Observation of the Braess Paradox in Electric Circuits

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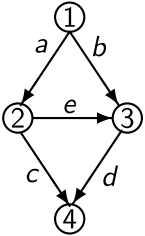
Braess Paradox in Transportation Networks

- First noted by Dietrich Braess in 1968.
- In a user-optimized transportation network, when a new link (road) is added, the change in equilibrium flows may result in a higher cost (travel time) to all travelers in the network, implying that users were better off without that link.

Examples of Braess Paradox

- Stuttgart, Germany - In 1969 a newly constructed road worsened traffic. Travel time decreased when the road was closed.
- New York City - Earth Day 1990 travel time decreased when 42nd St was closed.
- Seoul, Korea - A 6 lane road that was perpetually jammed was removed, traffic flow improved.

Classical Braess Paradox (1968) Transportation Network



f_a - flow on link a
O/D pair (1,4)
Demand $= 6$
3 paths: $p_1 = (a, c)$,
$p_2 = (b, d)$, $p_3 = (a, e, d)$.

Link cost functions:
$c_a(f_a) = 10f_a$,
$c_b(f_b) = f_b + 50$,
$c_c(f_c) = f_c + 50$,
$c_d(f_d) = 10f_d$,
$c_e(f_e) = f_e + 10$.

With link e, user-optimized flows on the paths p_1, p_2, and p_3 are each 2 and the user path costs are 92.

No user has any incentive to switch, since switching would result in a higher path cost.

Without link e, the user-optimized path flow pattern on the two original paths p_1 and p_2 is 3 for each path and the user path costs are 83.

Hence, the addition of link e makes all users of the network worse-off since the cost increases from 83 to 92!

Other Network Systems Exhibiting the Braess Paradox

The Braess paradox is also relevant to other network systems in which the users operate under decentralized (selfish) decision-making behavior.

- Spring Systems - Cohen and Horowitz *Nature* (1991), Penchina and Penchina *American Journal of Physics* (2003)
- The Internet - Nagurney, Parkes, and Daniele *Computational Management Science* (2007)
- Electric Power Generation and Distribution Networks - Bjorndal and Jornsten (2008), Witthaut and Timme *Phys.org* (2012)
- Biology - Motter *Nature Physics* (2014)
- Nanoscale Systems - Pala et al *Physical Review Letters* (2012)
- Wireless Systems - Altman et al (2008).

Can the Braess Paradox Exist in a Circuit Consisting Only of Passive Electrical Components?

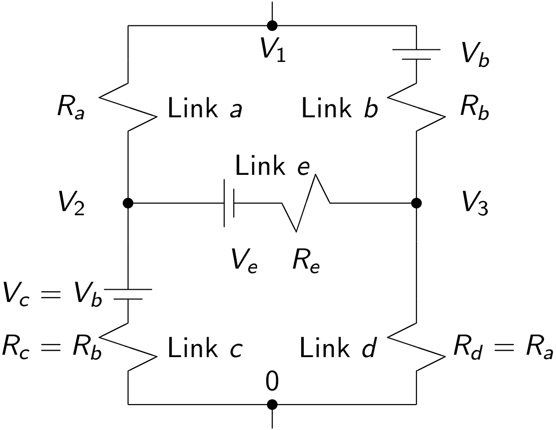
In a passive circuit, the conventional wisdom is that by adding a link (branch in circuit terminology), the resistance of such a circuit will decrease.

If the Braess Paradox would occur the *equivalent resistance* of the circuit would increase.

In terms of flow, for a fixed flow through the circuit, the voltage would rise (rather than decrease) when a branch was added. - Analogous to increase of cost in a transportation network.

An electrical network is an example of a **user-optimized network**, because all electrons move through the network with the same voltage drop.

Idealized Electrical Circuit Analogue for the Classical Braess Paradox



Because of the symmetry of the Braess Paradox example:

$$R_d = R_a, \qquad V_c = V_b, \qquad R_c = R_b.$$

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Idealized Electrical Circuit Analogue for the Classical Braess Paradox II

Let V_i; $i = 1, \ldots, n$, be the voltage at node i referenced to the reference/ground node of the circuit.

Let the demand through the electrical network be I and the flow through a link i be I_i.

In the electrical circuit, the voltage, V_1, is the equivalent of the cost for a user (electron) to flow through the circuit.

The Braess Paradox occurs if, by adding link e, the voltage V_1 increases.

Kirchhoff Nodal Analysis for the Braess Paradox Circuit

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = G^{-1} \begin{bmatrix} I + \frac{V_b}{R_b} \\ -\left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \\ \left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \end{bmatrix}$$

where G is the Conductance Matrix:

$$G = \begin{bmatrix} \frac{1}{R_a} + \frac{1}{R_b} & -\frac{1}{R_a} & -\frac{1}{R_b} \\ \frac{1}{R_a} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} & \frac{1}{R_e} \\ \frac{1}{R_a} & \frac{1}{R_e} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} \end{bmatrix}.$$

Classical Braess Paradox Example

In terms of voltages and currents, the classical Braess Paradox (1968) example has

$$V_b = 50V, \quad V_e = 10V, \quad R_a = 10\Omega, \quad R_b = R_e = 1\Omega, \quad \text{and} \quad I = 6.$$

With link e in the circuit, the Kirchhof matrix equation becomes

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1.1 & -.1 & -1 \\ .1 & -2.1 & 1 \\ .1 & 1 & -2.1 \end{bmatrix}^{-1} \begin{bmatrix} 56 \\ -60 \\ 60 \end{bmatrix} = \begin{bmatrix} 92 \\ 52 \\ 40 \end{bmatrix}.$$

Without link e, the equation becomes

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 50 + 11 \cdot \frac{6}{2} \\ 50 + \frac{6}{2} \\ 10 \cdot \frac{6}{2} \end{bmatrix} = \begin{bmatrix} 83 \\ 53 \\ 30 \end{bmatrix}.$$

Classical Braess Paradox Example - II

$V_1 = 83V$ without link e in the circuit and $V_1 = 92V$ when link e is in the network.

Voltage increases when link e is added.

This reproduces the transportation network example in the original Braess article.

Additional Insights from Kirchoff's Formulation - From the right-hand-side of nodal equations with link e in the circuit, one notes that V_e only occurs in the sum

$$V_b + V_e \frac{R_b}{R_e}.$$

This indicates that there might be networks that exhibit the Braess Paradox behavior without a fixed cost term in the added link e.

Zener Diode Formulation

In 1991 Cohen and Horowitz proposed that a Wheatstone Bridge topology circuit consisting of Zener diodes and resistors could exhibit the Braess Paradox.

Their circuit had *unrealistic values in practice, but convenient for illustration*.

Calculation of Realistic Component Values for a BP Electrical Circuit

The conductance matrix, G, can be made dimensionless by factoring out R_b^{-1} to become

$$G = \hat{G} R_b^{-1}.$$

The nodal equations become

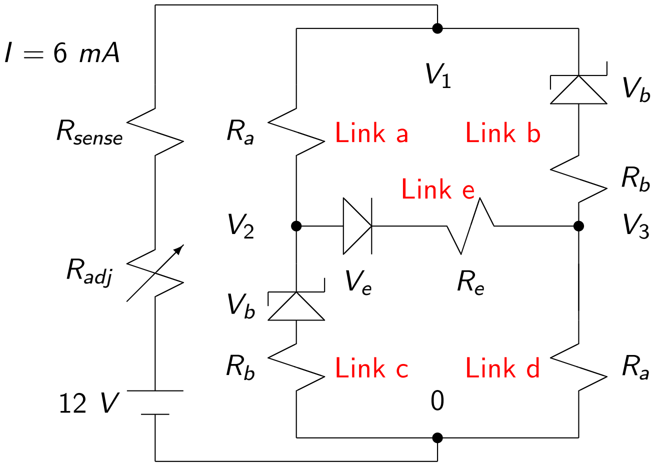
$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \hat{G}^{-1} \begin{bmatrix} IR_b + V_b \\ -\left(V_b + V_e \frac{R_b}{R_e}\right) \\ \left(V_b + V_e \frac{R_b}{R_e}\right) \end{bmatrix},$$

where $\hat{G}$ depends only on the ratios $\frac{R_b}{R_a}$ and $\frac{R_b}{R_e}$.

The matrix equation can be scaled by a constant to allow the choice of realistic component values, and current, I.

The batteries can be replaced by Zener diodes, which, to a good approximation, are voltage drops.

Electrical Circuit Using Zener Diodes



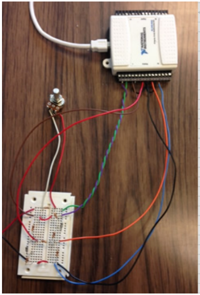
$V_b = 1N4733 = 5.1\ V, \qquad V_e = 1N4002 = .7\ V,$

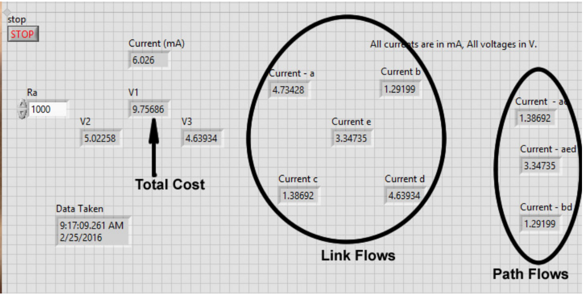
$R_a = 1000\Omega, \qquad R_b = R_e = 100\Omega.$

Experimental Setup

The voltages at nodes V_1, V_2, and V_3, and the voltage across R_{sense} are measured using the 4 analog input channels of a National Instruments USB-6009 Multifunction I/O Data Acquisition system, programmed using Labview running on a PC.

From these measurements and the knowledge of the resistor values R_a, R_b, and R_e, the link and path flows are calculated.





All currents are in mA, All voltages in V.

Current (mA): 6.026
Ra: 1000
V1: 9.75686
V2: 5.02258
V3: 4.63934
Data Taken: 9:17:09.261 AM 2/25/2016

Current - a: 4.73428
Current b: 1.29199
Current e: 3.34735
Current c: 1.38692
Current d: 4.63934

Current - a: 1.38692
Current - aed: 3.34735
Current - bd: 1.29199

Zener Diode Circuit Measurements

For this circuit, five measurements are made for cases corresponding, respectively, to:

- Case 1: link e absent
- Case 2: link e present with $V_e = .62\ V$ and $R_e = 100\ \Omega$ (analogous to the classical Braess example)
- Case 3: link e present with only $R_e = 100\Omega$
- Case 4: link e present with only $V_e = .62\ V$
- Case 5: link e is a short circuit, i.e., $R_e = 0$.

For all cases the cost functions on links $a - d$ are as below:

$$\begin{array}{rlll}
\text{Cost on link } a & c_a: & 1000f_a & = 1000I_a \\
\text{Cost on link } b & c_b: & 5.1 + 100f_b & = 5.1 + 100I_b \\
\text{Cost on link } c & c_c: & 5.1 + 100f_c & = 5.1 + 100I_c \\
\text{Cost on link } d & c_d: & 1000f_d & = 1000I_d.
\end{array}$$

Measured Voltage Across an Electrical Circuit Using Zener Diodes Exhibiting the Braess Paradox

Case	V_e	R_e	V_1	Form of Link e Cost Function
1	-	∞	8.13	Link *e* not in network
2	.62	100	9.21	$V_e + I_e R_e$
3	0	100	9.72	$I_e R_e$
4	.62	0	8.14	V_e
5	-	0	9.88	Link *e* is a short circuit

Interpretation of Zener Diode Results

- When link e is added, the voltage at node 1, V_1, increases, showing that the Braess Paradox occurs in the circuit.
- In electrical circuits one would normally expect the voltage to drop when a link is added. These multiple examples prove that, in contrast, the opposite can happen.
- The cost for the flow through the circuit is 8.13 V in the absence of link e and 9.21V in the presence of link e; thus, confirming the observation of the Braess Paradox in the circuit.

Interpretation of Zener Diode Results II

Cases 3-5 correspond to other functional forms of the cost functions for link e.

- For Case 3, the link e's cost is just proportional to the flow. From the nodal analysis, we note that if the Braess Paradox exists in a circuit for a set of values I, V_b, and V_e, one can choose another set of values, $V_b' = V_b + V_e$, $V_e' = 0$, and $I' = I - (V_e/R_e)$, without changing the RHS of the nodal equations. The Braess Paradox does occur in this modified circuit and is measured.

Interpretation of Zener Diode Results III

- For Case 4, link e is a fixed cost link. Because the fixed voltage drop is implemented as a diode, the voltage drop on a link will always depend weakly on the link current. This case only marginally illustrates the Braess Paradox.
- Case 5 corresponds to the case of a zero cost link e using a using a piece of wire for the link. This case may be analyzed as a circuit with a resistor in parallel with the series Zener diode-resistor combination. The measured V_1 in this case is less than either twice the Zener voltage $(10.2V)$ or the total current through the resistors, R_a $(12V)$, which may be interpreted by assuming non-ideal behavior of the reverse leakage current of a Zener diode.

Extension of the Braess Paradox Analysis to Other Forms of Cost Functions

Mathematically investigated in Leblanc (1975), Frank (1981), Bloy (2007).
The driving force for these investigations has been that realistic travel cost functions are based upon the Bureau of Public Roads (BPR) travel cost functions which model the cost on a link as

$$c_a(f_a) = t_a^0 \left(1 + k \left(\frac{f_a}{u_a}\right)^{\beta}\right),$$

where t_a^0, k, u_a, and β are positive constants. Often, $k = .15$, $\beta = 4$, and u_a is the practical capacity of link a.

While it is impossible to find a passive electrical component whose $I - V$ characteristics are identical in form to the BPR cost functions, the $I - V$ characteristics of a forward biased diode have an exponential shape.

Braess Paradox Circuit Using Diodes

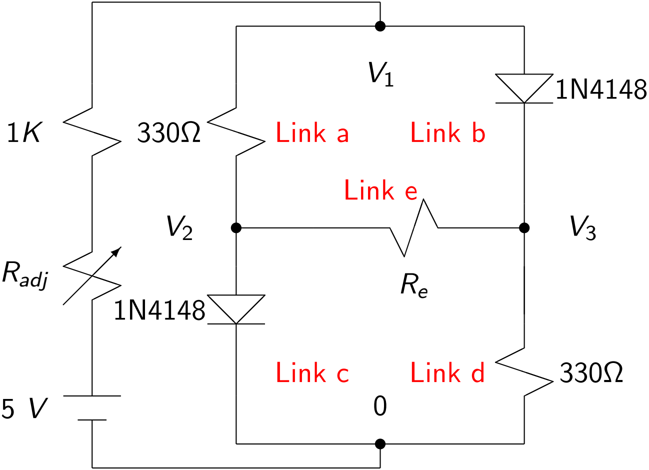
The first approximation to the Shockley model is the piecewise linear model, a voltage source in series with a resistor, identical in form to our earlier circuit.

The Shockley Diode model can be expanded as a power series in I producing higher order terms similar to those suggested as more complicated transportation cost functions.

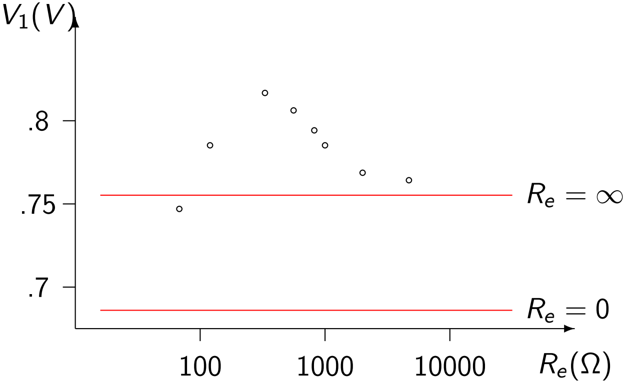
An electric circuit can be constructed with links b and c implemented by forward-biased silicon diodes. This topology was implied for transportation networks by Frank, Leblanc (1975).

Because it is not possible to write a direct matrix equation to analyze the circuit, the circuit is analyzed using SPICE to predict the occurrence of the Braess paradox.

Diode Resistor Circuit for Braess Paradox Measurement



Measured Node 1 Voltage for a Diode Resistor Circuit



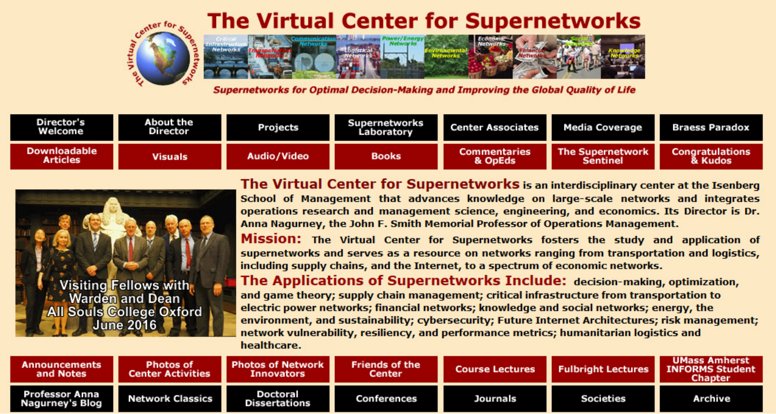
Summary

- We have explored the behavior of electrons flowing through an electrical circuit, which are governed by the same relationship that governs travelers driving in a road network and seeking their optimal routes of travel from origin nodes to destinations, acting independently.
- We proved that the Braess Paradox, originally proposed in user-optimized transportation networks, also can occur in electrical circuits, where the addition of a new link results in an increase in the voltage, rather than a decrease, as might be the expected.
- We provided examples in which cost functions are both linear as well as highly nonlinear and the same counterintuitive phenomenon is observed.

Summary

- From an electrical circuit perspective, the circuits constructed and described demonstrate the development of a circuit structure where the current and voltage at a node may be independently controlled.
- This result enables the development of alternative circuit structures that can be exploited in constructing more complex circuits, which can be embedded in macro, micro, and mesoscale electrical circuit systems.
- Because of these results, appropriately designed electrical circuits can be used as testbeds to further explore the properties and range of occurrence of the Braess Paradox in a variety of network systems, including transportation.

Thank you



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