An Efficiency Measure for Dynamic Networks with Application to the Internet and Vulnerability Analysis

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The Virtual Center for Supernetworks

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Outline

- Motivation
- Literature
- Evolutionary Variational Inequality (EVI) and the Internet
- Network Efficiency Measure & Network Component Importance for Dynamic Networks
- Dynamic Braess Paradox
- Conclusion

Motivation

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.
- For example:
  - Hurricane Katrina, August 23, 2005
  - The biggest blackout in North America, August 14, 2003
  - Two significant power outrages during the month of September 2003 – one in England and one in Switzerland and Italy
  - 9/11 Terrorist Attacks, September 11, 2001

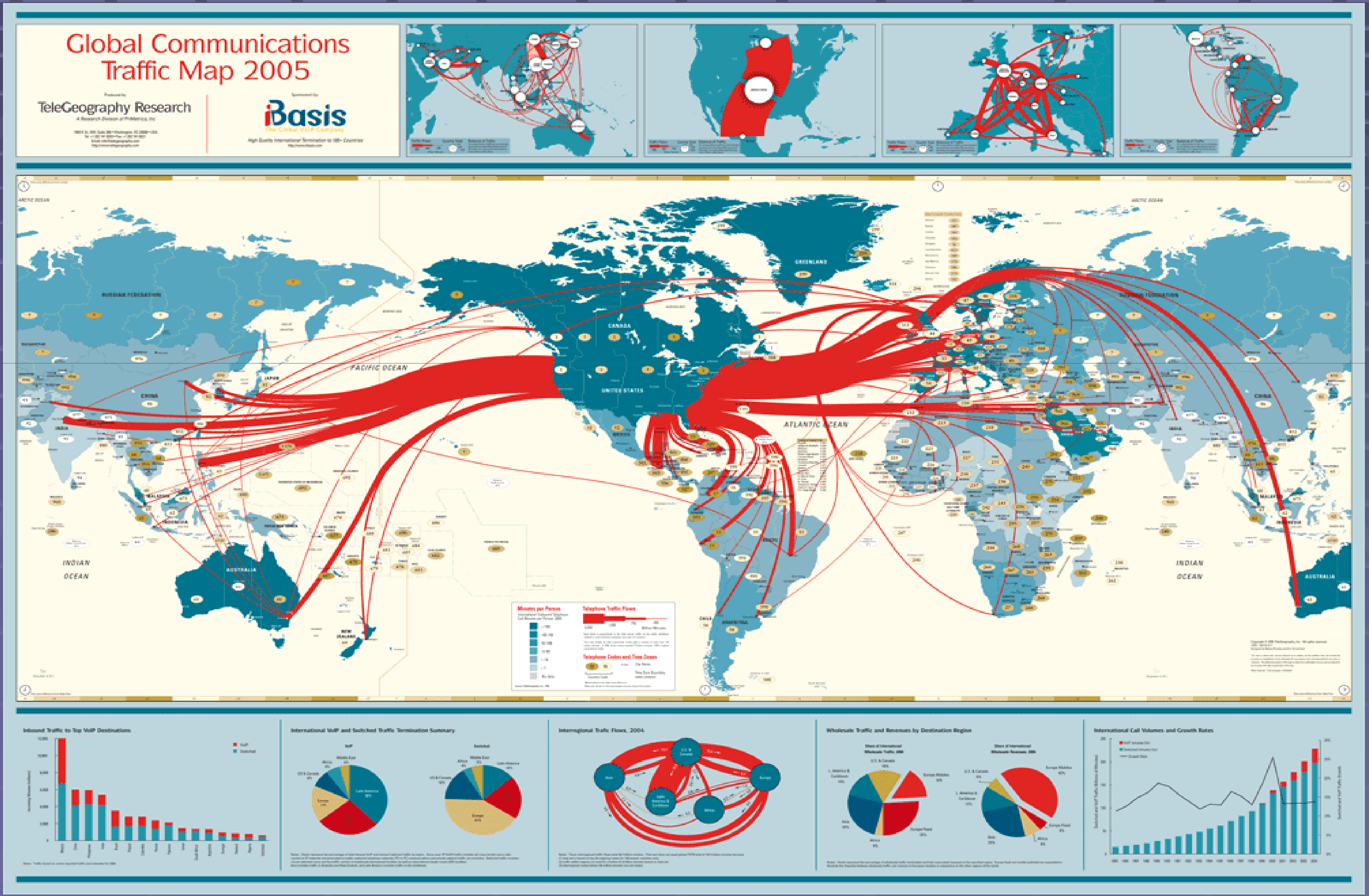
Motivation

- The Internet has revolutionized the way in which we work, interact, and conduct our daily activities. It has affected the young and the old as they gather information and communicate and has transformed business processes, financial investing and decision-making, and global supply chains. The Internet has evolved into a network that underpins our developed societies and economies.

Motivation

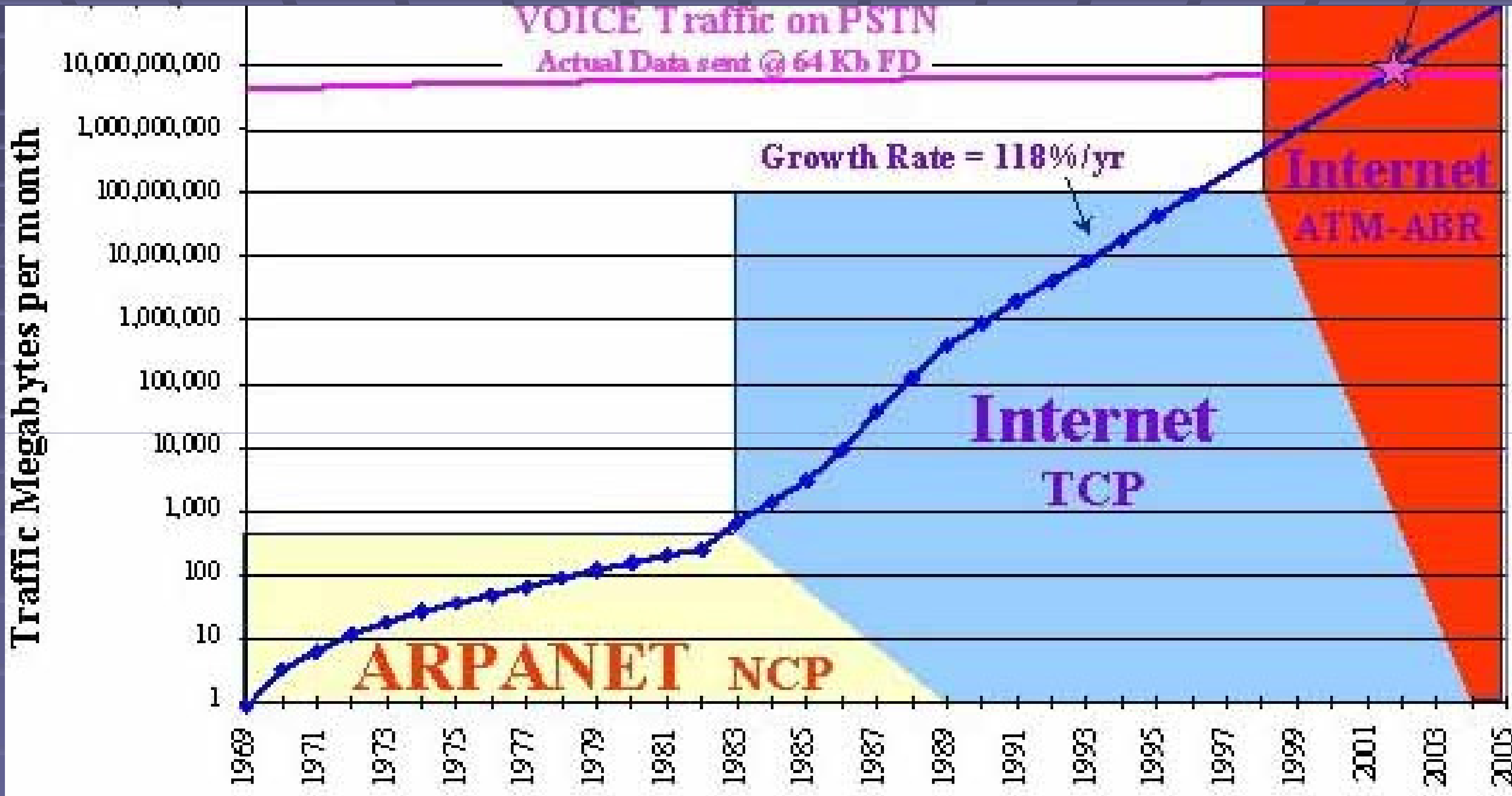
- "*A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks.*" (page 10 of Roughgarden (2005)).
- We can expect that a variety of time-dependent demand structures will occur on the Internet as individuals seek information and news online in response to major events or simply go about their daily activities whether at work or at home. Hence, it is relevant to study the vulnerability of in Internet networks with time-varying traffics.
- "... **traffic** over the **Internet doubling** every **100** days…" (Frances Hong (1999)).

Global Internet Communication Network



research.techkwondo.com

Global Internet Traffic Growth

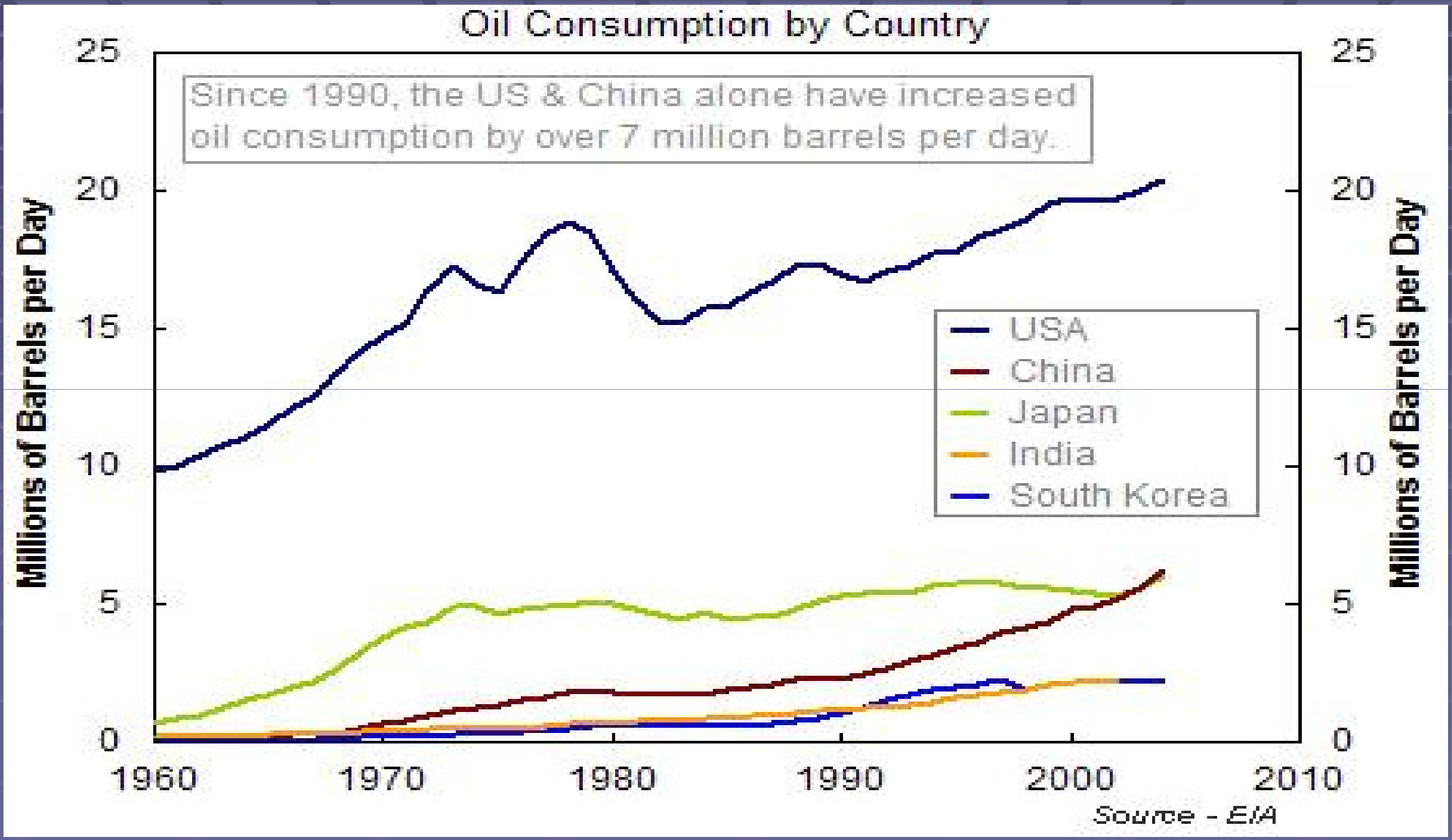


www.netvalley.com

Examples of Other Dynamic Networks

- Oil & Natural gas network
- Electricity generation and distribution network
- Supply chain network

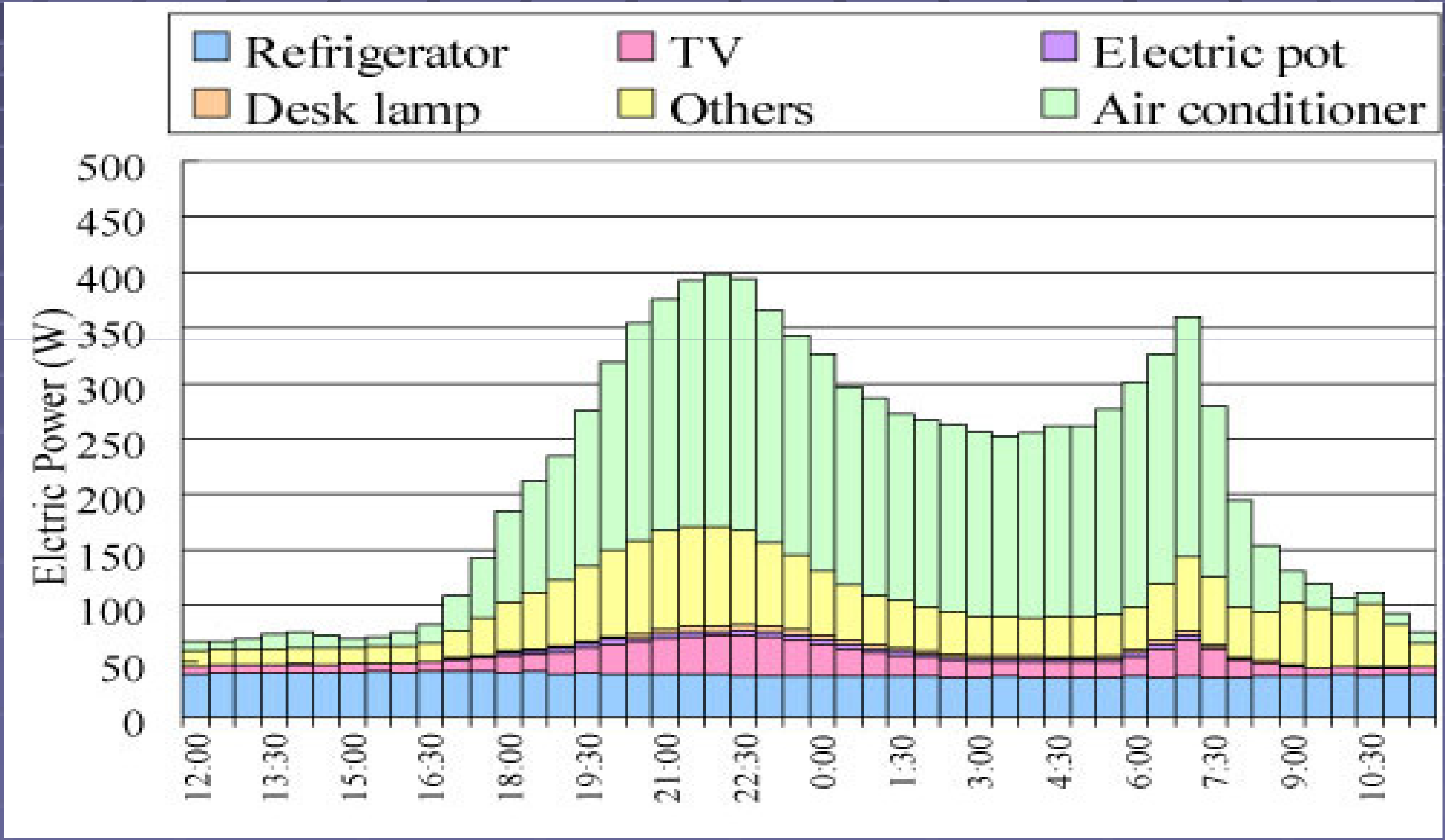
Varying Demand in Global Oil Demand



Energy Information Administration, U.S. Department of Energy

www.eia.doe.gov

Electricity Consumption Change in a Typical Day



www.terrapass.com

Literature on EVI and the Applications

- Daniele, Maugeri, and Oettli (1999)
- Daniele (2003, 2004, 2006)
- Nagurney et al. (2006)
- Nagurney, Liu, Cojocaru and Daniele (2006)
- Nagurney, Parkes, and Daniele (2006)

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Barrat, Barthélemy and Vespignani (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Chassin and Posse (2005)
- Holme, Kim, Yoon and Han (2002)
- Sheffi (2005)
- Taylor and D'este (2004)
- Jenelius, Petersen and Mattson (2006)
- Murray-Tuite and Mahmassani (2004)

EVI and the Internet

The Internet is modeled as a network $G = [N, L]$, consisting of the set of nodes N and the set of directed links L. The set of links L consists of n_L elements. The set of origin/destination (O/D) pairs of nodes is denoted by W and consists of n_W elements. We denote the set of routes (with a route consisting of links) joining the origin/destination (O/D) pair w by P_w. We assume that the routes are acyclic. Let P with n_P elements denote the set of all routes connecting all the O/D pairs in the Internet. Links are denoted by a, b, etc; routes by r, q, etc., and O/D pairs by w_1, w_2, etc. We assume that the Internet is traversed by a single class of "job" or "task."

EVI and the Internet

Let $d_w(t)$ denote the demand, that is, the traffic generated, between O/D pair w at time t. The flow on route r at time t, which is assumed to be nonnegative, is denoted by $x_r(t)$ and the flow on link a at time t by $f_a(t)$.

Since the demands over time are assumed known, the following conservation of flow equations must be satisfied at each t:

$$d_w(t) = \sum_{r \in P_w} x_r(t), \quad \forall w \in W,$$

that is, the demand associated with an O/D pair must be equal to the sum of the flows on the routes that connect that O/D pair. Also, we must have that

$$0 \leq x_r(t) \leq \mu_r(t), \quad \forall r \in P,$$

where $\mu_r(t)$ denotes the capacity on route r at time t.

The link flows are related to the route flows, in turn, through the following conservation of flow equations:

$$f_a(t) = \sum_{r \in P} x_r(t)\delta_{ar}, \quad \forall a \in L,$$

where $\delta_{ar} = 1$ if link a is contained in route r, and $\delta_{ar} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on routes that contain that link. All the link flows at time t are grouped into the vector $f(t)$, which is of dimension n_L.

The cost on route r at time t is denoted by $C_r(t)$ and the cost on a link a at time t by $c_a(t)$. We allow the cost on a link, in general, to depend upon the entire vector of link flows at time t, so that

$$c_a(t) = c_a(f(t)), \quad \forall a \in L.$$

The costs on routes are related to costs on links through the following equations:

$$C_r(x(t)) = \sum_{a \in L} c_a(x(t))\delta_{ar}, \quad \forall r \in P,$$

which means that the cost on a route at a time t is equal to the sum of costs on links that make up the route at time t. We group the route costs at time t into the vector $C(t)$, which is of dimension n_P.

We now define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0,T], R^{n_P})$ (where T denotes the time interval under consideration) given by

$$\mathcal{K} = \left\{ x \in L^2([0,T], R^{n_P}) : 0 \leq x(t) \leq \mu(t) \, \text{a.e. in} \, [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in} \, [0,T] \right\}$$

We assume that the capacities $\mu_r(t)$, for all r, are in $\mathcal{L}$ and that the demands, $d_w \geq 0$, for all w, are also in $\mathcal{L}$. Further, we assume that

$$0 \leq d(t) \leq \Phi\mu(t), \text{a.e. on} \, [0,T],$$

where Φ is the $n_W \times n_P$-dimensional O/D pair-route incidence matrix, with element (w,r) equal to 1 if route r is contained in P_w, and 0, otherwise. Due to the above assumption, the feasible set $\mathcal{K}$ is nonempty. As noted in Nagurney, Parkes, and Daniele (2006), $\mathcal{K}$ is also convex, closed, and bounded. Note that we are not restricted as to the form that the time-varying demands for the O/D pairs take since convexity is guaranteed even if the demands have a step-wise structure, or are piecewise continuous.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle \mathcal{G}, x \rangle\rangle := \int_0^T \langle \mathcal{G}(t), x(t) \rangle dt, \quad \mathcal{G} \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \to \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

Dynamic Network Equilibrium

A route flow pattern $x^* \in \mathcal{K}$ *is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle), if, at each time* t, *only the minimum cost routes not at their capacities are used (that is, have positive flow) for each O/D pair unless the flow on a route is at its upper bound (in which case those routes' costs can be lower than those on the routes not at their capacities). The state can be expressed by the following equilibrium conditions which must hold for every O/D pair* $w \in W$, *every route* $r \in P_w$, *and a.e. on* $[0, T]$:

$$C_r(x^*(t)) - \lambda_w^*(t) \begin{cases} \leq 0, & \textit{if} \quad x_r^*(t) = \mu_r(t), \\ = 0, & \textit{if} \quad 0 < x_r^*(t) < \mu_r(t), \\ \geq 0, & \textit{if} \quad x_r^*(t) = 0. \end{cases}$$

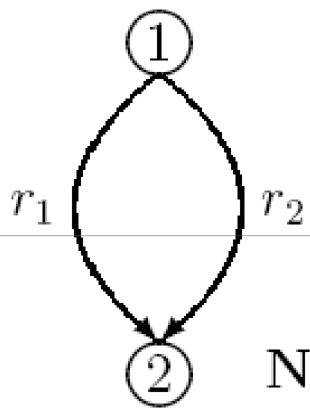
Theorem (Nagurney, Parkes, and Daniele (2006))

$x^* \in \mathcal{K}$ is an equilibrium flow according to Definition 1 if and only if it satisfies the evolutionary variational inequality:

$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

A Simple Numerical Example

Consider a network consisting of two nodes and two links. There is a single O/D pair *w*=(1,2).



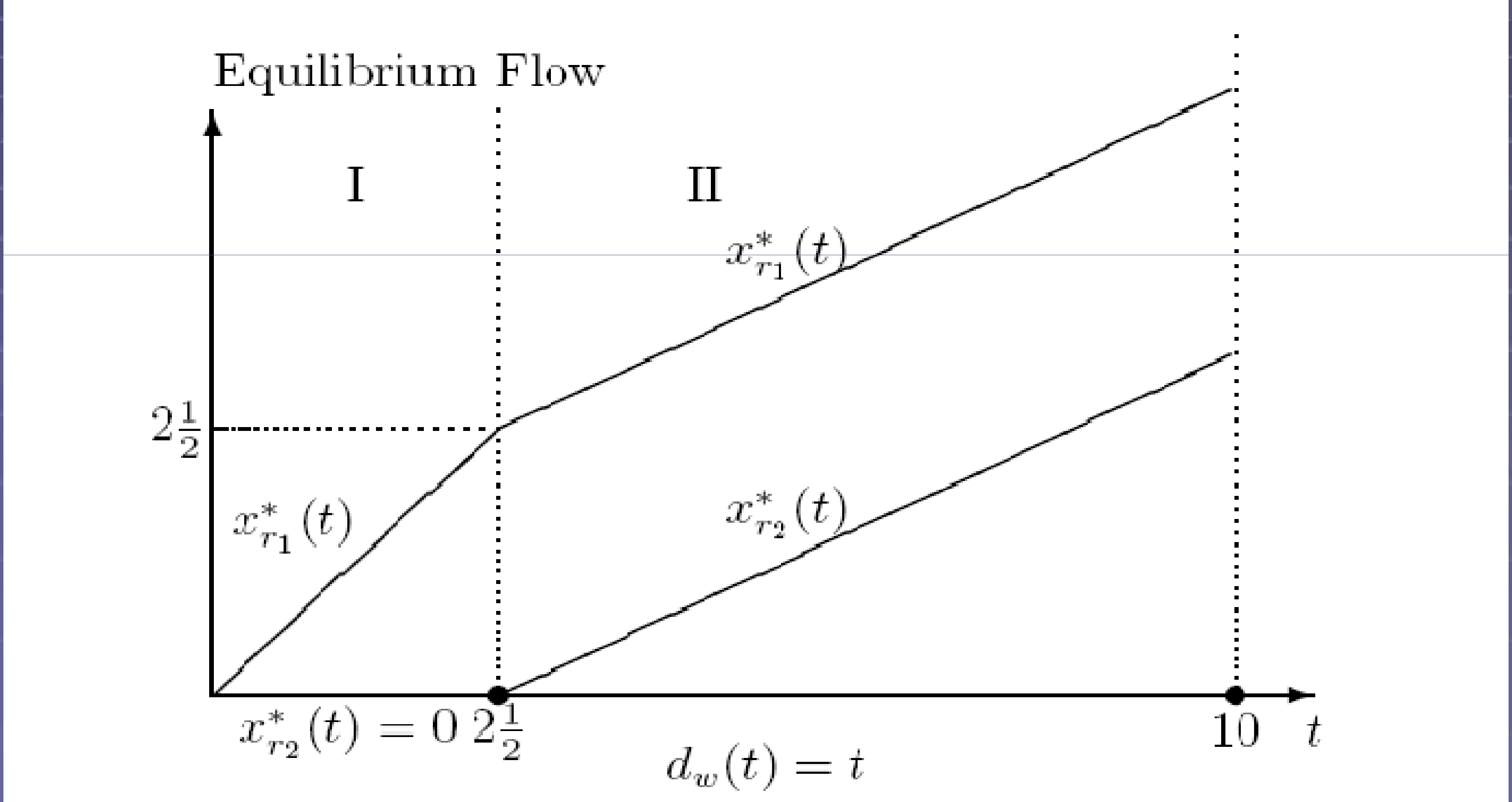
Network Structure of the Simple Numerical Example

$$C_{r_1}(x(t)) = 2x_{r_1}(t) + 5, \quad C_{r_2}(x(t)) = 2x_{r_2}(t) + 10.$$

Also, we assume that the route capacities are: $\mu_{r_1}(t) = \mu_{r_2}(t) = \infty$.

The time horizon is $[0, 10]$. The time-varying demand is assumed to be $d_w(t) = t$.

Equilibrium Trajectories of the Simple Numerical Example with Dynamic Demands



The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007) proposed a network efficiency measure (the N-Q Measure) for networks with fixed demand, which captures demand and flow information under the network equilibrium.

Definition : The N-Q Measure

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2007), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w

Network Efficiency Measure for Dynamic Networks - I

Dynamic Network Efficiency: Continuous Time Version

The network efficiency for the network G with time-varying demand d for $t \in [0, T]$, denoted by $\mathcal{E}(G, d, T)$, is defined as follows:

$$\mathcal{E}(G, d, T) = \frac{\int_0^T \left[\sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)}\right] / n_W \, dt}{T}.$$

Note the above measure is the average network performance over time of the dynamic network.

Network Efficiency Measure for Dynamic Networks - II

Let $d_w^1,\ d_w^2, ..., d_w^H$ denote demands for O/D pair w in H discrete time intervals, given, respectively, by: $[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$. We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair w at the H different time intervals by: $\lambda_w^1,\ \lambda_w^2, ..., \lambda_w^H$. The demand vector d, in this special discrete case, is a vector in $R^{nw \times H}$. The dynamic network efficiency measure in this case is as follows:

Dynamic Network Efficiency: Discrete Time Version

The network efficiency for the network (G, d) over H discrete time intervals: $[t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]$, where $t_H \equiv T$, and with the respective constant demands: $d_w^1,\ d_w^2, ..., d_w^H$ for all $w \in W$ is defined as follows:

$$\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H}[(\sum_{w \in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.$$

Special Case

Theorem

Assume that $d_w(t) = d_w$, for all O/D pairs $w \in W$ and for $t \in [0, T]$. Then, the dynamic network efficiency measure collapses to the Nagurney and Qiang (2007) measure:

$$\mathcal{E} = \frac{1}{n_W} \sum_{w \in W} \frac{d_w}{\lambda_w}.$$

Importance of a Network Component

Definition: Importance of a Network Component

The importance of network component g of network G with demand d over time horizon T is defined as follows:

$$I(g,d,T) = \frac{\mathcal{E}(G,d,T) - \mathcal{E}(G-g,d,T)}{\mathcal{E}(G,d,T)}$$

where $\mathcal{E}(G-g,d,T)$ is the dynamic network efficiency after component g is removed.

The Approach to Study the Importance of Network Components

The elimination of a link is represented in the N-Q measure by the removal of that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

Hence, our measure is well-defined even in the case of disconnected networks.

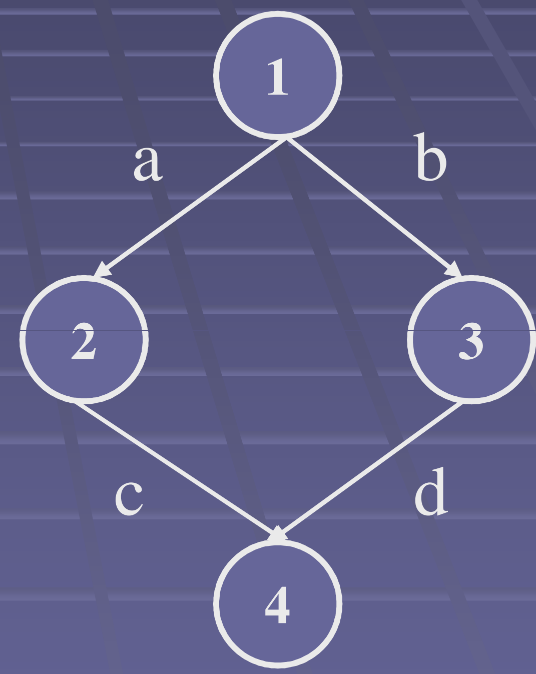
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of **6,** the equilibrium path flows are $\mathbf{x_{p_1}^{*} = x_{p_2}^{*} = 3}$ and

The equilibrium path travel cost is

$\mathbf{C_{p_1} = C_{p_2} = 83.}$



$\mathbf{c_a(f_a)=10\ f_a \quad c_b(f_b) = f_b+50}$

$\mathbf{c_c(f_c) = f_c+50 \quad c_d(f_d) = 10\ f_d}$

Adding a Link Increases Travel Cost for All!

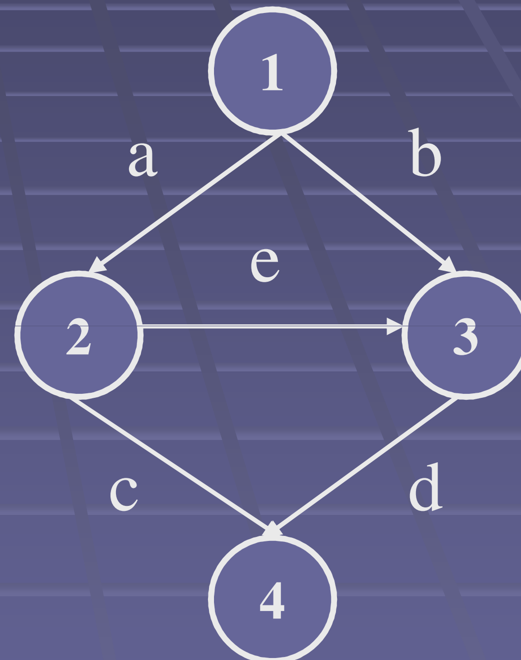
Adding a new link creates a new path **p_3=(a,e,d).**

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path **p_3, C_{p_3}=70.**

The new equilibrium flow pattern network is

$x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2$.

The equilibrium path travel costs: **$C_{p_1} = C_{p_2} = C_{p_3} = 92$.**



$c_e(f_e) = f_e + 10$

Dynamic Braess Network – Without Link *e*

We now construct time-dependent link costs, route costs, and demand for $t \in [0, T]$. It is important to emphasize that the case where time t is discrete, that is, $t = 0, 1, 2, \ldots, T$,

We consider, to start, the first network, consisting of links: a, b, c, d. We assume that the capacities $\mu_{r_1}(t) = \mu_{r_2}(t) = \infty$ for all $t \in [0, T]$. The link cost functions are assumed to be given and as follows for time $t \in [0, T]$:

$$c_a(f_a(t)) = 10 f_a(t), \quad c_b(f_b(t)) = f_b(t) + 50, \quad c_c(f_c(t)) = f_c(t) + 50, \quad c_d(f_d(t)) = 10 f_d(t).$$

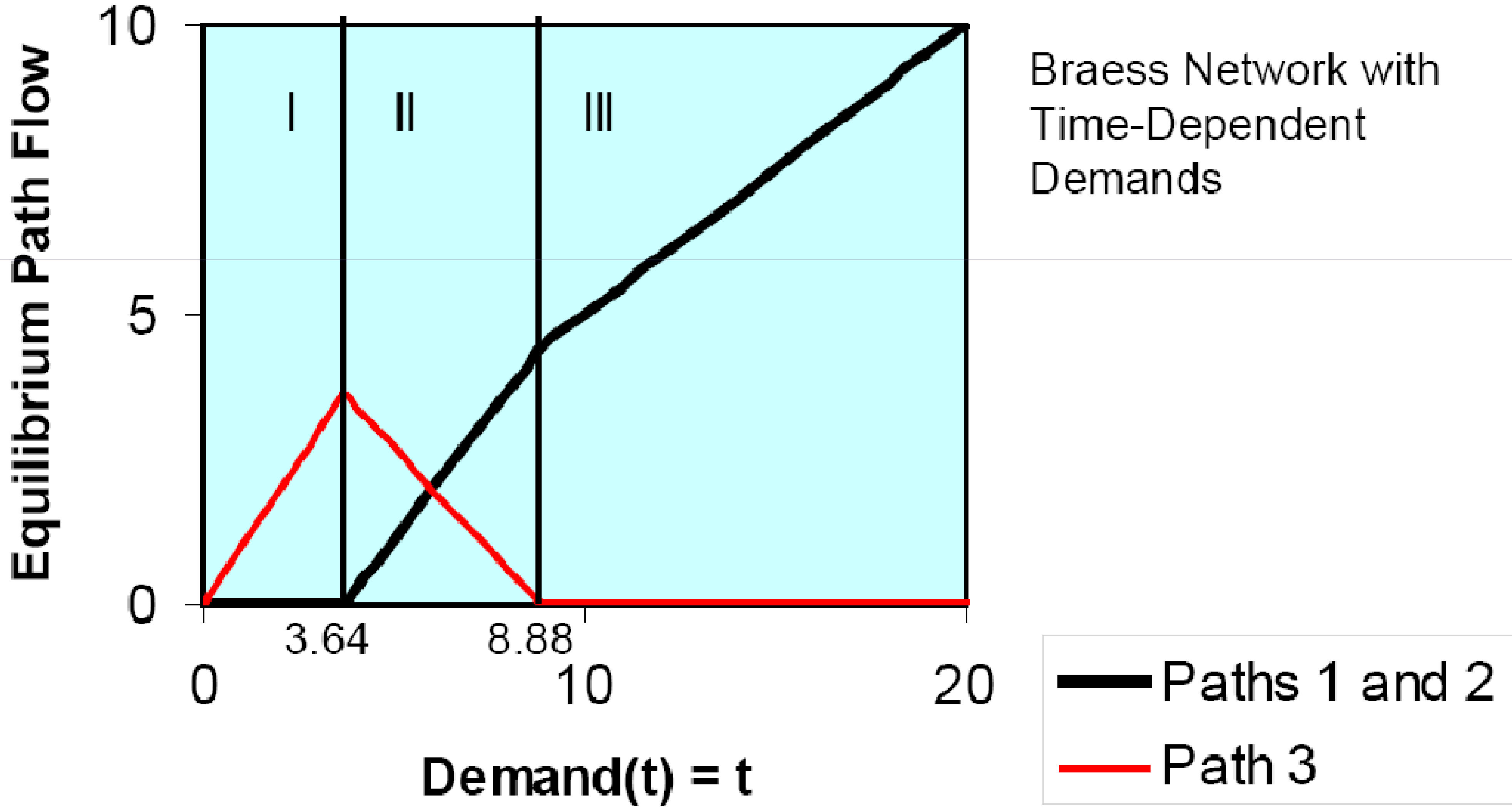
We assume a time-varying demand $d_w(t) = t$ for $t \in [0, T]$.

Dynamic Braess Network-Solution

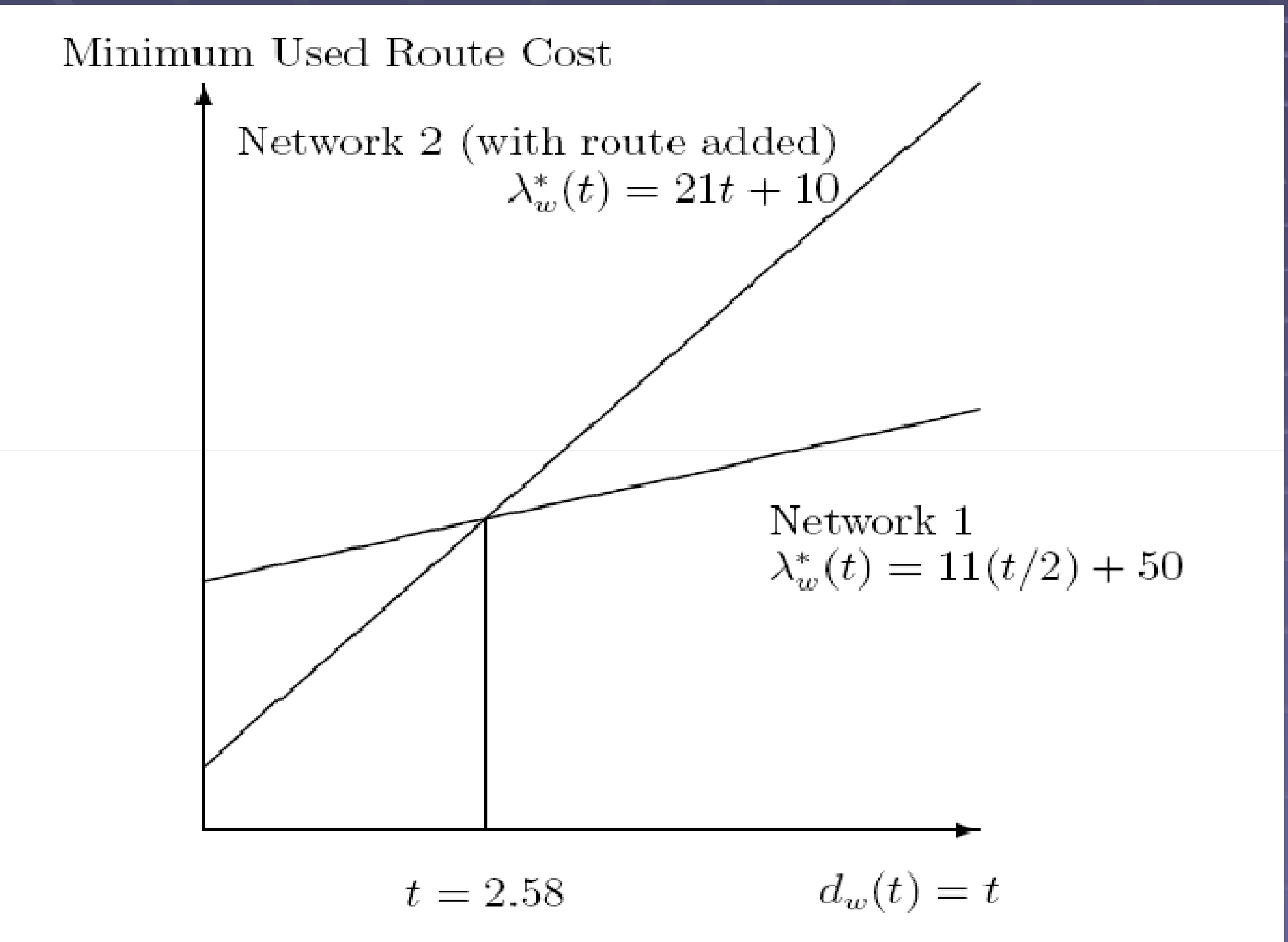
Solving EVI, we have the equilibrium path flow is $x^*_{r_1}(t) = \frac{t}{2}$ and $x^*_{r_2}(t) = \frac{t}{2}$ for $t \in [0, T]$.

The equilibrium route costs for $t \in [0, T]$ are given by: $C_{r_1}(x^*_{r_1}(t)) = 5\frac{1}{2}t + 50 = C_{r_2}(x^*_{r_2}(t)) = 5\frac{1}{2}t + 50$, and, clearly, equilibrium conditions hold for $t \in [0, T]$ a.e.

Dynamic Braess Network – Adding Link *e*



Dynamic Braess Network



Minimum Used Route Costs for Braess Networks 1 and 2

For demand in the range 2.58 < $d_w(t)$ = t < 8.89, the addition of the new route will result in everyone being worse off.

Importance of Nodes and Links in Dynamic Braess Network

Importance and Ranking of Links in the Dynamic Braess Network

Link	Importance Value	Importance Ranking
a	0.2604	1
b	0.1784	2
c	0.1784	2
d	0.2604	1
e	-0.1341	3

Importance and Ranking of Nodes in the Dynamic Braess Network

Node	Importance Value	Importance Ranking
1	1.0000	1
2	0.2604	2
3	0.2604	2
4	1.0000	1

Link e is never used after t=8.89 and in the range $t \in [2.58, 8.89]$ it increases the cost, so the fact that link e has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!

Conclusion

- The network efficiency measure captures user behavior, flows and costs on networks
- Extend our previous research on the network efficiency measure into the dynamic setting
- Applicable for varying demand in both continuous and discrete time
- The measure can be applied to other critical infrastructure networks

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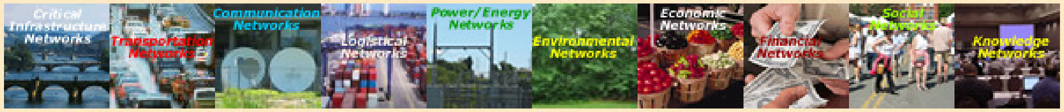

Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

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The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

Updated: April 12, 2007

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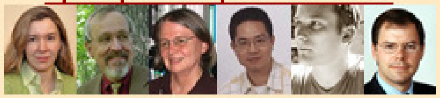

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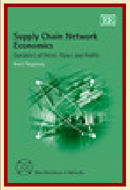

Supply Chain Network Economics

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