Dynamic Modeling of the Internet via Evolutionary Variational Inequalities with Vulnerability Analysis

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Outline

- Motivation
- Literature
- Evolutionary Variational Inequalities (EVIs) and the Internet
- The Network Efficiency Measure and Component Importance Identification for Dynamic Networks
- The Dynamic Braess Paradox
- Summary and Conclusions

Motivation: A Dynamic Model of the Internet

The Internet has revolutionized the way in which we work, interact, and conduct our daily activities. It has affected the young and the old as they gather information and communicate and has transformed business processes, financial investing and decision-making, and global supply chains. The Internet has evolved into a network that underpins our developed societies and economies.

Motivation: A Dynamic Model of the Internet

"A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks." (page 10 of Roughgarden (2005)).

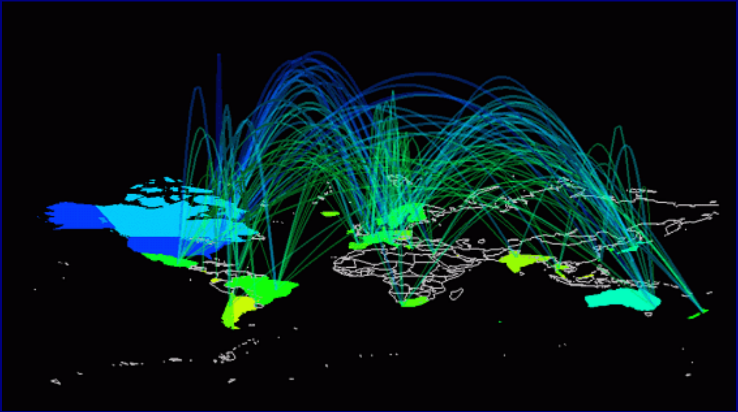
Motivation: A Dynamic Model of the Internet

- It has been shown that distributed routing, which is common in computer networks and, in particular, the Internet, and selfish (or source routing in computer networks) routing, as occurs in the case of user-optimized transportation networks, in which travelers select the minimum cost route between an origin and destination, are one and the same if the cost functions associated with the links that make up the paths/routes coincide with the lengths used to define the shortest paths.

- We assume that the costs on the links are congestion dependent, that is, they depend on the volume of the flow on the link.

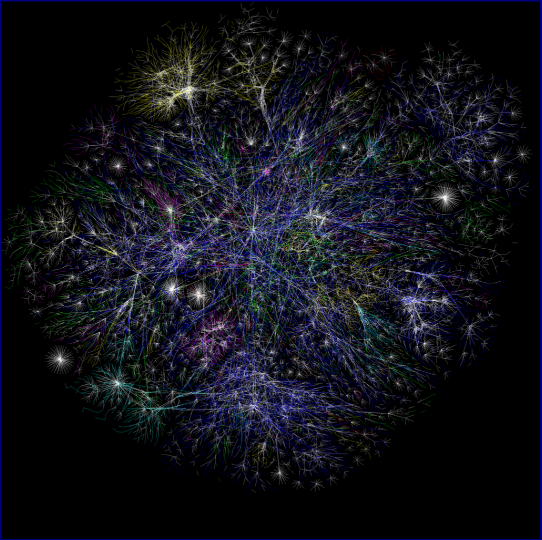
- We can expect that a variety of time-dependent demand structures will occur on the Internet as individuals seek information and news online in response to major events or simply go about their daily activities whether at work or at home. Hence, the development of this dynamic network model of the Internet is timely.

Internet Traffic Flows Between Fifty Countries



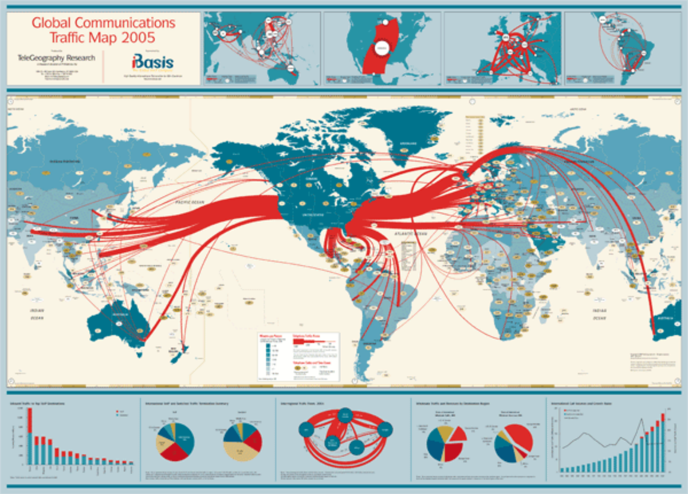
Source: Stephen Eick, Visual Insights

Visual Map of the Internet



Source: www.opte.org

Global Internet Communication Network



Source: www.telegeography.com

Motivation: Network Vulnerability Analysis

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.
- For example:
  - 9/11 Terrorist Attacks, September 11, 2001
  - The biggest blackout in North America, August 14, 2003
  - Two significant power outages during the month of September 2003 one in England and one in Switzerland and Italy
  - Hurricane Katrina, August 23, 2005
  - The Minneapolis Bridge Collapse, August 1, 2007
  - The Mediterranean Submarine Cable Disruption, January 30, 2008

Disasters in Transportation Networks

www.salem-news.com

www.boston.com

Disasters in Electric Power Networks

media.collegepublisher.com

www.cellar.org

www.crh.noaa.gov

Disasters in Communication Networks

www.tx.mb21.co.uk

www.w5jgv.com

www.wirelessestimator.com

Examples of Other Dynamic Networks

- Oil and natural gas networks
- Electricity generation and distribution networks
- Supply chain networks
- Transportation networks

Literature on EVIs and the Applications

- Daniele, P., Maugeri, A., Oettli, W. (1999). Time-dependent Traffic Equilibria, *Journal of Optimization Theory and its Applications* 103, 543-555.
- Daniele, P. (2003). Evolutionary Variational Inequalities and Economic Models for Demand Supply Markets, *Mathematical Models and Methods in Applied Sciences* 4, 471-489.
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- Cojocaru, M. G., Daniele, P., Nagurney, A. (2006). Double-Layered Dynamics: A Unified Theory of Projected Dynamical Systems and Evolutionary Variational Inequalities, *European Journal of Operational Research* 175, 494-507.

- Nagurney, A. Parkes, D., Daniele, P. (2007). The Internet, Evolutionary Variational Inequalities, and the Time-Dependent Braess Paradox, *Computational Management Science* 4, 355-375.

- Nagurney, A. (2006). *Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits*, Edward Elgar Publishing, Cheltenham, England.

- Nagurney, A., Liu, Z., Cojocaru, M. G., Daniele, P. (2007). Static and Dynamic Transportation Network Equilibrium Reformulations of Electric Power Supply Chain Networks with Known Demands, *Transportation Research E* 43, 624-646.

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Holme, Kim, Yoon and Han (2002)
- Murray-Tuite and Mahmassani (2004)
- Taylor and Deste (2004)
- Barrat, Barthlemy and Vespignani (2005)
- Criado, Flores, Hernández-Bermejo, Pello and Romance (2005)
- Chassin and Posse (2005)
- Sheffi (2005)
- DallAsta, Barrat, Barthlemy and Vespignani (2006)
- Jenelius, Petersen and Mattson (2006)

Our Research on Network Efficiency, Vulnerability, and Robustness

- Nagurney, A., Qiang, Q., 2007a. A Network Efficiency Measure for Congested Networks. *Europhysics Letters* 79, 38005.
- Nagurney, A., Qiang, Q., 2007b. A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs With Applications to Network Component Importance Identification and Vulnerability, *Proceedings of the 18th Annual POMS Conference*.
- Nagurney, A., Qiang, Q., 2007c. Robustness of Transportation Networks Subject to Degradable Links *Europhysics Letters* 80, 68001.
- Nagurney, A., Qiang, Q., 2007d. A Network Efficiency Measure With Application to Critical Infrastructure Networks. *Journal of Global Optimization* 40, 261-275.

Our Research on Network Efficiency, Vulnerability, and Robustness (Cont'd)

- Nagurney, A., Qiang, Q., 2008a. A Relative Total Cost Index For the Evaluation of Transportation Network Robustness In the Presence of Degradable Links and Alternative Travel Behavior. *International Transactions in Operational Research*, To appear.
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Online Access to the Papers Related to this Presentation

- Nagurney, A. Parkes, D., Daniele, P. (2007). The Internet, Evolutionary Variational Inequalities, and the Time-Dependent Braess Paradox, *Computational Management Science* 4, 355-375.
  http://supernet.som.umass.edu/articles/EVIsInternetBraessFinal.pdf
- Nagurney, A., Qiang, Q., 2008b. An Efficiency Measure for Dynamic Networks With Application to the Internet and Vulnerability Analysis. *Netnomics*, in press.

http://supernet.som.umass.edu/articles/DynamicNetworkEfficiency_rev.pdf

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The Dynamic Model of the Internet (Nagurney, Parkes, and Daniele (2007))

The Internet is modeled as a network $G = [N, L]$, consisting of the set of nodes N and the set of directed links L. The set of links L consists of n_L elements. The set of O/D pairs of nodes is denoted by W and consists of n_W elements. We denote the set of routes (with a route consisting of links) joining the O/D pair w by P_w. We assume that the routes are acyclic. Let P with n_P elements denote the set of all routes connecting all the O/D pairs in the Internet. Links are denoted by a, b, etc; routes by r, q, etc., and O/D pairs by w_1, w_2, etc. We assume that the Internet is traversed by "jobs" or "classes" of traffic and that there are K "jobs" with a typical job denoted by k.

Demands, Route Flows, and Link Flows

Let $d_w^k(t)$ denote the demand, that is, the traffic generated, between O/D pair w at time t by job class k. The flow on route r at time t of class k, which is assumed to be nonnegative, is denoted by $x_r^k(t)$ and the flow on link a of class k at time t by $f_a^k(t)$.

Conservation of Flow Between Demands and Route Flows

The following conservation of flow equations must be satisfied at each t:

$$d_w^k(t) = \sum_{r \in P_w} x_r^k(t), \quad \forall w \in W, \forall k$$

Route Capacities

Also, we must have that

$$0 \leq x_r^k(t) \leq \mu_r^k(t), \quad \forall r \in P, \forall k$$

where $\mu_r^k(t)$ denotes the capacity on route r of class k at time t.

Conservation of Flow Between Route Flows and Link Flows

$$f_a^k(t) = \sum_{r \in P} x_r^k(t)\delta_{ar}, \quad \forall a \in L, \forall k, \tag{3}$$

where $\delta_{ar} = 1$ if link a is contained in route r, and $\delta_{ar} = 0$, otherwise. Hence, the flow of a class on a link is equal to the sum of the flows of the class on routes that contain that link. All the link flows at time t are grouped into the vector $f(t)$, which is of dimension Kn_L.

Link Costs and Route Costs

The cost on route r at time t of class k is denoted by $C_r^k(t)$ and the cost on a link a of class k at time t by $c_a^k(t)$. For the sake of generality, we allow the cost on a link to depend upon the entire vector of link flows at time t, so that

$$c_a^k(t) = c_a^k(f(t)), \quad \forall a \in L, \forall k.$$

The costs on routes are related to costs on links through the following equations:

$$C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t))\delta_{ar}, \quad \forall r \in P, \forall k,$$

which means that the cost on a route of class k at a time t is equal to the sum of costs on links of that class that make up the route at time t. We group the route costs at time t into the vector $C(t)$, which is of dimension Kn_P.

Feasible Set

We consider the Hilbert space $\mathcal{L} = L^2([0, T], R^{Kn_P})$ (where $[0, T]$ denotes the time interval under consideration) given by

$$\mathcal{K} = \{x \in L^2([0, T], R^{Kn_P}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T];$$

$$\sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k \text{ a.e. in } [0, T]\}.$$

We assume that the capacities $\mu_r^k(t)$, for all r and k, are in $\mathcal{L}$, and that the demands, $d_w^k \geq 0$, for all w and k, are also in $\mathcal{L}$. Further, we assume that

$$0 \leq d(t) \leq \Phi\mu(t), \text{a.e. on } [0, T],$$

where Φ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element (kw, kr) equal to 1 if route r is contained in P_w, and 0, otherwise. Hence, the feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded. Note that we are not restricted as to the form that the time-varying demands for the O/D pairs take since convexity of $\mathcal{K}$ is guaranteed even if the demands have a step-wise structure, or are piecewise continuous.

Another Definition

The dual space of $\mathcal{L}$ is denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle \mathcal{G}, x \rangle\rangle := \int_0^T \langle \mathcal{G}(t), x(t) \rangle dt, \quad \mathcal{G} \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \longrightarrow \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

Dynamic Network Equilibrium

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle(cf. Wardrop (1952) and Beckmann, McGuire, and Winsten (1956))) if, at each time t, only the minimum cost routes for each class not at their capacities are used (that is, have positive flow) for each O/D pair unless the flow of that class on a route is at its upper bound (in which case those class routes' costs can be lower than those on the routes not at their capacities). The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every route $r \in P_w$, every class k; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if} \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if} \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if} \quad x_r^{k*}(t) = 0. \end{cases}$$

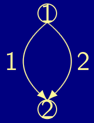
Theorem of Nagurney, Parkes, and Daniele (2007)

$x^* \in \mathcal{K}$ is an equilibrium flow according to the definition of dynamic network equilibrium if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t)\rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

A Multiclass Numerical Example

Consider a network (small subnetwork of the Internet) consisting of two nodes and two links as shown below. There is a single O/D pair $w = (1, 2)$. Since the routes connecting the O/D pair consist of single links we work with the routes r_1 and r_2 directly:



There are assumed to be two classes/jobs and the route costs are:
for Class 1:

$$C^1_{r_1}(x(t)) = 2x^1_{r_1}(t) + x^2_{r_1}(t) + 5, \quad C^1_{r_2}(x(t)) = 2x^2_{r_2}(t) + 2x^1_{r_2}(t) + 10,$$

for Class 2:

$$C^2_{r_1}(x(t)) = x^2_{r_1}(t) + x^1_{r_1}(t) + 5, \quad C^2_{r_2}(x(t)) = x^1_{r_2}(t) + 2x^2_{r_2}(t) + 5.$$

The time horizon is $[0, 10]$. The demands for the O/D pair are:

$$d^1_w(t) = 10 - t, \quad d^2_w(t) = t.$$

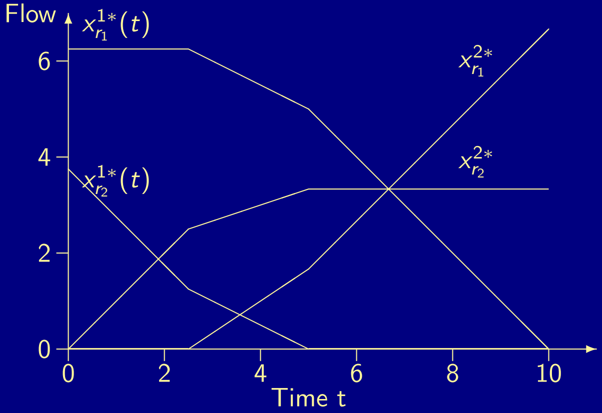
The upper bounds are: $\mu^1_{r_1} = \mu^1_{r_2} = \mu^1_{r_1} = \mu^2_{r_2} = \infty$.

Equilibrium Route Flows for the Multiclass Numerical Example

Equilibrium Multiclass Route Flows at time t					
Flow	$t=0$	$t=2.5$	$t=5$	$t=7.5$	$t=10$
$x^{1*}_{r_1}(t)$	6.25	6.25	5.00	2.50	0.00
$x^{1*}_{r_2}(t)$	3.75	1.25	0.00	0.00	0.00
$x^{2*}_{r_1}(t)$	0.00	0.00	$1.6\bar{6}$	$4.16\bar{6}$	$6.6\bar{6}$
$x^{2*}_{r_2}(t)$	0.00	2.50	$3.3\bar{3}$	$3.3\bar{3}$	$3.3\bar{3}$

We provide a graph of the equilibrium route trajectories, where we display also the interpolations between the discrete solutions. Since the route cost functions are strictly monotone over the time horizon $[0, 10]$ we know that the equilibrium trajectories are unique.

As the theory predicts, the trajectories are also continuous for this example. It is interesting to see that after time $t = 5$ route r_2 is never used by class 1, whereas route r_1 is not utilized for class 2 traffic until after $t = 2$.




Bellagio Research
Team Residency
March 2004

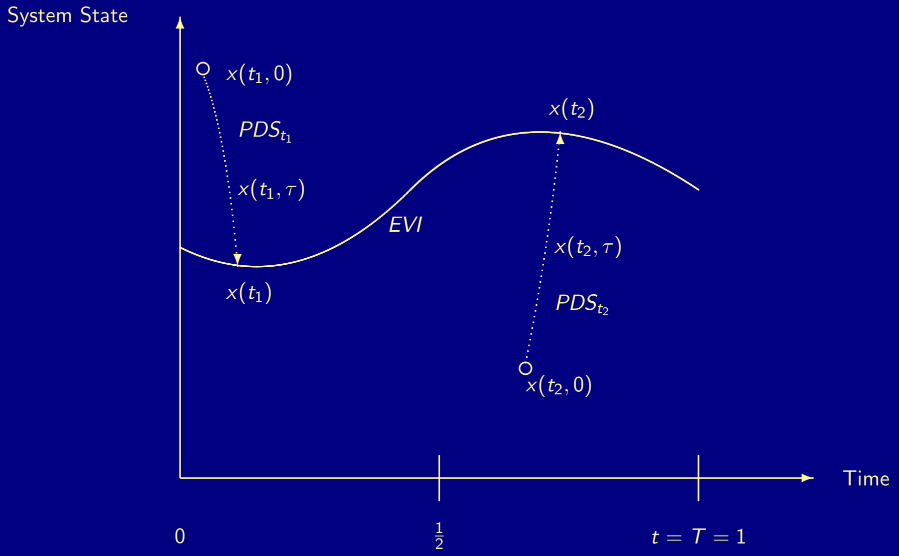
Double-Layered Dynamics

The unification of Evolutionary Variational Inequalities (EVIs) and Projected Dynamic Systems (PDSs) allows the modeling of dynamic networks over different time scales.

Papers:

- Cojocaru, M. G., Daniele, P., Nagurney, A. 2005. Projected Dynamical Systems and Evolutionary Variational Inequalities via Hilbert Spaces with Applications *Journal of Optimization Theory and Applications* 127, 1-15.
- Cojocaru, M. G., Daniele, P., Nagurney, A. 2006. Double-Layered Dynamics: A Unified Theory of Projected Dynamical Systems and Evolutionary Variational Inequalities *European Journal of Operational Research* 175, 494-507.

A Pictorial of the Double-Layered Dynamics



Equilibria of PDS and Variational Inequalities

An important feature of any PDS is that it is intimately related to a variational inequality problem (VI).

The equilibria of a PDS:

$$\frac{\partial}{\partial t}(x(t)) = \Pi_{\mathcal{K}}(x(t), -F(x(t))) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(x + \delta v) - x)}{\delta}, \;\; x(0) = x_0$$

where $P_{\mathcal{K}}$ *is the norm projection defined by*

$$P_{\mathcal{K}}(x) = \operatorname{argmin}_{x' \in \mathcal{K}} \|x^{'} - x\|.$$

that is, $x^* \in \mathcal{K}$ *such that* $\Pi_{\mathcal{K}}(x^*, -F(x^*)) = 0$ *are solutions to the* $VI(F, \mathcal{K})$*: find* $x^* \in \mathcal{K}$ *such that*

$$\langle F(x^*)^T, x - x^* \rangle \geq 0, \;\forall x \in \mathcal{K}.$$

and vice-versa, where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* $\mathcal{K}$*, where* $\mathcal{K}$ *is a Hilbert space.*

Definition: Standard Form of EVIs

Recall the following definition on $\mathcal{L} \times \mathcal{L}^*$:

$$\langle\langle \mathcal{G}, x \rangle\rangle := \int_0^T \langle \mathcal{G}(t), x(t) \rangle dt, \quad \mathcal{G} \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

The standard form of the EVI that we work with is: find $x^* \in \mathcal{K}$ such that $\ll F(x^*), x - x^* \gg$, $\forall x \in \mathcal{K}$, or equivalently, find $x^* \in \mathcal{K}$ such that

$$\int_0^T \langle F(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

EVIs and Infinite-dimensional PDS (Cojocaru, Daniele, and Nagurney (2005))

The solutions to the EVI problem are the same as the critical points of the PDS and vice versa, that is, the critical points of the PDS are the solutions to the EVI.

By choosing the Hilbert space to be $L^2([0,T],R^q)$*, we find the solutions to the EVI: find* $x^* \in \mathcal{K}$ *such that*

$$\int_0^T \langle F(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}$$

are the same as the critical points of the equation:

$$\frac{dx(t,\tau)}{d\tau} = \Pi_{\mathcal{K}}(x(t,\tau), -F(x(t,\tau))), \quad x(t,0) \in \mathcal{K},$$

that is, the points such that

$$\Pi_{\mathcal{K}}(x(t,\tau), -F(x(t,\tau))) \equiv 0, \ \ a.e. \ in \ [0,T],$$

which are obviously stationary with respect to τ*.*

The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007a, b, d) proposed a network efficiency measure for networks with fixed demands, which captures demand and flow information under the network equilibrium:

The network performance/efficiency measure, $\mathcal{E}(G, d)$, according to Nagurney and Qiang (2007a, b, d), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G, d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

Network Efficiency Measure for Dynamic Networks - Continuous Time

The network efficiency for the network G with time-varying demand d for $t \in [0, T]$, denoted by $\mathcal{E}(G, d, T)$, is defined as follows:

$$\mathcal{E}(G, d, T) = \frac{\int_0^T [\sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)}]/n_W \, dt}{T}.$$

Note that the above measure is the average network performance over time of the dynamic network.

Network Efficiency Measure for Dynamic Networks - Discrete Time

Let $d_w^1,\, d_w^2, ..., d_w^H$ denote demands for O/D pair w in H discrete time intervals, given, respectively, by:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$. We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair w at the H different time intervals by: $\lambda_w^1,\, \lambda_w^2, ..., \lambda_w^H$. The demand vector d, in this special discrete case, is a vector in $R^{n_W \times H}$. The dynamic network efficiency measure in this case is as follows:

Dynamic Network Efficiency: Discrete Time Version

The network efficiency for the network (G, d) over H discrete time intervals:
$[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, *where* $t_H \equiv T$, *and with the respective constant demands:* $d_w^1,\, d_w^2, ..., d_w^H$ *for all* $w \in W$ *is defined as follows:*

$$\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w\in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.$$

Special Case

Assume that $d_w(t) = d_w$, *for all O/D pairs* $w \in W$ *and for* $t \in [0, T]$. *Then, the dynamic network efficiency measure collapses to the Nagurney and Qiang (2007a, b, d) measure:*

$$\mathcal{E} = \frac{1}{n_W} \sum_{w \in W} \frac{d_w}{\lambda_w}.$$

Importance of a Network Component

The importance of network component g of network G with demand d over time horizon T is defined as follows:

$$I(g, d, T) = \frac{\mathcal{E}(G, d, T) - \mathcal{E}(G - g, d, T)}{\mathcal{E}(G, d, T)}$$

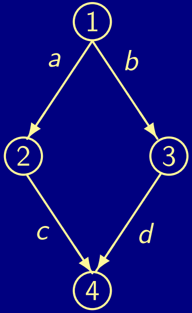
where $\mathcal{E}(G - g, d, T)$ is the dynamic network efficiency after component g is removed.

Our Approach to Identifying the Importance and Rankings of Network Components

The elimination of a link is represented in the Nagurney and Qiang (2008b) measure by the removal of that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Hence, our measure is well-defined even in the case of disconnected networks.

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $p_1 = (a, c)$ and $p_2 = (b, d)$. For a travel demand of 6, the equilibrium path flows are $x^*_{p_1} = x^*_{p_2} = 3$ and The equilibrium path travel costs are $C_{p_1} = C_{p_2} = 83$.

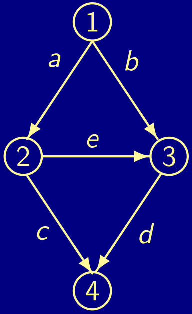


$$c_a(f_a) = 10f_a \quad c_b(f_b) = f_b + 50$$

$$c_c(f_c) = f_c + 50 \quad c_d(f_d) = 10f_d$$

Adding a Link Increases Travel Cost for All!

Adding a new link creates a new path $p_3 = (a, e, d)$. The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path p_3, C_{p_3}=70. The new equilibrium path flows are $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$. The equilibrium path travel costs are $C_{p_1} = C_{p_2} = C_{p_3} = 92$.



$c_e(f_e) = f_e + 10$

We now construct time-dependent link costs, route costs, and demand for $t \in [0, T]$. It is important to emphasize that the case where time t is discrete, that is, $t = 0, 1, 2, \ldots, T$, is trivially included in the equilibrium conditions and also captured in the EVI formulation.

We consider, to start, the first network, consisting of links: a, b, c, d. We assume that the capacities $\mu_{r_1}(t) = \mu_{r_2}(t) = \infty$ for all $t \in [0, T]$. The link cost functions are assumed to be given and as follows for time $t \in [0, T]$:

$$c_a(f_a(t)) = 10 f_a(t), \quad c_b(f_b(t)) = f_b(t) + 50,$$

$$c_c(f_c(t)) = f_c(t) + 50, \quad c_d(f_d(t)) = 10 f_d(t).$$

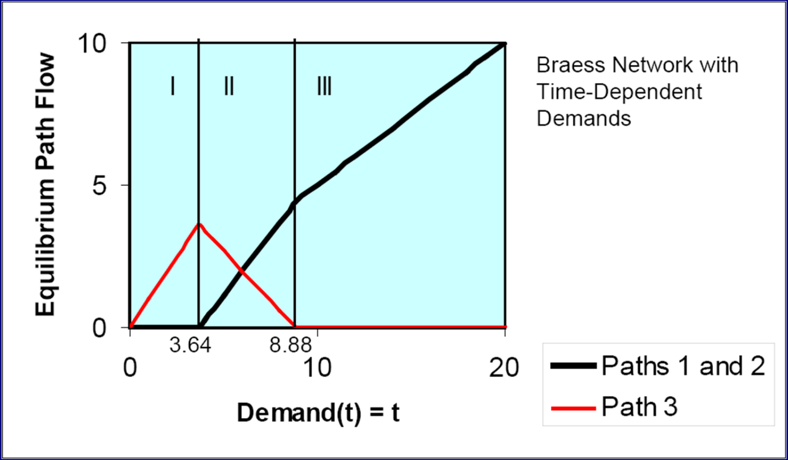
We assume a time-varying demand $d_w(t) = t$ for $t \in [0, T]$.

The Dynamic Braess Network - Solution

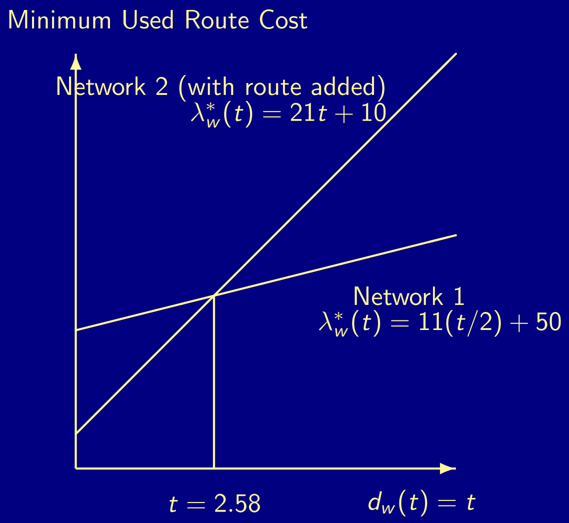
Solving the EVI, we have the equilibrium path flows are $x^*_{r_1}(t) = \frac{t}{2}$ and $x^*_{r_2}(t) = \frac{t}{2}$ for $t \in [0, T]$.

The equilibrium route costs for $t \in [0, T]$ are given by: $C_{r_1}(x^*_{r_1}(t)) = 5\frac{1}{2}t + 50 = C_{r_2}(x^*_{r_2}(t)) = 5\frac{1}{2}t + 50$, and, clearly, equilibrium conditions hold for $\in [0, T]$ a.e..

The Dynamic Braess Network Adding Link e



The Dynamic Braess Network



For demand in the range $2.58 < d_w(t) = t < 8.89$, the addition of the new route will result in everyone being worse off.

Minimum Used Route Costs for Braess Networks 1 and 2.

Importance of Nodes and Links in the Dynamic Braess Network Using the New Measure When $T = 10$

Link	Importance Value	Importance Ranking
a	0.2604	1
b	0.1784	2
c	0.1784	2
d	0.2604	1
e	-0.1341	3

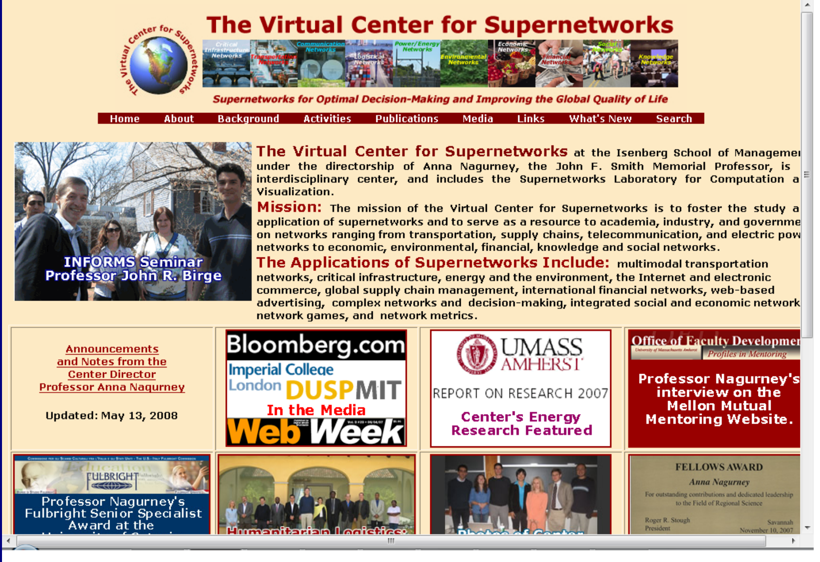
Node	Importance Value	Importance Ranking
1	1.0000	1
2	0.2604	2
3	0.2604	2
4	1.0000	1

Link e is never used after $t = 8.89$ and in the range $t \in [2.58, 8.89]$, it increases the cost, so the fact that link e has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!

http://supernet.som.umass.edu/fulbright-catania/FulbrightCataniaNagurney.htm

Summary and Conclusions

- We apply the EVI theory to study the dynamic model of the Internet.
- The proposed model captures the congestion and dynamic nature of the Internet.
- Our network efficiency measure captures user behavior, flows and costs on networks.
- The proposed measure extends our previous research on the network efficiency measure into the dynamic setting.
- The proposed network efficiency measure is applicable for varying demand in both continuous and discrete time.
- The proposed network efficiency measure can be applied to other critical infrastructure networks.
- The proposed network efficiency measure also has implication for the robustness and vulnerability of networks with partially disrupted network components (Nagurney and Qiang 2007c).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Thank You!

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