 0.00$, $f_{33}^* = 458.41$
- $c_{11}(f^*) = 0.00$, $c_{12}(f^*) = 20.00$, $c_{13}(f^*) = 30.00$
 $c_{2\bar{1}}(f^*) = 7,332.13$, $c_{22}(f^*) = 0.00$, $c_{23}(f^*) = 4,002.61$
 $c_{3\bar{1}}(f^*) = 3,329.52$, $c_{32}(f^*) = 60.00$, $c_{33}(f^*) = 0.00$
- $p_1^* = 12,000.02$, $p_2^* = 2,545.92$, $p_3^* = 1,454.06$
- $u_{\bar{1}}(p^*) = 16,727.02$, $u_2(p^*) = 2,908.15$, $u_3(p^*) = 6,910.76$
- $\delta_1^* = 16,727.02$, $\delta_2^* = 2,908.15$, $\delta_3^* = 6,910.76$
- $\mu_{\bar{1}}^* = 6,486.74$

Multiclass Example without Regulations

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Example 2

3 countries and 2 classes of migrants

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Example 2

3 countries and 2 classes of migrants

Data

- $u_1^1(p) = -p_1^1 - .5p_2^1 - .5p_1^2 + 30,000$, $u_2^1(p) = -2p_2^1 - p_1^1 - p_2^2 + 20,000$, $u_3^1(p) = -3p_3^1 + .5p_2^1 - p_3^2 + 10,000$
 $u_1^2(p) = -2p_1^2 - p_1^1 + 25,000$, $u_2^2(p) = -3p_2^2 - p_2^1 + 15,000$, $u_3^2(p) = -p_3^2 - .5p_1^1 + 20,000$
- $\bar{p}_1^2 = 5,000$, $\bar{p}_2^2 = 3,000$, and $\bar{p}_3^2 = 500$
- $c_{ii}^k = 0$, $\forall i$, and for $k = 1, 2$
- $c_{12}^2(f) = 2f_{12}^2 + 10$, $c_{13}^2(f) = f_{13}^2 + 20$,
 $c_{21}^2(f) = 3f_{21}^2 + 10$, $c_{23}^2(f) = 2f_{23}^2 + 30$,
 $c_{31}^2(f) = f_{31}^2 + 25$, $c_{32}^2(f) = 2f_{32}^2 + 15$

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Equilibrium migration pattern

- $f_{11}^{1*} = 10,000.00$, $f_{12}^{1*} = 0.00$, $f_{13}^{1*} = 0.00$,
 $f_{21}^{1*} = 2,649.57$, $f_{22}^{1*} = 1,547.75$, $f_{23}^{1*} = 802.68$,
 $f_{31}^{1*} = 1,000.00$, $f_{32}^{1*} = 0.00$, $f_{33}^{1*} = 0.00$
- $f_{11}^{2*} = 2,343.67$, $f_{12}^{2*} = 182.49$, $f_{13}^{2*} = 2,473.85$,
 $f_{21}^{2*} = 0.00$, $f_{22}^{2*} = 1,955.57$, $f_{23}^{2*} = 1,044.43$,
 $f_{31}^{2*} = 0.00$, $f_{32}^{2*} = 0.00$, $f_{33}^{2*} = 500.00$
- $c_{11}^1(f^*) = 0.00$, $c_{12}^1(f^*) = 20.00$, $c_{13}^1(f^*) = 30.00$,
 $c_{21}^1(f^*) = 13,287.86$, $c_{22}^1(f^*) = 0.00$, $c_{23}^1(f^*) = 3,230.72$,
 $c_{31}^1(f^*) = 6,080.09$, $c_{32}^1(f^*) = 60.00$, $c_{33}^1(f^*) = 0.00$

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Equilibrium migration pattern

- $c_{11}^2(f^*) = 0.00$, $c_{12}^2(f^*) = 374.98$, $c_{13}^2(f^*) = 2,493.85$,
 $c_{21}^2(f^*) = 10.00$, $c_{22}^2(f^*) = 0.00$, $c_{23}^2(f^*) = 2,118.86$,
 $c_{31}^2(f^*) = 25.00$, $c_{32}^2(f^*) = 15.00$, $c_{33}^2(f^*) = 0.00$
- $p_1^{1*} = 13,649.59$, $p_2^{1*} = 1,547.75$, $p_3^{1*} = 802.68$,
 $p_1^{2*} = 2,343.67$, $p_2^{2*} = 2,138.06$, $p_3^{2*} = 4,018.28$
- $u_1^1(p^*) = 14,404.70$, $u_2^1(p^*) = 1,116.84$, $u_3^1(p^*) = 4,347.56$,
 $u_1^2(p^*) = 6,663.08$, $u_2^2(p^*) = 7,038.06$, $u_3^2(p^*) = 9,156.92$
- $\delta_1^{1*} = 14,404.70$, $\delta_2^{1*} = 1,116.84$, $\delta_3^{1*} = 8,324.62$,
 $\delta_1^{2*} = 6,663.08$, $\delta_2^{2*} = 7,038.06$, $\delta_3^{2*} = 9,156.92$

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Regulation

$$f_{13}^1 + f_{23}^1 + f_{13}^2 + f_{23}^2 \leq 2,000$$

Equilibrium migration pattern

- $f_{11}^{1*} = 10,000.00$, $f_{12}^{1*} = 0.00$, $f_{13}^{1*} = 0.00$,
 $f_{21}^{1*} = 2,746.92$, $f_{22}^{1*} = 1,788.86$, $f_{23}^{1*} = 464.22$,
 $f_{31}^{1*} = 1,000.00$, $f_{32}^{1*} = 0.00$, $f_{33}^{1*} = 0.00$
- $f_{11}^{2*} = 3,581.93$, $f_{12}^{2*} = 232.53$, $f_{13}^{2*} = 1,185.54$,
 $f_{21}^{2*} = 0.00$, $f_{22}^{2*} = 2,649.76$, $f_{23}^{2*} = 350.24$,
 $f_{31}^{2*} = 0.00$, $f_{32}^{2*} = 0.00$, $f_{33}^{2*} = 500.00$
- $c_{11}^1(f^*) = 0.00$, $c_{12}^1(f^*) = 20.00$, $c_{13}^1(f^*) = 30.00$,
 $c_{21}^1(f^*) = 13,774.62$, $c_{22}^1(f^*) = 0.00$, $c_{23}^1(f^*) = 1,876.90$,
 $c_{31}^1(f^*) = 6.080.02$, $c_{32}^1(f^*) = 60.00$, $c_{33}^1(f^*) = 0.00$

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Equilibrium migration pattern

- $c_{11}^2(f^*) = 0.00$, $c_{12}^2(f^*) = 475.06$, $c_{13}^2(f^*) = 1,205.54$,
 $c_{21}^2(f^*) = 10.00$, $c_{22}^2(f^*) = 0.00$, $c_{23}^2(f^*) = 730.47$,
 $c_{31}^2(f^*) = 25.00$, $c_{32}^2(f^*) = 15.00$, $c_{33}^2(f^*) = 0.00$
- $p_1^{1*} = 13,746.93$, $p_2^{1*} = 1,788.86$, $p_3^{1*} = 464.22$,
 $p_1^{2*} = 3,581.93$, $p_2^{2*} = 2,882.29$, $p_3^{2*} = 2,035.78$
- $u_1^1(p^*) = 13,567.68$, $u_2^1(p^*) = -206.93$, $u_3^1(p^*) = 7,465.98$,
 $u_1^2(p^*) = 4,089.21$, $u_2^2(p^*) = 4,564.27$, $u_3^2(p^*) = 11,090.76$
- $\delta_1^{1*} = 13,567.69$, $\delta_2^{1*} = -206.94$, $\delta_3^{1*} = 7,487.66$,
 $\delta_1^{2*} = 4,089.20$, $\delta_2^{2*} = 4,564.27$, $\delta_3^{2*} = 11,090.75$,
- $\mu_3^* = 5,796.02$

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Conclusions

- Challenges such as climate **change and associated disruptions**, along with **wars, conflicts, and strife**, are acting as push forces for humans to seek locations of greater safety and security

Conclusions

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Conclusions

- Challenges such as climate **change and associated disruptions**, along with **wars, conflicts, and strife**, are acting as push forces for humans to seek locations of greater safety and security
- Governments are being forced to deal with **increases in migratory flows** across national boundaries