A Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief: Effective Product Delivery Under Competition for Financial Funds

Anna Nagurney[1] Patrizia Daniele[2] Emilio Alvarez Flores[3] Valeria Caruso[2]

[1]Radcliffe Institute for Advanced Study, Harvard University

Isenberg School of Management, University of Massachusetts Amherst

[2]Department of Mathematics and Computer Science, University of Catania, Italy

[3]Cisco Services, Research Triangle Park, North Carolina

3rd International Conference on Dynamics of Disasters, Kalamata, Greece, July 5-9, 2017

Acknowledgments

The first author thanks the Radcliffe Institute for Advanced Study at Harvard University for support through a Summer Fellowship in 2017. Support from the John F. Smith Memorial Fund at the University of Massachusetts Amherst is also acknowledged.

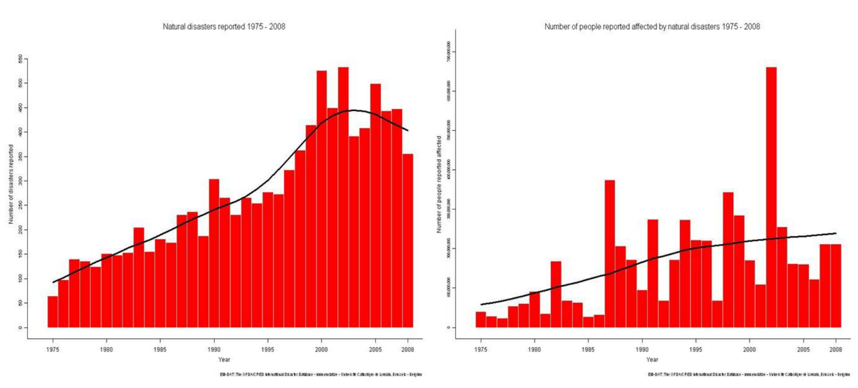
This paper is dedicated to the memory of Professor Martin J. Beckmann, Professor Emeritus at Brown University, who passed away on April 11, 2017 at the age of 92. He was a renowned scholar in transportation science, regional science, and operations research, and his work on network equilibria have had a profound impact on both theory and practice.

Outline

- Background and Motivation
- The Variational Equilibrium Network Framework for Humanitarian Organizations
- Lagrange Theory and Analysis of the Marginal Utilities
- The Algorithm and Case Study
- Summary and Conclusions

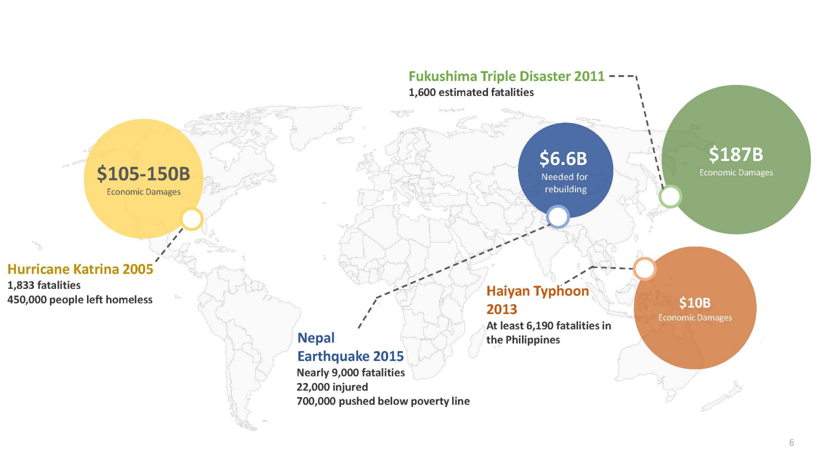
Background and Motivation

Natural Disasters (1975–2008)



Disasters have a catastrophic effect on human lives and a region's or even a nation's resources. A total of 2.3 billion people were affected by natural disasters from 1995-2015 (UN Office of Disaster Risk (2015)).

Some Recent Disasters



Hurricane Katrina in 2005

Hurricane Katrina has been called an *"American tragedy,"* in which essential services failed completely.

The Triple Disaster in Japan on March 11, 2011

Superstorm Sandy and Power Outages

Manhattan without power October 30, 2012 as a result of the devastation wrought by Superstorm Sandy.

Challenges Associated with Disaster Relief

• Timely delivery of relief items is challenged by damaged and destroyed infrastructure (transportation, telecommunications, hospitals, etc.).

• Shipments of the wrong supplies create congestion and materiel convergence (sometimes referred to as the second disaster).

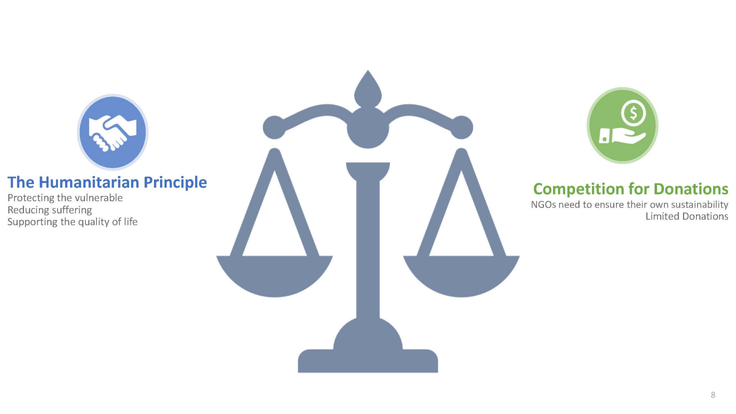
• • Within three weeks following the 2010 earthquake in Haiti, 1,000 NGOs were operating in Haiti. News media attention of insufficient water supplies resulted in immense donations to the Dominican Red Cross to assist its island neighbor. Port-au-Price was saturated with both cargo and gifts-in-kind.

• • After the Fukushima disaster, there were too many blankets and items of clothing shipped and even broken bicycles.

• • After Katrina, even tuxedos were delivered to victims.

Better coordination among NGOs is needed.

Challenges Associated with Disaster Relief - The NGO Balancing Act



There were 1.5 million registered NGOs in the US in 2012. $300 billion in donations given yearly to US charities.

Challenges Associated with Disaster Relief - Driving Forces

Need for Game Theory Network Models for Disaster Relief

Therefore

there is a need to *develop appropriate analytical tools* that can assist NGOs, as well as governments in modeling the complex interactions in disaster relief to improve outcomes.

Game Theory and Disaster Relief

We developed the first Generalized Nash Equilibrium (GNE) model for post-disaster humanitarian relief, which contains both a financial component and a supply chain component. The Generalized Nash Equilibrium problem is a generalization of the Nash Equilibrium problem (cf. Nash (1950, 1951)).

"A Generalized Nash Equilibrium Network Model for Post-Disaster Humanitarian Relief," Anna Nagurney, Emilio Alvarez Flores, and Ceren Soylu, *Transportation Research E* **95** (2016), pp 1-18.

Game Theory and Disaster Relief

The GNE model that we constructed had a structure that allowed us to reformulate the equilibrium conditions, which included shared constraints, as an optimization problem; typically, Generalized Nash Equilibrium problems are formulated as quasi-variational inequalities, which can pose challenges for computations.

The new model, which we are presenting at this conference, has the following extended features:

1. The financial funds functions, which capture the amount of donations to each NGO, given their visibility through media of the supplies of relief items delivered at demand points, and under competition, need not take on a particular structure.

2. The altruism or benefit functions, also included in each NGO's utility function, need not be linear.

3. The competition associated with logistics is captured through total cost functions that depend not only on a particular NGO's relief item shipments but also on those of the other NGOs.

Game Theory and Disaster Relief

In order to guarantee effective product delivery at the demand points, we retain the lower and upper bounds on demand points for the relief item supplies as introduced in Nagurney, Alvarez Flores, and Soylu (2016).

This feature of shared, or common, constraints among competing decision-makers makes the problem a Generalized Nash Equilibrium problem rather than just a Nash Equilibrium one.

Moreover, we make use of a *Variational Equilibrium* and, hence, we do not need to utilize quasi-variational inequalities in the formulation and computations but can apply the more advanced variational inequality theory.

Some Literature

Our disaster relief game theory framework entails competition for donors as well as media exposure plus supply chain aspects. We now highlight some of the related literature on these topics.

• Natsios (1995) contends that the cheapest way for relief organizations to fundraise is to provide early relief in highly visible areas.

• Balcik et al. (2010) note that the media is a critical factor affecting relief operations with NGOs seeking visibility to attract more resources from donors. They also review the challenges in coordinating humanitarian relief chains and describe the current and emerging coordination practices in disaster relief.

Some Literature

- Olsen and Carstensen (2003) confirmed the frequently repeated argument that media coverage is critical in relation to emergency relief allocation in a number of cases that they analyzed.

- Van Wassenhove (2006) also emphasizes the role of the media in humanitarian logistics and states that following appeals in the media, humanitarian organizations are often flooded with unsolicited donations that can create bottlenecks in the supply chain.

- Zhuang, Saxton, and Wu (2014) develop a model that reveals the amount of charitable contributions made by donors is positively dependent on the amount of disclosure by the NGOs. They also emphasize that there is a dearth of existing game-theoretic research on nonprofit organizations. Our model attempts to help to fill this void.

Game Theory and Disaster Relief

Although there have been quite a few optimization models developed for disaster relief there are very few game theory models.

Toyasaki and Wakolbinger (2014) constructed the first models of financial flows that captured the strategic interaction between donors and humanitarian organizations using game theory and also included earmarked donations.

Muggy and Stamm (2014), in turn, provide an excellent review of game theory in humanitarian operations and emphasize that there are many untapped research opportunities for modeling in this area.

Additional references to disaster relief and humanitarian logistics can be found in our papers.

The Variational Equilibrium Network Framework for Humanitarian Organizations

The Network Structure of the Model

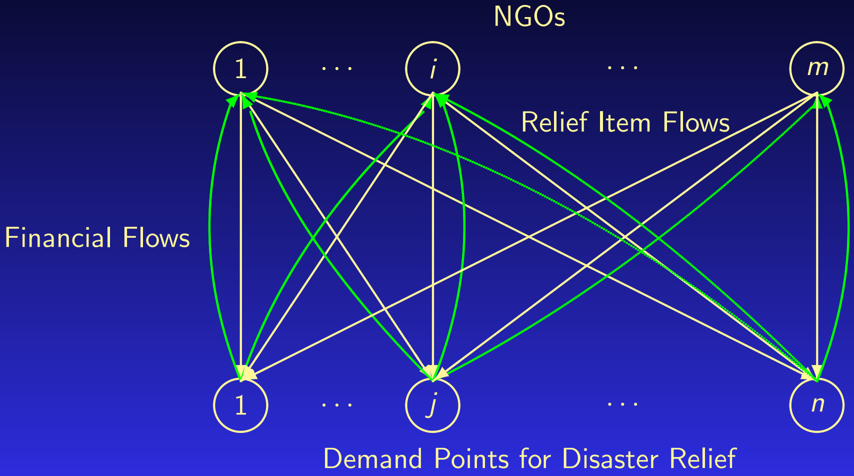


Figure 1: The Network Structure of the Game Theory Model

The Game Theory Model

We consider m humanitarian organizations, that is, nongovernmental organizations (NGOs), with a typical NGO denoted by i, seeking to deliver relief supplies, post a disaster, to n demand points, with a typical demand point denoted by j. The relief supplies can be water, food, or medicine.

We denote the volume of the relief item shipment (flow) delivered by NGO i to demand point j by q_{ij}. We group the nonnegative relief item flows from each NGO i; $i = 1, \ldots, m$, into the vector $q_i \in R^n_+$ and then we group the relief item flows of all the NGOs to all the demand points into the vector $q \in R^{mn}_+$. The vector q_i is the vector of strategies of NGO i.

The Game Theory Model

Each NGO i encumbers a cost, c_{ij}, associated with shipping the relief items to location j, where we assume that

$$c_{ij} = c_{ij}(q), \quad j = 1, \dots n, \tag{1}$$

with these cost functions being convex and continuously differentiable. The cost functions (1) are associated with the logistics aspects.

Each NGO i; $i = 1, \dots, m$, based on the media attention and the visibility of NGOs at demand point j; $j = 1, \dots, n$, receives financial funds from donors given by the expression

$$\sum_{j=1}^{n} P_{ij}(q), \tag{2}$$

where $P_{ij}(q)$ denotes the financial funds in donation dollars given to NGO i due to visibility of NGO i at location j.

The Game Theory Model

Since the NGOs are humanitarian organizations involved in disaster relief, each NGO i also derives some utility from delivering the needed relief supplies. We, hence, introduce an altruism/benefit function B_i; $i = 1, \ldots, m$, such that

$$B_i = B_i(q), \tag{3}$$

and each benefit function is assumed to be concave and continuously differentiable.

Each NGO i; $i = 1, \ldots, m$, has an amount s_i of the relief item that it can allocate post-disaster, which must satisfy:

$$\sum_{j=1}^{n} q_{ij} \leq s_i. \tag{4}$$

We assume that the relief supplies have been prepositioned so that they are in stock and available, since time is of the essence.

In addition, the relief item flows for each i; $i = 1, \ldots, m$, must be nonnegative, that is:

$$q_{ij} \geq 0, \quad j = 1, \ldots, n. \tag{5}$$

The Game Theory Model

Each NGO i; $i = 1, \ldots, m$, seeks to maximize its utility, U_i, with the utility consisting of the financial gains due to its visibility through media of the relief item flows, plus the utility associated with the logistical (supply chain) aspects of delivery of the supplies, which consists of the weighted altruism/benefit function minus the logistical costs.

Without the imposition of demand bound constraints (which will follow), the optimization problem faced by NGO i; $i = 1, \ldots, m$, is, thus,

$$\text{Maximize} \quad U_i(q) = \sum_{j=1}^{n} P_{ij}(q) + \omega_i B_i(q) - \sum_{j=1}^{n} c_{ij}(q) \tag{6}$$

subject to constraints (4) and (5).
The above model is a Nash Equilibrium problem, which can be formulated as a variational inequality problem (cf. Gabay and Moulin (1980) and Nagurney (1999)).

The Common Constraints

The common constraints are imposed by an authority ensure that the needs of the disaster victims are met, while recognizing the negative effects of waste and material convergence. The imposition of such constraints in terms of effectiveness and even gains for NGOs was demonstrated in Nagurney, Alvarez Flores, and Soylu (2016).

The two sets of common imposed constraints, at each demand point j; $j = 1, \ldots, n$, are as follows:

$$\sum_{i=1}^{m} q_{ij} \geq \underline{d}_j, \tag{7}$$

and

$$\sum_{i=1}^{m} q_{ij} \leq \bar{d}_j, \tag{8}$$

where $\underline{d}_j$ is the lower bound on the amount of the relief item needed at demand point j and $\bar{d}_j$ is the upper bound.

We assume that

The Game Theory Model

We assume that

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} \underline{d}_j. \tag{9}$$

Hence, the total supply of the relief item of the NGOs is sufficient to meet the needs at all the demand points.
We define the feasible set K_i for each NGO i as:

$$K_i \equiv \{q_i | \, (4) \text{ and } (5) \text{ hold}\} \tag{10}$$

and we let $K \equiv \prod_{i=1}^{m} K_i$.
In addition, we define the feasible set $\mathcal{S}$ consisting of the shared constraints as:

$$\mathcal{S} \equiv \{q | \, (7) \text{ and } (8) \text{ hold}\}. \tag{11}$$

Definition 1: Disaster Relief Generalized Nash Equilibrium

A relief item flow pattern $q^* \in K = \prod_{i=1}^{m} K_i$, $q^* \in \mathcal{S}$, *constitutes a disaster relief Generalized Nash Equilibrium if for each NGO* i; $i = 1, \ldots, m$:

$$\hat{U}_i(q_i^*, \hat{q}_i^*) \geq U_i(q_i, \hat{q}_i^*), \quad \forall q_i \in K_i, \forall q \in \mathcal{S}, \tag{12}$$

where $\hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*)$.

An equilibrium is established if no NGO can unilaterally improve upon its utility by changing its relief item flows in the disaster relief network, given the relief item flow decisions of the other NGOs, and subject to the supply constraints, the nonnegativity constraints, and the shared/coupling constraints. We remark that both K and $\mathcal{S}$ are convex sets.

Variational Inequality Formulation of the Nash Equilibrium Counterpart

If there are no coupling, that is, shared, constraints in the above model, then q and q^* in Definition 1 need only lie in the set K, and, under the assumption of concavity of the utility functions and that they are continuously differentiable, we know that (cf. Gabay and Moulin (1980) and Nagurney (1999)) the solution to what would then be a Nash equilibrium problem (see Nash (1950, 1951)) would coincide with the solution of the following variational inequality problem: determine $q^* \in K$, such that

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} \hat{U}_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \tag{13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the corresponding Euclidean space and $\nabla_{q_i} \hat{U}_i(q)$ denotes the gradient of $\hat{U}_i(q)$ with respect to q_i.

Variational Equilibrium

As emphasized in Nagurney, Yu, and Besik (2017), a refinement of the Generalized Nash Equilibrium is what is known as a variational equilibrium and it is a specific type of GNE (see Kulkarni and Shabhang (2012)).

Specifically, in a GNE defined by a variational equilibrium, the Lagrange multipliers associated with the common/shared/coupling constraints are all the same.

This feature provides a fairness interpretation and is reasonable from an economic standpoint.

More precisely, we have the following definition:

Definition 2: Variational Equilibrium
A strategy vector q^* *is said to be a variational equilibrium of the above Generalized Nash Equilibrium game if* $q^* \in K, q^* \in \mathcal{S}$ *is a solution of the variational inequality:*

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} U_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \forall q \in \mathcal{S}. \tag{14}$$

Variational Equilibrium

We now expand the terms in variational inequality (14).

Specifically, we have that (14) is equivalent to the variational inequality: determine $q^* \in K, q^* \in \mathcal{S}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}-\sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}-\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right]\times\left[q_{ij}-q^*_{ij}\right]\geq 0,$$

$$\forall q \in K, \forall q \in \mathcal{S}. \qquad (15)$$

Standard Form

We now put variational inequality (15) into standard variational inequality form (see Nagurney (1999)), that is: determine $X^* \in \mathcal{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{16}$$

where F is a given continuous function from $\mathcal{K}$ to R^N, $\mathcal{K}$ is a closed and convex set, with both the vectors $F(X)$ and X being column vectors, and $N = mn$.
We define $X \equiv q$ and $F(X)$ where component (i,j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, of $F(X)$, $F_{ij}(X)$, is given by

$$F_{ij}(X) \equiv \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} \right] \tag{17}$$

and $\mathcal{K} \equiv K \cap \mathcal{S}$. Then, clearly, (14) takes on the standard form (16).

Existence and Uniqueness of an Equilibrium Solution

A solution q^* of disaster relief item flows to the variational inequality problem (15) is guaranteed to exist since the function $F(X)$ in (16) is continuous under the imposed assumptions and the feasible set $\mathcal{K}$ comprised of the constraints is compact.

It follows from the classical theory of variational inequalities (cf. Kinderlehrer and Stampacchia (1980) and Nagurney (1999)) that if $F(X)$ is strictly monotone, that is:

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K}, X^1 \neq X^2,$$

then the solution to the variational inequality (16) is unique, and we have a unique equilibrium product shipment pattern q^* from the NGOs to the demand points.

Lagrange Theory and Analysis of Marginal Utilities

Lagrange Theory and Analysis of Marginal Utilities

We now explore the Lagrange theory associated with variational inequality (15) and we provide an analysis of the marginal utilities at the equilibrium solution.

For an application of Lagrange theory to other models, see: Daniele (2001) (spatial economic models), Barbagallo, Daniele, and Maugeri (2012) (financial networks), Toyasaki, Daniele, and Wakolbinger (2014) (end-of-life products networks), Daniele and Giuffrè (2015) (random traffic networks), Caruso and Daniele (2016) (transplant networks), Nagurney and Dutta (2016) (competition for blood donations).

By setting:

$$C(q)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}-\sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}-\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right](q_{ij}-q_{ij}^*), \tag{18}$$

variational inequality (15) can be rewritten as a minimization problem as follows:

$$\min_{\mathcal{K}} C(q)=C(q^*)=0. \tag{19}$$

Lagrange Theory and Analysis of Marginal Utilities

Under the previously imposed assumptions, we know that all the involved functions in (19) are continuously differentiable and convex. We set:

$$\begin{aligned}
a_{ij} &= -q_{ij} \leq 0, & \forall i,\ \forall j, \\
b_i &= \sum_{j=1}^{n} q_{ij} - s_i \leq 0, & \forall i, \\
c_j &= \underline{d}_j - \sum_{i=1}^{m} q_{ij} \leq 0, & \forall j, \\
e_j &= \sum_{i=1}^{m} q_{ij} - \overline{d}_j \leq 0, & \forall j,
\end{aligned} \tag{20}$$

and

$$\Gamma(q) = (a_{ij}, b_i, c_j, e_j)_{i=1,\dots,m;\, j=1,\dots,n}\,. \tag{21}$$

As a consequence, we remark that $\mathcal{K}$ can be rewritten as

$$\mathcal{K} = \{q \in R^{mn} : \Gamma(q) \leq 0\}. \tag{22}$$

We now consider the following Lagrange function:

$$\begin{aligned}\mathcal{L}(q,\alpha,\delta,\sigma,\varepsilon) &= \sum_{j=1}^{n} c_{ij}(q) - \sum_{j=1}^{n} P_{ij}(q) - \omega_i B_i(q) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} a_{ij} + \sum_{i=1}^{m} \delta_i b_i + \sum_{j=1}^{n} \sigma_j c_j + \sum_{j=1}^{n} \varepsilon_j e_j,\end{aligned} \tag{23}$$

$$\forall q \in R_+^{mn},\ \forall \alpha \in R_+^{mn},\ \forall \delta \in R_+^{m},\ \forall \sigma \in R_+^{n},\ \forall \varepsilon \in R_+^{n},$$

where α is the vector with components: $\{\alpha_{11},\ldots,\alpha_{mn}\}$; δ is the vector with components $\{\delta_1,\ldots,\delta_m\}$; σ is the vector with elements: $\{\sigma_1,\ldots,\sigma_n\}$, and ϵ is the vector with elements: $\{\epsilon_1,\ldots,\epsilon_n\}$.

Lagrange Theory and Analysis of Marginal Utilities

It is easy to prove that the feasible set $\mathcal{K}$ is convex and that the Slater condition is satisfied. Then, if q^* is a minimal solution to problem (19), there exist $\alpha^* \in R^{mn}_+$, $\delta^* \in R^m_+$, $\sigma^* \in R^n_+$, $\varepsilon^* \in R^n_+$ such that the vector $(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ is a saddle point of the Lagrange function (23); namely:

$$\mathcal{L}(q^*, \alpha, \delta, \sigma, \varepsilon) \leq \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*) \leq \mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*), \quad (24)$$

$$\forall q \in R^{mn}_+,\ \forall \alpha \in R^{mn}_+,\ \forall \delta \in R^m_+,\ \forall \sigma \in R^n_+,\ \forall \varepsilon \in R^n_+,$$

and

$$\alpha^*_{ij} a^*_{ij} = 0,\ \forall i,\ \forall j,$$

$$\delta^*_i b^*_i = 0,\ \forall i,$$

$$\sigma^*_j c^*_j = 0, \quad \varepsilon^*_j e^*_j = 0,\ \forall j. \quad (25)$$

Lagrange Theory and Analysis of Marginal Utilities

From the right-hand side of (24), it follows that $q^* \in R^{mn}_+$ is a minimal point of $\mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ in the whole space R^{mn}, and hence, for all $i = 1, \ldots, m$, and for all $j = 1, \ldots, n$, we have that:

$$\frac{\partial \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)}{\partial q_{ij}}$$

$$= \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} - \alpha^*_{ij} + \delta^*_i - \sigma^*_j + \varepsilon^*_j = 0, \tag{26}$$

together with conditions (25).

Conditions (25) and (26) represent an equivalent formulation of variational inequality (15). Indeed, if we multiply (26) by $(q_{ij} - q_{ij}^*)$ and sum up with respect to i and j, we get:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} \right] (q_{ij} - q_{ij}^*)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}^* q_{ij} - \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}^* q_{ij}^*}_{=0} - \sum_{i=1}^{m} \left(\delta_i^* \sum_{j=1}^{n} q_{ij} - \underbrace{\delta_i^* \sum_{j=1}^{n} q_{ij}^*}_{=\delta_i^* s_i} \right)$$

$$+ \sum_{j=1}^{n} \left(\sigma_j^* \sum_{i=1}^{m} q_{ij} - \underbrace{\sigma_j^* \sum_{i=1}^{m} q_{ij}^*}_{=\sigma_j^* \underline{d}_j} \right) - \sum_{j=1}^{n} \left(\varepsilon_j^* \sum_{i=1}^{m} q_{ij} - \underbrace{\varepsilon_j^* \sum_{i=1}^{m} q_{ij}^*}_{=\varepsilon_j^* \overline{d}_j} \right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\underbrace{\alpha_{ij}^* q_{ij}}_{\geq 0} - \sum_{i=1}^{m}\delta_i^*\left(\underbrace{\sum_{j=1}^{n} q_{ij} - s_i}_{\leq 0}\right) + \sum_{j=1}^{n}\sigma_j^*\left(\underbrace{\sum_{i=1}^{m} q_{ij} - \underline{d}_j}_{\geq 0}\right)$$

$$- \sum_{j=1}^{n}\varepsilon_j^*\left(\underbrace{\sum_{i=1}^{m} q_{ij} - \overline{d}_j}_{\leq 0}\right) \geq 0. \quad (27)$$

Interpretation of the Lagrange Multipliers

We now discuss the meaning of some of the Lagrange multipliers. We focus on the case where $q^*_{ij} > 0$; namely, the relief item flow from NGO i to demand point j is positive; otherwise, if $q^*_{ij} = 0$, the problem is not interesting. Then, from the first line in (25), we have that $\alpha^*_{ij} = 0$.

Let us consider the situation when the constraints are not active, that is, $b^*_i < 0$ and $\underline{d}_j < \sum_{i=1}^{m} q^*_{ij} < \overline{d}_j$.

Interpretation of the Lagrange Multipliers

Specifically, $b_i^* < 0$ means that $\sum_{j=1}^{n} q_{ij}^* < s_i$; that is, the sum of relief items sent by the i-th NGO to all demand points is strictly less than the total amount s_i at its disposal. Then, from the second line in (25), we get: $\delta_i^* = 0$.

At the same time, from the last line in (25), $\underline{d}_j < \sum_{i=1}^{m} q_{ij}^* < \overline{d}_j$, leads to: $\sigma_j^* = \varepsilon_j^* = 0$.

Lagrange Theory and Analysis of Marginal Utilities

Hence, (26) yields:

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \alpha_{ij}^* - \delta_i^* + \sigma_j^* - \varepsilon_j^* = 0$$

$$\iff \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}}. \qquad (28)$$

In this case, the marginal utility associated with the financial donations plus altruism is equal to the marginal costs.

Lagrange Theory and Analysis of Marginal Utilities

If, on the other hand, $\sum_{i=1}^{m} q_{ij}^* = \underline{d}_j$, then $\sigma_j^* > 0$. Hence, we get:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \sigma_j^* = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}}, \text{ with } \sigma_j^* > 0, \quad (29)$$

and, therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} > \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}, \quad (30)$$

which means that the marginal costs are greater than the marginal utility associated with the financial donations plus altruism and this is a very bad situation.

Finally, if $\sum_{i=1}^{m} q_{ij}^* = \overline{d}_j$, then $\varepsilon_j^* > 0$, we have that:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \varepsilon_j^*, \text{ with } \varepsilon_j^* > 0. \quad (31)$$

Therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} < \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}. \quad (32)$$

In this situation, the relevant marginal utility exceeds the marginal cost and this is a desirable situation.

Lagrange Theory and Analysis of Marginal Utilities

Analogously, if we assume that the conservation of flow equation is active; that is, if $\sum_{j=1}^{n} q_{ij}^* = s_i$, then $\delta_i^* > 0$. As a consequence, we obtain:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \delta_i^*, \text{ with } \delta_i^* > 0, \quad (33)$$

which means that, once again, the marginal utility associated with the financial donations plus altruism exceeds the marginal cost and this is the desirable situation.

From the above analysis of the Lagrange multipliers and marginal utilities at the equilibrium solution, we can conclude that the most convenient situation, in terms of the marginal utilities, is the one when $\sum_{i=i}^{m} q_{ij}^* = \bar{d}_j$ and $\sum_{j=1}^{n} q_{ij}^* = s_i$.

Lagrange Theory and Analysis of Marginal Utilities

Taking into account the Lagrange multipliers, an equivalent variational formulation of problem (6) under constraints (4), (5), (7), and (8) is the following one:

$$\text{Find } (q^*, \delta^*, \sigma^*, \varepsilon^*) \in R_+^{mn+m+2n}:$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \delta_i^* - \sigma_j^* + \varepsilon_j^*\right]$$

$$(q_{ij} - q_{ij}^*) + \sum_{i=1}^{m}\left(s_i - \sum_{j=1}^{n} q_{ij}^*\right)(\delta_i - \delta_i^*)$$

$$+\sum_{j=1}^{n}\left(\sum_{i=1}^{m} q_{ij}^* - \underline{d}_j\right)(\sigma_j - \sigma_j^*) + \sum_{j=1}^{n}\left(\overline{d}_j - \sum_{i=1}^{m} q_{ij}^*\right)(\varepsilon_j - \varepsilon_j^*) \geq 0, \tag{34}$$

$$\forall q \in R_+^{mn},\ \forall \delta \in R_+^{m},\ \forall \sigma \in R_+^{n},\ \forall \varepsilon \in R_+^{n}.$$

The Algorithm and Case Study

The Algorithm

The algorithm that we apply to compute the numerical examples comprising the case study is the Euler method of Dupuis and Nagurney (1993).

As established therein, for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Conditions for convergence for a variety of network-based problems can be found in Nagurney and Zhang (1996) and Nagurney (2006).

Specifically, at iteration τ, the Euler method yields the following closed form expressions for the relief item flows and the Lagrange multipliers.

Explicit Formulae for the Euler Method Applied to the Game Theory Model

In particular, we have the following closed form expression for the relief item flows $i = 1, \ldots, m; j = 1, \ldots, n$, at each iteration:

$$q_{ij}^{\tau+1}$$

$$= \max\{0, q_{ij}^{\tau} + a_{\tau}(\sum_{k=1}^{n} \frac{\partial P_{ik}(q^{\tau})}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^{\tau})}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial c_{ik}(q^{\tau})}{\partial q_{ij}} - \delta_i^{\tau} + \sigma_j^{\tau} - \epsilon_j^{\tau})\}; \tag{35}$$

The Algorithm

The following are the closed form expressions for the Lagrange multipliers associated with the supply constraints (4), respectively, for $i = 1, \ldots, m$:

$$\delta_i^{\tau+1} = \max\{0, \delta_i^{\tau} + a_{\tau}(-s_i + \sum_{j=1}^{n} q_{ij}^{\tau})\}; \tag{36}$$

The following are the closed form expressions for the Lagrange multipliers associated with the lower bound demand constraints (7), respectively, for $j = 1, \ldots, n$:

$$\sigma_j^{\tau+1} = \max\{0, \sigma_j^{\tau} + a_{\tau}(-\sum_{i=1}^{m} q_{ij}^{\tau} + \underline{d}_j)\}. \tag{37}$$

The Algorithm

The following are the closed form expressions for the Lagrange multipliers associated with the upper bound demand constraints (8), respectively, for $j = 1, \ldots, n$:

$$\epsilon_j^{\tau+1} = \max\{0, \epsilon_j^{\tau} + a_{\tau}(-\bar{d}_j + \sum_{i=1}^{m} q_{ij}^{\tau})\}. \quad (38)$$

The Case Study - Tornados Strike Massachusetts

Our case study is inspired by a disaster consisting of a series of tornados that hit western Massachusetts on June 1, 2011. The largest tornado was measured at EF3. It was the worst tornado outbreak in the area in a century (see Flynn (2011)). A wide swath from western to central MA of about 39 miles was impacted.

The tornado killed 4 persons, injured more than 200 persons, damaged or destroyed 1,500 homes, left over 350 people homeless in Springfield's MassMutual Center arena, left 50,000 customers without power, and brought down thousands of trees.

The Case Study - Tornados Strike Massachusetts

FEMA estimated that 1,435 residences were impacted with the following breakdowns: 319 destroyed, 593 sustaining major damage, 273 sustaining minor damage, and 250 otherwise affected. FEMA estimated that the primary impact was damage to buildings and equipment with a cost estimate of $24,782,299.

Total damage estimates from the storm exceeded $140 million, the majority from the destruction of homes and businesses.

Especially impacted were the city of Springfield and the towns of Monson and Brimfield. It has been estimated that, in the aftermath, the Red Cross served about 11,800 meals and the Salvation Army about 20,000 meals (cf. Western Massachusetts Regional Homeland Security Advisory Council (2012)).

Example 1 of the Case Study

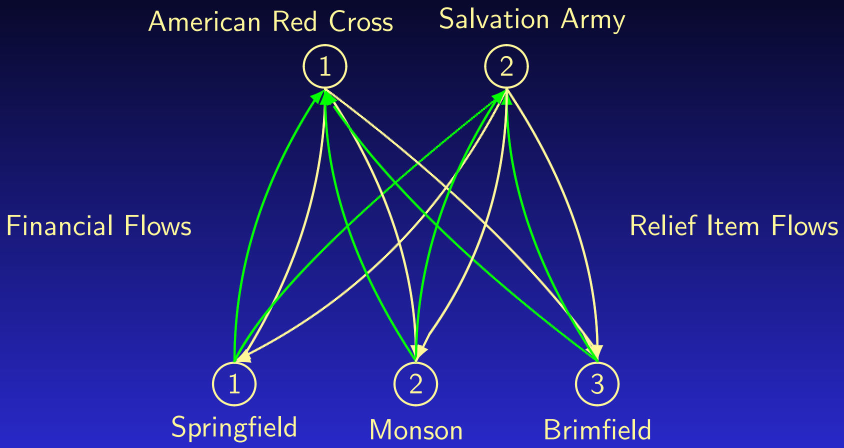


Figure 2: The Network Topology for the Case Study, Example 1

Example 1 of the Case Study

The data for our case study, Example 1, are given below. The supplies of meals available for delivery to the victims are:

$$s_1 = 25,000, \quad s_2 = 25,000,$$

with the weights associated with the altruism benefit functions of the NGOs given by:

$$\omega_1 = 1, \quad \omega_2 = 1.$$

The financial funds functions are:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22})},$$

$$P_{13}(q) = 400\sqrt{(2q_{13} + q_{23})},$$

$$P_{21}(q) = 800\sqrt{(4q_{21} + q_{11})}, \quad P_{22}(q) = 400\sqrt{(2q_{22} + q_{12})},$$

$$P_{23}(q) = 200\sqrt{(2q_{23} + q_{13})}.$$

The altruism functions are:

$$B_1(q) = 300q_{11} + 200q_{12} + 100q_{13}, \quad B_2(q) = 400q_{21} + 300q_{22} + 200q_{23}.$$

Example 1 of the Case Study

The cost functions, which capture distance from the main storage depots in Springfield, are:

$c_{11}(q) = .15q_{11}^2 + 2q_{11}$, $c_{12}(q) = .15q_{12}^2 + 5q_{12}$, $c_{13}(q) = .15q_{13}^2 + 7q_{13}$,

$c_{21}(q) = .1q_{21}^2 + 2q_{21}$, $c_{22}(q) = .1q_{22}^2 + 5q_{22}$, $c_{23}(q) = .1q_{23}^2 + 7q_{23}$.

Example 1 of the Case Study

The demand lower and upper bounds at the three demand points are:

$$\underline{d}_1 = 10000, \quad \bar{d}_1 = 20000,$$

$$\underline{d}_2 = 1000, \quad \bar{d}_2 = 10000,$$

$$\underline{d}_3 = 1000, \quad \bar{d}_3 = 10000.$$

The Euler method was implemented in FORTRAN and a Linux system at the University of Massachusetts Amherst was used for the computations.

The algorithm was initialized as follows: all Lagrange multipliers were set to 0.00 and the initial relief item flows to a given demand point were set to the lower bound divided by the number of NGOs, which here is two.

Example 1 of the Case Study

The Euler method yielded the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q_{11}^* = 3800.24, \quad q_{12}^* = 668.64, \quad q_{13}^* = 326.66,$$

$$q_{21}^* = 6199.59, \quad q_{22}^* = 1490.52, \quad q_{23}^* = 974.97.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00.$$

Since the demand at the first demand point, which is the city of Springfield, is essentially at its lower bound, we have that:

$$\sigma_1^* = 835.22,$$

with

$$\sigma_2^* = 0.00, \quad \sigma_3^* = 0.00.$$

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon_1^* = \epsilon_2^* = \epsilon_3^* = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 180,713.23, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 168,996.78.$$

This is reasonable since the American Red Cross tends to have greater visibility post disasters than the Salvation Army through the media and that was the case post the Springfield tornadoes.

Example 1 of the Case Study - Nash Equilibrium Version

We then proceeded to solve the Nash equilibrium counterpart of the above Generalized Nash Equilibrium problem formulated as a variational equilibrium. The variational inequality for the Nash equilibrium is given in (13) and does not include the upper and lower bound demand constraints. We solved it using the Euler method but over the feasible set K as in (13).

The computed equilibrium relief item flows for the Nash equilibrium are:

$$q^*_{11} = 1040.22, \quad q^*_{12} = 668.64, \quad q^*_{13} = 326.66,$$

$$q^*_{21} = 2054.51, \quad q^*_{22} = 1490.52, \quad q^*_{23} = 974.97.$$

The Lagrange multipliers associated with the supply constraints are:

$$\delta^*_1 = 0.00, \quad \delta^*_2 = 0.00.$$

Example 1 of the Case Study - Nash Equilibrium Version

Observe that, without the imposition of the bounds on the demands, Springfield, which is demand point 1 and is a big city, receives only about one third of the volume of supplies (in this case, meals) as needed, and as determined by the Generalized Nash Equilibrium solution.

The American Red Cross now garners financial donations of: 119,985.66, whereas the Salvation Army stands to receive financial donations equal to: 110,683.60. These values are significantly lower than the analogous ones for the Generalized Nash equilibrium model above.

NGOs, by coordinating their deliveries of needed supplies, such as meals, can gain in terms of financial donations and attend to the victims' needs better by delivering in the amounts that have been estimated to be needed in terms of lower and upper bounds.

Example 1 of the Case Study

This more general model, for which an optimization reformulation does not exist, in contrast to the model of Nagurney, Alvarez Flores, and Soylu (2016), nevertheless, supports the numerical result findings in the case study for Katrina therein.

Example 2 of the Case Study

We now investigate the possible impact of the addition of a new disaster relief organization, such as a church-based one, or the Springfield Partners for Community Action, which also assisted in disaster relief, providing meals post the tornadoes. Hence, the network topology for case study, Example 2, is as in Figure 3. We refer to the added NGO as "Other." It is based in Springfield.

Example 2 of the Case Study

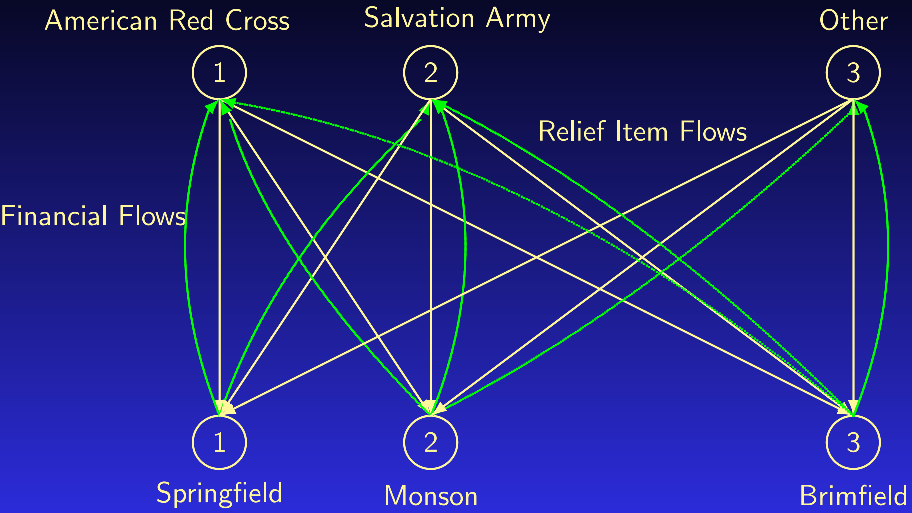


Figure 3: The Network Topology for the Case Study, Example 2

The data are as in Example 1 but with the original $P_{ij}(q)$ functions for the America Red Cross and the Salvation Army expanded as per below and the added data for the "Other" NGO also as given below.

The financial funds functions for are now:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21} + q_{31})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22} + q_{32})}$$

$$P_{21}(q) = 800\sqrt{(4q_{21} + q_{11} + q_{31})}, \quad P_{22}(q) = 400\sqrt{(2q_{22} + q_{12} + q_{32})},$$

with those for the new NGO:

$$P_{31}(q) = 400\sqrt{(2q_{31} + q_{11} + q_{21})}, \quad P_{32}(q) = 200\sqrt{(2q_{32} + q_{12} + q_{22})},$$

Example 2 of the Case Study

The weight $\omega_3 = 1$ and the altruism/benefit function for the new NGO is:

$$B_3(q) = 200q_{31} + 100q_{32} + 100q_{33}.$$

The cost functions associated with the added NGO are:

$$c_{31}(q) = .1q_{31}^2 + q_{31}, \quad c_{32}(q) = .2q_{32}^2 + 5q_{32}, \quad c_{33}(q) = .2q_{33}^2 + 7q_{33}.$$

Example 2 of the Case Study

The Euler method converged to the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q_{11}^* = 2506.97, \quad q_{12}^* = 667.85, \quad q_{13}^* = 325.59,$$

$$q_{21}^* = 4259.59, \quad q_{22}^* = 1489.98, \quad q_{23}^* = 974.45,$$

$$q_{31}^* = 3233.35, \quad q_{32}^* = 242.42, \quad q_{33}^* = 235.52.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00, \quad \delta_3^* = 0.00.$$

The demand at the first demand point, which is the city of Springfield, is at the lower bound of 10000.00. Hence, we have that: $\sigma_1^* = 446.70$, with $\sigma_2^* = 0.00$, and $\sigma_3^* = 0.00$.

Example 2 of the Case Study

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon_1^* = \epsilon_2^* = \epsilon_3^* = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 173,021.70, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 155,709.50,$$

$$\sum_{j=1}^{3} P_{3j}(q^*) = 60,504.14.$$

Example 2 of the Case Study

The volumes of relief items from the American Red Cross and the Salvation Army to Springfield are greatly reduced, as compared to the respective volumes in Example 1 and both original NGOs in Example 1 now experience a reduction in financial donations because of the increased competition for financial donations.

Example 2 of the Case Study - Nash Equilibrium Version

For completeness, we also solved the Nash equilibrium counterpart for Example 2.

The Nash equilibrium relief item flows are:

$$q^*_{11} = 1036.27, \quad q^*_{12} = 667.85, \quad q^*_{13} = 325.59,$$

$$q^*_{21} = 2051.17, \quad q^*_{22} = 1489.98, \quad q^*_{23} = 974.45,$$

$$q^*_{31} = 1009.61, \quad q^*_{32} = 242.42, \quad q^*_{33} = 235.52.$$

The financial donations of the NGOs are now the following:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 129{,}037.42, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 115{,}964.80,$$

$$\sum_{j=1}^{3} P_{3j}(q^*) = 43{,}07.16.$$

Example 2 of the Case Study

In Example 2 of our case study, we, again, see that the NGOs garner greater financial funds through the Generalized Nash Equilibrium solution, rather than the Nash equilibrium one.

Moreover, the needs of the victims are met under the Generalized Nash Equilibrium solution.

Summary and Conclusions

Summary and Conclusions

• We constructed a new Generalized Nash Equilibrium (GNE) model for disaster relief, which contains both logistical as well as financial funds aspects. The NGOs compete for financial funds through their visibility in the response to a disaster and provide needed supplies to the victims. A coordinating body imposes upper bounds and lower bounds for the supplies at the various demand points to guarantee that the victims receive the amounts at the points of demand that are needed, and without excesses that can add to the congestion and materiel convergence.

• We use a variational equilibrium formulation of the Generalized Nash Equilibrium, which is then amenable to solution via variational inequality algorithms. We provide qualitative properties of the equilibrium pattern and also utilize Lagrange theory for the analysis of the NGOs' marginal utilities.

Summary and Conclusions

- The computational scheme yields closed form expressions, at each iteration, for the product flows and the Lagrange multipliers.

- The algorithm is then applied to a case study, inspired by rare tornadoes that caused devastation in parts of western and central Massachusetts in 2011. For completeness, we also compute the solution to the Nash equilibrium counterparts of the two examples making up the case study, in which the common demand bound constraints are removed.

Summary and Conclusions

- The case study reveals that victims may not receive the required amounts of supplies, without the imposition of the demand bounds. These results provide further support for the need for greater coordination in disaster relief.

- Moreover, by delivering the required amounts of supplies the NGOs can also garner greater financial donations.

THANK YOU!

For more information, see:
http://supernet.isenberg.umass.edu