TUTORIAL ON SUPERNETWORKS

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This is a tutorial on the topic of supernetworks. It is introductory and provides a brief overview of the subject. It is drawn from the book, **Supernetworks: Decision-Making for the Information Age**, by A. Nagurney and J. Dong, Edward Elgar Publishers, where additional topics and more complete coverage can be found.

1. Background

Network systems provide the infrastructure and foundation for the functioning of today's societies and economies. They come in many forms and include physical networks such as: transportation and logistical networks, communication networks, energy and power networks, as well as more *abstract* networks comprising: economic and financial networks, environmental networks, social, and knowledge networks.

For example, transportation networks give us the means to cross physical distance in order to conduct our daily activities. They provide us with access to both food as well as to consumer products and come in a myriad of forms: road, air, rail, or waterway. According to the U. S. Department of Transportation, the significance of transportation in dollar value alone as spent by US consumers, businesses, and governments was $950 billion in 1998.

Communication networks, in turn, allow us to communicate with friends and colleagues and to conduct the necessary transactions of life. They, through such innovations as the Internet, have transformed the manner in which we live, work, and conduct business today. Communication networks allow the transmission of voice, data/information, and/or video and can involve telephones, computers, as well as satellites, and microwaves. The trade publication *Purchasing* reports that corporate buyers alone spent \$517.6 billion on telecommunications goods and services in 1999.

Energy networks, in addition, are essential to the very existence of the *Network Economy* and help to fuel not only transportation networks but in many settings also communication networks. They provide electricity to run the computers and to light our businesses, oil and gas to heat our homes and to power vehicles, and water for our very survival. In 1995, according to the U. S. Department of Commerce, the energy expenditures in the United States were \$515.8 billion.

Financial networks (see Nagurney and Siokos (1997)) supply businesses with the resources to expand, to innovate, and to satisfy the needs of consumers. They allow individuals to invest and to save for the future for themselves and for their children and for governments to provide for their citizens and to develop and enhance communities.

The advent of the Information Age with the increasing availability of new computer and communication technologies, along with the Internet, have transformed the ways in which individuals work, travel, and conduct their daily activities, with profound implications for existing and future networks. Moreover, the decision-making process itself has been altered due to the addition of alternatives and options which were not, heretofore, possible or even feasible. The boundaries for decision-making have been redrawn as individuals can now work from home or purchase products from work. Managers can now locate raw materials and other inputs from suppliers through information networks in order to maximize profits while simultaneously ensuring timely delivery of finished goods. Financing for their businesses can be obtained online. Individuals, in turn, can obtain information about products from their homes and make their purchasing decisions accordingly.

The reality of today's networks include: large-scale nature and complexity, increasing congestion, alternative behaviors of users of the networks, as well as interactions between the networks themselves, notably, between transportation and telecommunication networks. The decisions made by the users of the networks, in turn, affect not only the users themselves but others, as well, in terms of profits and costs, timeliness of deliveries, the quality of the environment, etc.

In this tutorial, the foundations of the theory of *supernetworks* are laid down in order to formalize decision-making in the Information Age. “Super” networks are networks that are “above and beyond” existing networks, which consist of nodes, links, and flows, with nodes corresponding to locations in space, links to connections in the form of roads, cables, etc., and flows to vehicles, data, etc. Supernetworks are conceptual in scope, graphical in perspective, and, with the accompanying theory, predictive in nature.

In particular, the supernetwork framework, captures, in a unified fashion, decision-making facing a variety of economic agents including consumers and producers as well as distinct intermediaries in the context of today’s networked economy. The decision-making process may entail weighting trade-offs associated with the use of transportation versus telecommunication networks. The behavior of the individual agents is modeled as well as their interactions on the complex network systems with the goal of identifying the resulting equilibrium flows and prices.

For definiteness, Table 1 presents some basic *classical* networks and the associated nodes, links, and flows. By *classical* network is meant a network in which the nodes correspond to physical locations in space and the links to physical connections between the nodes.

The topic of networks and the management thereof dates to ancient times with examples including the publicly provided Roman road network and the “time of day” chariot policy, whereby chariots were banned from the ancient city of Rome at particular times of day. The formal study of networks, consisting of nodes, links, and flows, in turn, involves: how to model such applications (as well as numerous other ones) as mathematical entities, how to study the models qualitatively, and how to design algorithms to solve the resulting models effectively. The study of networks is necessarily interdisciplinary in nature due to their breadth of appearance and is based on scientific techniques from applied mathematics, computer science, and engineering with applications as varied as finance and even biology. Network models and tools which are widely used by businesses, industries, as well as governments today (cf. Ahuja, Magnanti, and Orlin (1993), Nagurney and Siokos (1997), Nagurney (1999, 2000a), and the references therein).

Table 1: Examples of Classical Networks

Network System	Nodes	Links	Flows
Transportation			
Urban	Intersections, Homes, Places of Work	Roads	Autos
Air	Airports	Airline Routes	Planes
Rail	Railyards	Railroad Track	Trains
Manufacturing and Logistics	Distribution Points, Processing Points	Routes Assembly Line	Parts, Products
Communication	Computers	Cables	Messages
	Satellites	Radio	Messages
	Phone Exchanges	Cables, Microwaves	Voice, Video
Energy	Pumping Stations	Pipelines	Water
	Plants	Pipelines	Gas, Oil

Basic examples of network problems are: *the shortest path problem*, in which one seeks to determine the most efficient path from an origin node to a destination node; *the maximum flow problem*, in which one wishes to determine the maximum flow that one can send from an origin node to a destination node, given that there are capacities on the links that cannot be exceeded, and *the minimum cost flow problem*, where there are both costs and capacities associated with the links and one must satisfy the demands at the destination nodes, given supplies at the origin nodes, at minimal total cost associated with shipping the flows, and subject to not exceeding the arc capacities. Applications of the shortest path problem are found in transportation and telecommunications, whereas the maximum flow problem arises in machine scheduling and network reliability settings, with applications of the minimum cost flow problem ranging from warehousing and distribution to vehicle fleet planning and scheduling.

Networks also appear in surprising and fascinating ways for problems, which initially may not appear to involve networks at all, such as a variety of financial problems and in knowledge production and dissemination. Hence, the study of networks is not limited to only

Table 2: Examples of Supernetworks

Financial Networks with Intermediation
Supply Chain Networks with Electronic Commerce
Teleshopping/Shopping Networks
Telecommuting/Commuting Networks
Combined Transportation and Location Networks

physical networks where nodes coincide with locations in space but applies also to abstract networks. The ability to harness the power of a network formalism provides a competitive advantage since:

• many present-day problems are concerned with flows be they, material, human, capital, or informational over space and time and, hence, ideally suited as an application domain for network theory;

• one may avail oneself of a graphical or visual depiction of different problems;

• one may identify similarities and differences in distinct problems through their underlying network structure, and

• one may apply efficient network algorithms for problem solution.

Supernetworks may be comprised of such networks as transportation, telecommunication, logistical and financial networks, among others. They may be *multilevel* as when they formalize the study of supply chain networks or *multitiered* as in the case of financial networks with intermediation. Furthermore, decision-makers on supernetworks may be faced with multiple criteria and, hence, the study of supernetworks also includes the study of multicriteria decision-making. In Table 2, some examples of supernetworks are given.

In particular, the supernetwork framework allows one to formalize the alternatives available to decision-makers, to model their individual behavior, typically, characterized by particular criteria which they wish to optimize, and to, ultimately, compute the flows on the supernetwork, which may consist of product shipments, travelers between origins and destinations, financial flows, as well as the associated "prices." Hence, the concern is with human decision-making and how the supernetwork concept can be utilized to crystallize and inform in this dimension.

The origins of the theme of this web-site, as well as this tutorial, come from the invited essay of Nagurney (2000a) in *OR/MS Today*. In that essay, Nagurney set out to capture the interrelationships among the foundational networks in our economies and societies today. The essay, in turn, was based on Nagurney's Distinguished Faculty Lecture given at the University of Massachusetts on April 5, 2000 (see Nagurney (2000b)). Subsequent reseach yielded the first book on supernetworks, co-authored by Nagurney and Dong, from which this tutorial is drawn.

Below the theme of supernetworks is further elaborated upon and, in particular, the origins of the concept and the term *supernetworks* identified.

There are many books on networks, both methodological as well as historical. Here our interests focus on nonlinear, multidimensional, in the form of multiple tiers or multiple levels or multiple criteria, abstract networks in the form of supernetworks.

2. The Origins of Supernetworks

In this part of the tutorial a discussion of the three foundational classes of networks: transportation, telecommunication, and economic and financial networks is given. Such networks have served not only as the basis for the origins of the term "supernetwork," but, also, they arise as critical subnetworks in the applications that are relevant to decision-making in the Information Age today. In addition, the use of the term "supernetworks" in genetics and biology is given. The focus of supernetworks here, however, is on *human* decision-making in the Information Age. As mentioned earlier, the term "supernetwork" refers to networks that are above and beyond existing networks, and the foundations of supernetworks are network systems that have not only been historically crucial to the development of economies and societies but that have also been subjected to rigorous analysis due to their practical importance.

2.1 Transportation Networks

Transportation networks are complex network systems in which the decisions of the individual travelers affect the efficiency and productivity of the entire network system. Transportation networks, as noted in Table 1, come in many forms: notably, urban networks, freight networks, and airline networks. The "supply" in such a network system is represented by the network topology and the underlying cost characteristics, whereas the "demand" is represented by the users of the network system, that is, the travelers. The study of transportation networks and their efficient management dates to ancient times. Indeed, Romans imposed controls over chariot traffic during different times of the day in order to deal with the congestion (cf. Banister and Button (1993)).

In 1972, Dafermos demonstrated in a paper, through a formal model, how a *multiclass* traffic network could be cast into a single-class traffic network through the construction of an expanded (and *abstract*) network consisting of as many copies of the original network as there were classes. She clearly identified the origin/destination pairs, demands, link costs, and flows on the abstract network. The applications of such networks she stated, "arise not only in street networks where vehicles of different types share the same roads (e.g., trucks and passenger cars) but also in other types of transportation networks (e. g., telephone networks)." Hence, she not only recognized that abstract networks could be used to handle multimodal transportation networks but also telecommunication networks! Moreover, she considered both user-optimizing and system-optimizing behavior, terms which she had coined with Sparrow in a paper in 1969.

In 1976, Dafermos proposed an integrated traffic network equilibrium model in which one could visualize and formalize the entire transportation planning process (consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal fashion) as path choices over an appropriately constructed *abstract* network. The genesis and formal treatment of decisions more complex than route choices as *path* choices on abstract networks, that is, supernetworks, were, hence, reported as early as 1972 and 1976.

The importance and wider relevance of such abstract networks in decision-making, with a focus on transportation planning were accentuated through the term "hypernetwork" used by Sheffi (1978), and Sheffi and Daganzo (1978, 1980), which was later retermed as "supernetwork" by Sheffi (1985).

For example, Sheffi and Daganzo (1978) described a framework for discussing many transportation supply-demand equilibrium problems, where "the sequence of choices that the individual faces when he or she is about to make a travel (or not-to-travel) decision as a case choice of a route on an abstract network (hypernetwork)." They recognized the contribution of Dafermos (1976) and also considered probabilistic choice models. Thus, they explicitly considered that decision-making in a transportation context could be modeled as a "route" selection over an abstract network. The route, henceforth, referred to as a "path" to emphasize the generality of the concept, would, thus, correspond to a choice and the links to parts and pieces of the complete decision.

The recognition and appropriate construction of *abstract* networks was pivotal in that it allowed for the incorporation of transportation-related decisions (where as noted by Dafermos (1972), transportation applied also to communication networks) which were not based solely on route selection in a classical sense, that is, what route should one take from one's origin, say, place of residence, to one's destination, say, place of employment. Hence, abstract networks, with origins and destinations corresponding to appropriately defined nodes, links connecting nodes having associated disutilities (that is, costs), and paths comprised of links (directed) connecting the origins and destinations, could capture such travel alternatives as not simply just a route but, also, the "mode" of travel, that is, for example, whether one chose to use private or public transportation. Furthermore, with the addition of not only added abstract links and paths, but abstract origin and destination nodes as well one could include the selection of such locational decisions as the origins and destinations themselves within the same decision-making framework.

For example, in order to fix ideas, in Figure 1, a supernetwork topology for an example of a simple mode/route choice problem is presented. In this example, it is recognized, at the outset, that the routes underlying the different modes may be distinct and, hence, rather than making copies of the network according to Dafermos (1972), the supernetwork construction is done with the path choices directly on the supernetwork itself.

In the network in Figure 1, travelers seek to determine their "best" paths from the origin node 1 to the destination node 4, where a path consists of both the selection of the mode

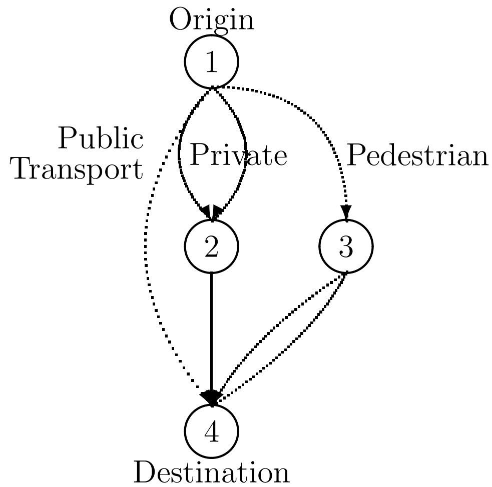

Figure 1: Example Mode and Route Choice Supernetwork Topology

of travel as well as the route of travel. Link 1 corresponds to the use of public transit, and there is only one route choice using this mode of travel. On the other hand, if one selects private transportation (typically, the automobile), one could take either of two routes: with the first route consisting of the first link joining nodes 1 and 2 and then the link joining node 2 to node 4, and the second route consisting of the second link joining nodes 1 and 2 and then onto node 4. Finally, one could choose either of two pedestrian routes to travel from node 1 to node 4, with the pedestrian routes differing by their second component links.

Another simple example is now provided, which illustrates simultaneous route and destination choice in a supernetwork framework, as conceived by Dafermos (1976), whose abstract network framework also captured other transportation/location decisions (cf. Figure 2). Assume that there is a single origin, which corresponds to the place of residence and is denoted by node 1 in Figure 2. Assume that there are three places of (potential) employment, denoted, respectively, by nodes 2, 3, and 4. There are two available routes of travel from the origin to each employment node. Dafermos proposed, in this case, to construct a single abstract node, denoted by node 5 in Figure 2, which serves as the abstract destination, and to connect each of the nodes 2 through 4 with node 5. Hence, the paths connecting node 1

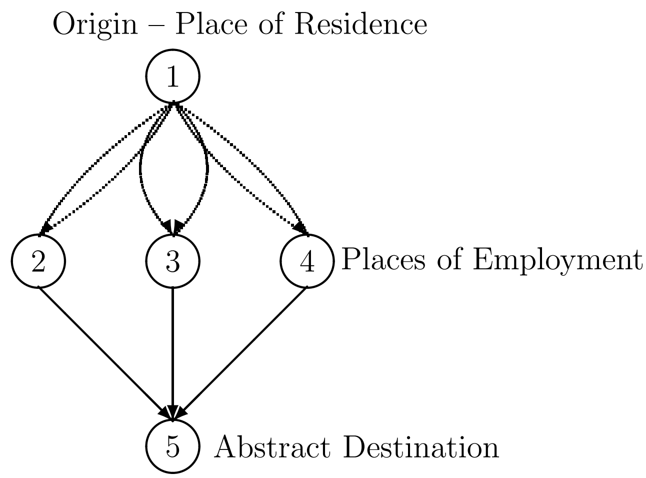

Figure 2: Example Route and Destination Choice Supernetwork Topology

with node 5 represent both route and destination choices.

The behavioral principle utilized by Dafermos (1976) and by Sheffi and Daganzo (1978, 1980) (who also formulated stochastic models) was that decision-makers select the "cost-minimizing" routes among all their available choices. Dafermos (1972) considered both system-optimization as well as user-optimization. According to Sheffi and Daganzo (1978), the selection of cost-minimizing routes "this is consistent with the principle of utility maximization of choice theory." Moreover, they stated that: "Although hypernetworks enable us to visualize choice problems in a unified way... their main advantage is that they enable us to perform supply-demand equilibrium analysis on a mathematically consistent basis with disaggregate demand models." We further elaborate on the behavioral principle, known as user-optimization, later in this tutorial. Additional references to supernetworks and transportation can be found in the book by Nagurney and Dong (2001).

2.2 Telecommunication Networks

We now turn to a discussion of the use of the term "supernetworks" in the context of telecommunication networks. In the *American Scientist*, Denning (1985) continued his discussion of the internal structure of computer networks, which had appeared in a volume of the same journal earlier that year, and emphasized how "protocol software can be built as a series of layers. Most of this structure is hidden from the users of the network." He then proceeded to ask the question, "What should the users see?" Denning, subsequently, in the article, answered the question in the context of the then National Science Foundation's Advanced Scientific Computing Initiative to make national supercomputer centers accessible to the entire scientific community. He said that such a system would be a network of networks, that is, a "supernetwork," and a powerful tool for science. Interestingly, he emphasized the importance of location-independent naming, so that if a physical location of a resource would change, none of the supporting programs or files would need to be edited or recompiled. Hence, in a sense, his view of supernetworks is in concert with that of ours in that nodes do not need to correspond to locations in space and may have an abstract association.

Earlier, Schubert, Goebel, and Cercone (1979) had used the term in the context of knowledge representation as follows: "In the network approach to knowledge representation, concepts are represented as nodes in a network. Networks are compositional: a node in a network can be some other network, and the same subnetwork can be a subnetwork of several larger supernetworks,..."

In 1997, the Illinois Bar Association considered the following to be an accepted definition of the Internet: "the Internet is a supernetwork of computers that links together individual computers and computer networks located at academic, commercial, government and military sites worldwide, generally by ordinary local telephone lines and long-distance transmission facilities. Communications between computers or individual networks on the Internet are achieved through the use of standard, nonproprietary protocols." The reference to the Internet as a supernetwork was also made by Fallows (1996) who stated in *The Atlantic Monthly* that "The Internet is the supernetwork that links computer networks around the world."

Mr. Vinton G. Cerf, the co-developer of the computer networking protocal, TCP/IP, used for the Internet, in his keynote address to the Internet/Telecom 95 Conference (see Telecom95 (1995)), noted that at that time there were an estimated 23 million users of the

Internet, and that vast quantities of the US Internet traffic "pass through internet MCI's backbone." He then went on to say in the same source that "Just a few months back, MCI rolled out a supernetwork for the National Science Foundation known as the very broadband network service or VNBS...VBNS is being used as an experimental platform for developing new national networking applications."

Decision-making on transportation and telecommunication networks can be done simultaneously through the supernetwork concept. For example, as demonstrated in the book **Supernetworks: Decision-Making for the Information Age** (cf. Nagurney and Dong (2001)), supply chain networks with electronic commerce, financial networks with intermediation, teleshopping versus shopping, telecommuting versus commuting, as well as transportation and location decisions in the Information Age can also be formulated and solved within the supernetwork theoretical umbrella. Later in this tutorial we present some illustrative examples.

2.3 Economic and Financial Networks

The concept of a network in economics was implicit as early as in the classical work of Cournot (1838), who not only seems to have first explicitly stated that a competitive price is determined by the intersection of supply and demand curves, but had done so in the context of two spatially separated markets in which the cost of transporting the good between markets was considered. Pigou (1920) also studied a network system in the setting of a transportation network consisting of two routes and noted that the "system-optimized" solution was distinct from the "user-optimized" solution.

Nevertheless, the first instance of an abstract network or supernetwork in the context of economic applications, was actually due to Quesnay (1758), who visualized the circular flow of funds in an economy as a network. Since that very early contribution there have been numerous economic and financial models that have been constructed over abstract networks. In particular, we note the work of Dafermos and Nagurney (1985) who identified the isomorphism between traffic network equilibrium problems and spatial price equilibrium problems, whose development had been originated by Samuelson (1952) (who, interestingly, focused on the bipartite network structure of the spatial price equilibrium problem) and

Takayama and Judge (1971).

Zhao (1989) (see also Zhao and Dafermos (1991) and Zhao and Nagurney (1993)) identified the general economic equilibrium problem known as Walrasian price equilibrium as a network equilibrium problem over an abstract network with very simple structure. The structure consisted of a single origin/destination pair of nodes and single links joining the two nodes. This structure was then exploited for computational purposes. Nagurney (1989), in turn, proposed a migration equilibrium problem over an abstract network with an identical structure. A variety of abstract networks in economics were studied in the book by Nagurney (1999), which also contains extensive references to the subject. In the book on supernetworks, we have also demonstrated that the abstract network concept also captures the interactions between/among the underlying networks of economies and societies. As noted by Nagurney (2000a): "The interactions among transportation networks, telecommunication networks, as well as financial networks is creating supernetworks ..." .

2.4 Supernetworks in Genetics

Interestingly, the term *supernetworks* has also been applied in biology, notably, in genetics. According to Noveen, Hartenstein, and Chuong (1998), many interacting genes give rise to a gene network, with many interacting gene networks giving rise to a gene "supernetwork." They go on to further state: "The function of a gene supernetwork is more complicated than a gene network. A gene supernetwork, for example, may be involved in determining the development of an entire limb while a gene network, working within the supernetwork, may be involved in setting up one of the axes of the limb bud." According to the same source, a gene supernetwork is defined as "a collection of gene networks which participate with each other during the morphogenesis of a specific structure, for example an organ, a segment, or an appendage." The authors then go on to discuss duplication, divergence, and conservation of a gene supernetwork and note that, as with gene networks, gene supernetworks can be duplicated during evolution, "thus giving rise to new structures which are the same as or different from the original structure."

3. Characteristics of Supernetworks

Supernetworks are a conceptual and analytical formalism for the study of a variety of decision-making problems on networks. Hence, their characteristics include characteristics of the foundational networks. The characteristics of today's networks include: large-scale nature and complexity of network topology; congestion; alternative behavior of users of the network, which may lead to paradoxical phenomena, and the interactions among networks themselves such as in transportation versus telecommunications networks. Moreover, policies surrounding networks today may have a major impact not only economically but also socially.

Large-Scale Nature and Complexity

Many of today's networks are characterized by both a large-scale nature and complexity of the underlying network topology. For example, in Chicago's Regional Transportation Network, there are 12,982 nodes, 39,018 links, and 2,297,945 origin/destination (O/D) pairs (see Bar-Gera (1999)), whereas in the Southern California Association of Governments model there are 3,217 origins and/or destinations, 25,428 nodes, and 99,240 links, plus 6 distinct classes of users (cf. Wu, Florian, and He (2000)).

In terms of the size of existing telecommunications networks, AT&T's domestic network has 100,000 origin/destination pairs (cf. Resende (2000)), whereas in their detail graph applications in which nodes are phone numbers and edges are calls, there are 300 million nodes and 4 billion edges (cf. Abello, Pardalos, and Resende (1999)).

Congestion

Congestion is playing an increasing role in not only transportation networks but also in telecommunication networks. For example, in the case of transportation networks in the United States alone, congestion results in $100 billion in lost productivity, whereas the figure in Europe is estimated to be $150 billion. The number of cars is expected to increase by 50% by 2010 and to double by 2030 (see Nagurney (2000c)).

In terms of the Internet, with 275 million present users, the Federal Communications Commission reports that the volume of traffic is doubling every 100 days, which is remarkable given that telephone traffic has typically increased only by about 5 percent a year (cf.

Labaton (2000)). As individuals increasingly access the Internet through wireless communication such as through handheld computers and cellular phones, experts fear that the heavy use of airwaves will create additional bottlenecks and congestion that could impede the further development of the technology.

System-Optimization versus User-Optimization

In many of today's networks, not only is congestion a characteristic feature leading to nonlinearities, but the behavior of the users of the networks themselves may be that of noncooperation. For example, in the case of urban transportation networks, travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time, which although "optimal" from an individual's perspective (user-optimization) may not be optimal from a societal one (system-optimization) where one has control over the flows on the network and, in contrast, seeks to minimize the total cost in the network and, hence, the total loss of productivity. Consequently, in making any kind of policy decisions in such networks one must take into consideration the users of the particular network. Indeed, this point is vividly illustrated through a famous example known as the Braess paradox, in which it is assumed that the underlying behavioral principle is that of user-optimization. In the Braess (1968) network, the addition of a new road with no change in the travel demand results in all travelers in the network incurring a higher travel cost and, hence, being worse off.

The increase in travel cost on the paths is due, in part, to the fact that in this network two links are shared by distinct paths and these links incur an increase in flow and associated cost. Hence, Braess's paradox is related to the underlying topology of the networks. One may show, however, that the addition of a path connecting an O/D pair that shares no links with the original O/D pair will never result in Braess's paradox for that O/D pair.

Interestingly, as reported in the *New York Times* by Kolata (1990), this phenomenon has been observed in practice both in the case of New York City when in 1990, 42nd Street was closed for Earth Day and the traffic flow actually improved. Just to show that it is not a purely New York or US phenomena concerning drivers and their behavior an analogous situation was observed in Stuttgart where a new road was added to the downtown but the traffic flow worsened and following complaints, the new road was torn down (see Bass (1992)).

This phenomenon is also relevant to telecommunications networks (see Korilis, Lazar, and Orda (1999)) and, in particular, to the Internet which is another example of a "non-cooperative network" and, therefore, network tools have wide application in this setting as well especially in terms of congestion management and network design (see also Cohen and Kelly (1990)).

Network Interactions

Clearly, one of the principal facets of the Network Economy is the interaction among the networks themselves. For example, the increasing use of electronic commerce especially in business to business transactions is changing not only the utilization and structure of the underlying logistical networks but is also revolutionizing how business itself is transacted and the structure of firms and industries. Cellular phones are being using as vehicles move dynamically over transportation networks resulting in dynamic evolutions of the topologies themselves. The unifying concept of supernetworks with associated methodologies allows one to explore the interactions among such networks as transportation networks, telecommunication networks, as well as financial networks.

4. Decision-Making Concepts

As the above discussion has revealed, networks in the Information Age are complex, typically large-scale systems and the study of their efficient operation, often through some outside intervention, has attracted much interest from economists, computer scientists, engineers, as well as transportation and urban planners and operations researchers.

In particular, the underlying behavior of the users of the network system is essential in studying their operation. Importantly, Wardrop (1952) explicitly recognized alternative possible behaviors of users of transportation networks and stated two principles, which are commonly named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel whereas the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire, and Winsten (1956) were the first to rigorously formulate these conditions mathematically, as had Samuelson (1952) in the framework of spatial price equilibrium problems in which there were, however, no congestion effects. Specifically, Beckmann, McGuire, and Winsten (1956) established the equivalence between the *traffic network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker conditions of an appropriately constructed optimization problem (cf. Bazaraa, Sherali, and Shetty (1993)), under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their approach made the formulation, analysis, and subsequent computation of solutions to traffic network problems based on actual transportation networks realizable.

Dafermos and Sparrow (1969) coined the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system is minimized. In the latter problem, marginal costs rather than average costs are equilibrated. The former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle.

See Table 3 for the two distinct behavioral principles underlying transportation networks. The concept of "system-optimization" is also relevant to other types of "routing models" in transportation, as well as in communications (cf. Bertsekas and Gallager (1992)), including those concerned with the routing of freight and computer messages, respectively. Dafermos and Sparrow (1969) also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing (in order to handle congestion) function of the flow on the particular

Table 3: Distinct Behavior on Transportation Networks

User-Optimization	**System-Optimization**
$\Downarrow$	$\Downarrow$
Equilibrium Principle:	**Optimality Principle:**
User travel costs on used paths for each O/D pair are equalized and minimal.	Marginals of the total travel cost on used paths for each O/D pair are equalized and minimal.

link and linear.

4.1 System-Optimization Versus User-Optimization

The basic network models are now reviewed, under distinct assumptions of their operation and distinct behavior of the users of the network. The models are classical and were developed in the context of transportation. They are due to Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969). We later present more general models.

For definiteness, and for easy reference, we present the classical system-optimized network model and then the classical user-optimized network model. Although these models were first developed for transportation networks, here they are presented in the broader setting of network systems, since they are as relevant in other application settings, in particular, in telecommunication networks and, more generally, in supernetworks.

More general models are then outlined, in which the user link cost functions are no longer separable and are also asymmetric. We provide the variational inequality formulations of the governing equilibrium conditions (see Kinderlehrer and Stampacchia (1980) and Nagurney (1993)), since, in this case, the conditions can no longer be reformulated as the Kuhn-Tucker conditions of a convex optimization problem. Finally, we present the variational inequality formulations in the case of elastic demands.

4.1.1 The System-Optimized Problem

Consider a general network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes, and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. In transportation networks (see also Table 1), nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting, thus, is a sequence of "roads" which comprise a route from an origin to a destination. In the telecommunication context, however, nodes can correspond to switches or to computers and links to telephone lines, cables, microwave links, etc. In the supernetwork setting, a path is viewed more broadly and need not be limited to a route-type decision.

Let P_ω denote the set of paths connecting the origin/destination (O/D) pair of nodes ω. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the flow on path p and let f_a denote the flow on link a. The path flows on the network are grouped into the column vector $x \in R_+^{n_P}$, where n_P denotes the number of paths in the network. The link flows, in turn, are grouped into the column vector $f \in R_+^n$, where n denotes the number of links in the network.

The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (1) states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if one lets d_ω denote the demand associated with O/D pair ω, then one must have that

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{2}$$

where $x_p \geq 0$, $\forall p \in P$; that is, the sum of all the path flows between an origin/destination pair ω must be equal to the given demand d_ω.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p.

Assume that the user link cost function is given by the *separable* function

$$c_a = c_a(f_a), \quad \forall a \in \mathcal{L}, \tag{3}$$

where c_a is assumed to be an increasing function of the link flow f_a in order to model the effect of the link flow on the cost.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, hence, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L}, \tag{4}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. Here the cost is interpreted in a general sense. From a transportation engineering perspective, however, the cost on a link is assumed to coincide with the travel time on a link. Later in this tutorial, we consider generalized cost functions of the links which are constructed using weights and different criteria.

In the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a), \tag{5}$$

where the total cost on a link is given by expression (4).

The system-optimization problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a) \tag{6}$$

subject to:

$$\sum_{p \in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{7}$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in \mathcal{L}, \tag{8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{9}$$

The constraints (7) and (8), along with (9), are commonly referred to in network terminology as *conservation of flow equations*. In particular, they guarantee that the flow in the network, that is, the users (whether these are travelers or computer messages, for example) do not "get lost."

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P, \tag{10}$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}. \tag{11}$$

In view of (8), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x)x_p \tag{12}$$

subject to constraints (7) and (9).

System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function in the S-O problem is convex, and the feasible set consisting of the linear constraints is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: For each O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, the flow pattern x (and link flow pattern f), satisfying (7)–(9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_\omega, & \text{if} \quad x_p > 0 \\ \geq \mu_\omega, & \text{if} \quad x_p = 0, \end{cases} \tag{13}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a}\delta_{ap}, \tag{14}$$

and in (13) it is evaluated at the solution.

Note that in the S-O problem, according to the optimality conditions (13), it is the marginal of the total cost on each used path connecting an O/D pair which is equalized and minimal. Indeed, conditions (13) state that a system-optimized flow pattern is such that for each origin/destination pair the incurred marginals of the total cost on all used path are equal and minimal (see also Table 3).

4.1.2 The User-Optimized Problem

We now describe the user-optimized network problem, also commonly referred to in the transportation literature as the *traffic assignment* problem or the *traffic network equilibrium* problem. Again, as in the system-optimized problem of Section 4.1.1, the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, as well as the user link cost functions are assumed as given. Recall that user-optimization follows Wardrop's first principle.

Network Equilibrium Conditions

Now, however, one seeks to determine the path flow pattern x^* (and link flow pattern f^*) which satisfies the conservation of flow equations (7), (8), and the nonnegativity assumption on the path flows (9), and which also satisfies the network equilibrium conditions given by the following statement.

For each O/D pair $\omega \in \Omega$ and each path $p \in P_\omega$:

$$C_p \begin{cases} = \lambda_\omega, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega, & \text{if} \quad x_p^* = 0. \end{cases} \tag{15}$$

Hence, in the user-optimization problem there is no explicit optimization concept, since now users of the network system act independently, in a noncooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, conditions (15) are simply a restatement of Wardrop's (1952) first principle mathematically and mean that only those paths connecting an O/D pair will be used which have equal and minimal user costs. Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost. User-optimization represents decentralized decision-making, whereas system-optimization represents centralized decision-making. See also Table 3.

In order to obtain a solution to the above problem, Beckmann, McGuire, and Winsten (1956) established that the solution to the equilibrium problem, in the case of user link cost functions (cf. (3)) in which the cost on a link only depends on the flow on that link could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}} \int_0^{f_a} c_a(y)dy \tag{16}$$

subject to:

$$\sum_{p\in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{17}$$

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{18}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{19}$$

Note that the conservation of flow equations are identical in both the user-optimized network problem (see (17)–(19)) and the system-optimized problem (see (7) – (9)). The behavior of the individual decision-makers termed "users", however, is different. Users of the network system, which generate the flow on the network now act independently, and are not controlled by a centralized controller.

The objective function given by (16) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It does not possess the economic meaning of the objective function encountered in the system-optimization problem given by (6), equivalently, by (12).

4.2 Models with Asymmetric Link Costs

There has been much dynamic research activity in the past several decades in both the modeling and the development of methodologies to enable the formulation and computation of more general network equilibrium models, with a focus on traffic networks. Examples of general models include those that allow for multiple modes of transportation or multiple classes of users, who perceive cost on a link in an individual way. We now consider network models in which the user cost on a link is no longer dependent solely on the flow on that link. Other network models, including dynamic traffic models, can be found in Mahmassani et al. (1993), and in the books by Ran and Boyce (1996) and Nagurney and Zhang (1996), and the references therein.

We now consider user link cost functions which are of a general form, that is, in which the cost on a link may depend not only on the flow on the link but on other link flows on the network, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}. \tag{20}$$

In the case where the symmetry assumption exists, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (15) as the solution to an optimization problem (cf. Dafermos (1972) and the references therein), albeit, again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory*.

Indeed, it was in the problem domain of traffic network equilibrium problems that the theory of finite-dimensional variational inequalities realized its earliest success, beginning with the contributions of Smith (1979) and Dafermos (1980). For an introduction to the subject, as well as applications ranging from traffic network equilibrium problems to financial equilibrium problems, see the book by Nagurney (1999). The methodology of finite-dimensional variational inequalities has been utilized in order to develop a spectrum of supernetwork models (see Nagurney and Dong (2001)).

The system-optimization problem, in turn, in the case of nonseparable (cf. (20)) user link cost functions becomes (see also (6)–(9)):

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f), \tag{21}$$

subject to (7)–(9), where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as in (13), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b \in \mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P. \tag{22}$$

Variational Inequality Formulations of Fixed Demand Problems

As mentioned earlier, in the case where the user link cost functions are no longer symmetric, one cannot compute the solution to the U-O, that is, to the network equilibrium, problem using standard optimization algorithms. Such cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

Since equilibrium is such a fundamental concept in terms of supernetworks and since variational inequality theory is one of the basic ways in which to study such problems we now, for completeness, also give variational inequality formulations of the network equilibrium conditions (15). These formulations are presented without proof (for derivations, see Smith (1979) and Dafermos (1980), as well as Florian and Hearn (1995) and the book by Nagurney (1999)).

First, the definition of a variational inequality problem is recalled. We then give both the variational inequality formulation in path flows as well as in link flows of the network equilibrium conditions. Subsequently, in this tutorial, we extend these concepts to multicriteria, multiclass network equilibrium problems.

Specifically, the variational inequality problem (finite-dimensional) is defined as follows:

Definition 1: Variational Inequality Problem

The finite-dimensional variational inequality problem, VI$(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K}$ *such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{23}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^N*.*

Variational inequality (23) is referred to as being in *standard form*. Hence, for a given problem, typically an *equilibrium* problem, one must determine the function F that enters the variational inequality problem, the vector of variables X, as well as the feasible set $\mathcal{K}$.

The variational inequality problem contains, as special cases, such well-known problems as systems of equations, optimization problems, and complementarity problems. Thus, it is a powerful unifying methodology for equilibrium analysis and computation.

Theorem 1: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version

A vector $x^* \in K^1$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (15) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{24}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K^1, \tag{25}$$

where C *is the* n_P*-dimensional column vector of path user costs and* K^1 *is defined as:* $K^1 \equiv \{x \geq 0,$ *such that* (17) *holds*$\}$.

Theorem 2: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A vector $f^* \in K^2$ *is a network equilibrium link flow pattern if and only if it satisfies the* *variational inequality problem:*

$$\sum_{a\in\mathcal{L}} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2, \tag{26}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K^2, \tag{27}$$

where c is the n-dimensional column vector of link user costs and K^2 is defined as: $K^2 \equiv \{f \,|\, \text{there exists an}\, x \geq 0 \text{ and satisfying} (17) \text{ and} (18)\}$.

Note that one may put variational inequality (25) in standard form (23) by letting $F \equiv C$, $X \equiv x$, and $\mathcal{K} \equiv K^1$. Also, one may put variational inequality (27) in standard form where now $F \equiv c$, $X \equiv f$, and $\mathcal{K} \equiv K^2$.

Alternative variational inequality formulations of a problem are useful in devising other models, including dynamic versions, as well as for purposes of computation using different algorithms.

Variational Inequality Formulations of Elastic Demand Problems

The general network equilibrium model with elastic demands due to Dafermos (1982) is now recalled. Specifically, it is assumed that now one has associated with each O/D pair ω in the network a disutility λ_ω, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which are no longer fixed, but are now variables, that is,

$$\lambda_\omega = \lambda_\omega(d), \quad \forall \omega \in \Omega, \tag{28}$$

where d is the J-dimensional column vector of the demands.

The notation, otherwise, is as described earlier, except that here we also consider user link cost functions which are general, that is, of the form (20). The conservation of flow equations (see also (1) and (2)), in turn, are given by

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{29}$$

$$d_\omega = \sum_{p\in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{30}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{31}$$

Hence, in the elastic demand case, the demands in expression (30) are now variables and no longer given, as was the case for the fixed demand expression in (2).

Network Equilibrium Conditions in the Case of Elastic Demand

The network equilibrium conditions (see also (15)) now take on in the elastic demand case the following form: For every O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, a vector of path flows and demands (x^*, d^*) satisfying (30)–(31) (which induces a link flow pattern f^* through (29)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_\omega(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{32}$$

Equilibrium conditions (32) state that the costs on used paths for each O/D pair are equal and minimal and equal to the disutility associated with that O/D pair. Costs on unutilized paths can exceed the disutility.

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (32) are presented. These are analogues of the formulations (24) and (25), and (26) and (27), respectively, for the fixed demand model.

Theorem 3: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version

A vector $(x^*, d^*) \in K^3$ is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (32) if and only if it satisfies the variational inequality problem:

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d_\omega^*) \geq 0, \quad \forall (x, d) \in K^3, \tag{33}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \quad \forall (x, d) \in K^3, \tag{34}$$

where λ *is the* J*-dimensional vector of disutilities and* K^3 *is defined as:* $K^3 \equiv \{x \geq 0,$ *such that* (30) *holds*$\}$.

Theorem 4: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A vector $(f^*, d^*) \in K^4$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f_a^*) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d_\omega^*) \geq 0, \quad \forall (f, d) \in K^4, \tag{35}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \quad \forall (f, d) \in K^4, \tag{36}$$

where $K^4 \equiv \{(f, d),$ *such that there exists an* $x \geq 0$ *satisfying* (29), (31)$\}$

Note that, under the symmetry assumption on the disutility functions, that is, if $\frac{\partial \lambda_w}{\partial d_\omega} = \frac{\partial \lambda_\omega}{\partial d_w}$, for all w, ω, in addition to such an assumption on the user link cost functions (see following (20)), one can obtain (see Beckmann, McGuire, and Winsten (1956)) an optimization reformulation of the network equilibrium conditions (32), which in the case of separable user link cost functions and disutility functions is given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \int_0^{f_a} c_a(y) dy - \sum_{\omega \in \Omega} \int_0^{d_\omega} \lambda_\omega(z) dz \tag{37}$$

subject to: (29)–(31).

An example of a simple elastic demand network equilibrium problem is now given.

Example 1

Consider the network depicted in Figure 3 in which there are three nodes: 1, 2, 3; three links: a, b, c; and a single O/D pair $\omega_1 = (1, 3)$. Let path $p_1 = (a, c)$ and path $p_2 = (b, c)$.

Assume that the user link cost functions are:

$$c_a(f) = 5f_a + 13, \quad c_b(f) = 7f_b + f_a + 5, \quad c_c(f) = 3f_c + f_a + f_b + 12,$$

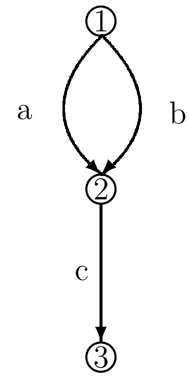

Figure 3: An Elastic Demand Example

and the disutility (or inverse demand) function is given by:

$$\lambda_{\omega_1}(d_{\omega_1}) = -2d_{\omega_1} + 104.$$

Observe that in this example, the user link cost functions are non-separable for links b and c and asymmetric and, hence, the equilibrium conditions (cf. (32)) cannot be reformulated as the solution to an optimization problem, but, rather, as the solution to the variational inequalities (33) (or (34)), or (35) (or (36)).

The U-O flow and demand pattern that satisfies equilibrium conditions (32) is: $x^*_{p_1} = 5$, $x^*_{p_2} = 4$, and $d^*_{\omega_1} = 9$, with associated link flow pattern: $f^*_a = 5$, $f^*_b = 4$, $f^*_c = 9$.

The incurred user costs on the paths are: $C_{p_1} = C_{p_2} = 86$, which is precisely the value of the disutility λ_{ω_1}. Hence, this flow and demand pattern satisfies equilibrium conditions (32). Indeed, both paths p_1 and p_2 are utilized and their user paths costs are equal to each other. In addition, these costs are equal to the disutility associated with the origin/destination pair that the two paths connect.

5. Multiclass, Multicriteria Supernetworks

In this part of the tutorial, we describe how the concept of a multicriteria supernetwork can be utilized to address decision-making in the Information Age. We then present specific applications, in particular, telecommuting and teleshopping. This section is expository and the theoretical foundations can be found in the supernetworks book.

The term "multicriteria" captures the multiplicity of criteria that decision-makers are often faced with in making their choices, be they regarding consumption, production, transportation, location, or investment. Criteria which are considered as part of the decision-making process may include: cost minimization, time minimization, opportunity cost minimization, profit maximization, as well as risk minimization, among others.

Indeed, the Information Age with the increasing availability of new computer and communication technologies, along with the Internet, have transformed the ways in which many individuals work, travel, and conduct their daily activities today. Moreover, the decision-making process itself has been altered through the addition of alternatives which were not, heretofore, possible or even feasible. As stated in a recent issue of *The Economist* (2000), "The boundaries for employees are redrawn... as people work from home and shop from work."

It is our belief that a network equilibrium framework is natural since not only are now many of the relevant decisions taking place on networks but also the concept of a supernetwork – as shall be demonstrated in this section of the tutorial – is sufficiently general in an abstract and mathematical setting to also capture many of the salient features comprising decision-making today. Further applications, as well as extensions and variations to the work set forth in this chapter, are presented in the supernetworks book.

The first publications in the area of multicriteria decision-making on networks focused on transportation networks, and were by Schneider (1968) and Quandt (1967). However, they assumed fixed travel times and travel costs. Here, in contrast, these functions (as well as any other appropriate criteria functions) are flow-dependent. The first flow-dependent such model was by Dafermos (1981), who considered an infinite number of decision-makers, rather than a finite number as is done here. Furthermore, she assumed two criteria, whereas we consider a finite number, where the number can be as large as necessary. Moreover, the modeling framework set out in this chapter can also handle elastic demands. The first general elastic demand multicriteria network equilibrium model was developed by Nagurney and Dong (2000), who considered two criteria and fixed weights but allowed the weights to be class- and link-dependent. The models in this chapter, in contrast, allow the particular application to be handled with as many finite criteria as are relevant and retain the flexible feature of allowing the weights associated with the criteria to be both class- and link-dependent.

Table 4 contains a list of additional references. All the citations in Table 4 consider two criteria only, typically, time and cost, which are of particular relevance to route selection on transportation networks, when the behavior of the travelers is that of user-optimization. In particular, Table 4 describes, briefly, the type of multicriteria traffic network models, qualitative properties, etc., that are studied in the specific citation. It notes whether separable functions, that is, functions that depend only on the flow on a single link are used, or whether general ones are treated in the formulations. The citations are distinguished as to the type of formulation used, that is, whether the model is an infinite (inf.) or finite-dimensional (fin.) variational inequality (VI) one or an optimization one. It also notes the type of qualitative properties obtained and whether an algorithm is included.

In this section, we recall the multiclass, multicriteria network equilibrium models with elastic demand and with fixed demand, respectively. Each class of decision-maker is allowed to have weights associated with the criteria which are also permitted to be link-dependent for modeling flexibility purposes. In Section 5.1, the governing equilibrium conditions along with the variational inequality formulations are presented.

The usefulness of the multicriteria, multiclass network equilibrium framework is then illustrated in Section 5.2 by applying it to two distinct areas: telecommuting versus commuting decision-making and teleshopping versus shopping decision-making. The discussion of the qualitative properties of the solutions as well as computational procedures are beyond the focus of this tutorial. We refer the reader to **Supernetworks: Decision-Making for the Information Age** for such topics as well as numerous other applications.

Table 4: Some Multicriteria Traffic Network Equilibrium Contributions Post 1968

Citation	Flow Dependence	Type of Demand	Formulation	Algorithm	Qualitative Properties
Dial (1979)	No	fixed	optimization	Yes	No
Dafermos (1981)	Yes;	fixed; class-dependent	inf.-dim. VI	Yes; for network of special structure	existence; uniqueness in special case
Leurent (1993a)	time only;	elastic; separable, not class-dependent	optimization	Yes	existence; uniqueness
Leurent (1993b)	time only; general functions	elastic; separable, not class-dependent	fin.-dim. VI	Yes	Yes
Marcotte, Nguyen, and Tanguay (1996); see also Marcotte and Zhu (1994)	Yes; general functions	fixed; class-dependent	inf.-dim. VI	Yes	Yes
Leurent (1996)	time only; general functions	elastic; not class-dependent	fin.-dim. VI	Yes; optimization case only	existence; uniqueness
Dial (1996)	Yes; separable functions	fixed; class-dependent	optimization; inf.- and fin.-dim. VI	Yes; for special case	Yes
Marcotte and Zhu (1997)	Yes; general functions	fixed; class-dependent	inf.-dim VI	Yes	existence uniqueness in special case
Marcotte (1998)	Yes; general functions	fixed; class-dependent	inf.-dim. VI; fin.-dim. VI	Yes	existence;
Dial (1999)	Yes; separable functions	fixed; class-dependent	optimization; inf.- and fin.-dim. VI	No	Yes
Nagurney (2000d)	Yes; general functions	fixed; class-dependent	fin.-dim. VI	Yes	existence; uniqueness in special case

5.1 The Multiclass, Multicriteria Network Equilibrium Models

In this section, the multiclass, multicriteria network equilibrium models are developed. The elastic demand model is presented first and then the fixed demand model. The equilibrium conditions are, subsequently, shown to satisfy finite-dimensional variational inequality problems.

Consider a general network $G = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes in the network and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes and let p denote a path, assumed to be acyclic, consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. There are n links in the network and n_P paths. Let Ω denote the set of J O/D pairs. The set of paths connecting the O/D pair ω is denoted by P_ω and the entire set of paths in the network by P.

Note that in the supernetwork framework a link may correspond to an actual physical link of transportation or to an abstract or virtual link corresponding to telecommunications. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may consist also of a set of links corresponding to a combination of physical and virtual choices. A path, hence, in the supernetwork framework, abstracts a decision as a sequence of links or possible choices from an origin node, which represents the beginning of the decision, to the destination node, which represents its completion.

Assume that there are now k classes of decision-makers in the network with a typical class denoted by i. Let f_a^i denote the flow of class i on link a and let x_p^i denote the nonnegative flow of class i on path p. The relationship between the link flows by class and the path flows is:

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \quad \forall i, \quad \forall a \in \mathcal{L}, \tag{38}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a class of decision-maker on a link is equal to the sum of the flows of the class on the paths that contain that link.

In addition, let f_a denote the total flow on link a, where

$$f_a = \sum_{i=1}^{k} f_a^i, \quad \forall a \in \mathcal{L}. \tag{39}$$

Thus, the total flow on a link is equal to the sum of the flows of all classes on that link. Group the class link flows into the kn-dimensional column vector $\tilde{f}$ with components: $\{f_1^1, \ldots, f_n^1, \ldots, f_1^k, \ldots, f_n^k\}$ and the total link flows: $\{f_1, \ldots, f_n\}$ into the n-dimensional column vector f. Also, group the class path flows into the kn_P-dimensional column vector $\tilde{x}$ with components: $\{x_{p_1}^1, \ldots, x_{p_{n_P}}^k\}$. The demand associated with origin/destination (O/D) pair ω and class i will be denoted by d_ω^i. Group the demands into a column vector $d \in R^{kJ}$. Clearly, the demands must satisfy the following conservation of flow equations:

$$d_\omega^i = \sum_{p \in P_\omega} x_p^i, \quad \forall i, \forall \omega, \tag{40}$$

that is, the demand for an O/D pair for each class is equal to the sum of the path flows of that class on the paths that join the O/D pair.

The functions associated with the links are now described. In particular, assume that there are H criteria which the decision-makers may utilize in their decision-making with a typical criterion denoted by h. Assume that C_{ha} denotes criterion h associated with link a, where

$$C_{ha} = C_{ha}(f), \quad \forall a \in \mathcal{L}, \tag{41}$$

where C_{ha} is assumed to be a continuous function.

For example, criterion 1 may be time, in which case we would have

$$C_{1a} = C_{1a}(f) = t_a(f), \quad \forall a \in \mathcal{L}, \tag{42}$$

where $t_a(f)$ denotes the time associated with traversing link a. In the case of a transportation link, one would expect the function to be higher than for a telecommunications link. Another relevant criterion may be cost, that is,

$$C_{2a} = C_{2a}(f) = c_a(f), \quad \forall a \in \mathcal{L}, \tag{43}$$

which might reflect (depending on the link a) an access cost in the case of a telecommunications link, or a transportation or shipment cost in the case of a transportation link. One can expect both time and cost to be relevant criteria in decision-making in the Information Age especially since telecommunications is at times a substitute for transportation and it is typically associated with higher speed and lower cost (cf. Mokhtarian (1990)).

In addition, another relevant criterion in evaluating decision-making in the Information Age is opportunity cost since one may expect that this cost would be high in the case of teleshopping, for example (since one cannot physically experience and evaluate the product), and lower in the case of shopping. Furthermore, in the case of telecommuting, there may be perceived to be a higher associated opportunity cost by some classes of decision-makers who may miss the socialization provided by face-to-face interactions with coworkers and colleagues. Hence, a third possible criterion may be opportunity cost, where

$$C_{3a} = C_{3a}(f) = o_a(f), \quad \forall a \in \mathcal{L}, \tag{44}$$

with $o_a(f)$ denoting the opportunity cost associated with link a. Finally, a decision-maker may wish to associate a safety cost in which case the fourth criterion may be

$$C_{4a} = C_{4a}(f) = s_a(f), \quad \forall a \in \mathcal{L}, \tag{45}$$

where $s_a(f)$ denotes a security or safety cost measure associated with link a. In the case of teleshopping, for example, decision-makers may be concerned with revealing personal or credit information, whereas in the case of transportation, commuters may view certain neighborhood roads as being dangerous.

We assume that each class of decision-maker has a potentially different perception of the tradeoffs among the criteria, which are represented by the nonnegative weights: $w^i_{1a}, \ldots, w^i_{Ha}$. Hence, w^i_{1a} denotes the weight on link a associated with criterion 1 for class i, w^i_{2a} denotes the weight associated with criterion 2 for class i, and so on. Observe that the weights are link-dependent and can incorporate specific link-dependent factors which could include for a particular class factors such as convenience and sociability. A typical weight associated with class i, link a, and criterion h is denoted by w^i_{ha}.

Nagurney and Dong (2000) were the first to model link-dependent weights but only considered two criteria. Nagurney, Dong, and Mokhtarian (2000), in turn, used link-dependent weights but assumed only three criteria, in particular, travel time, travel cost, and opportunity cost in their integrated multicriteria network equilibrium models for telecommuting versus commuting.

Here, a generalized cost function is proposed and defined as follows.

Definition 2: Generalized Link Cost Function

A generalized link cost of class i associated with link a and denoted by C^i_a is given by:

$$C^i_a = \sum_{h=1}^{H} w^i_{ha} C_{ha}, \quad \forall i, \quad \forall a \in \mathcal{L}. \tag{46}$$

For example, (46) states that each class of decision-maker i when faced by H distinct criteria on each link a assigns his own weights $\{w^i_{ha}\}$ to the links and criteria.

In lieu of (39) – (46), one can write

$$C^i_a = C^i_a(\tilde{f}), \quad \forall i, \quad \forall a \in \mathcal{L}, \tag{47}$$

and group the generalized link costs into the kn-dimensional column vector C with components: $\{C^1_1, \ldots, C^1_n, \ldots, C^k_1, \ldots, C^k_n\}$.

For example, if there are four criteria associated with decision-making and they are given by (42) through (45), then the generalized cost function on a link a as perceived by class i would have the form:

$$C^i_a = w^i_{1a} C_{1a}(\tilde{f}) + w^i_{2a} C_{2a}(\tilde{f}) + w^i_{3a} C_{3a}(\tilde{f}) + w^i_{4a} C_{4a}(\tilde{f}). \tag{48}$$

Let now C^i_p denote the generalized cost of class i associated with path p in the network where

$$C^i_p = \sum_{a \in \mathcal{L}} C^i_a(\tilde{f}) \delta_{ap}, \quad \forall i, \quad \forall p. \tag{49}$$

Hence, the generalized cost associated with a class and a path is that class's weighted combination of the various criteria on the links that comprise the path.

Note from the structure of the criteria on the links as expressed by (41) and the generalized cost structure assumed for the different classes on the links according to (46) and (47), that it is explicitly being assumed that the relevant criteria are functions of the total flows on the links, where recall that the total flows (see (39)) correspond to the total number of decision-makers of all classes that selects a particular link. This is not unreasonable since one can expect that the greater the number of decision-makers that select a particular link (which comprises a part of a path), the greater the congestion on that link and, hence, one can expect the time of traversing the link as well as the cost to increase.

In the case of the elastic demand model, assume, as given, the inverse demand functions λ^i_ω for all classes i and all O/D pairs ω, where:

$$\lambda^i_\omega = \lambda^i_\omega(d), \quad \forall i, \forall \omega, \tag{50}$$

where these functions are assumed to be smooth and continuous. Group the inverse demand functions into a column vector $\lambda \in R^{kJ}$.

The Behavioral Assumption

Assume that the decision-making involved in the problem is repetitive in nature such as, for example, in the case of commuting versus telecommuting, or shopping versus teleshopping. The behavioral assumption that is proposed, hence, is that decision-makers select their paths so that their generalized costs are minimized.

Specifically, the behavioral assumption utilized is similar to that underlying traffic network assignment models (cf. (15) and (32)) (see, e.g., Beckmann, McGuire, and Winsten (1956), Dafermos and Sparrow (1969), and Dafermos (1982)) in that it is assumed that each class of decision-maker in the network selects a path so as to minimize the generalized cost on the path, given that all other decision-makers have made their choices. Such an idea has also been used in the context of multiclass, multicriteria traffic networks, as noted in Table 4. The generalized path costs in our model (cf. (46) and (49)), however, are more general than those in the models featured in Table 4.

In particular, the following are the network equilibrium conditions for the problem outlined above:

Multiclass, Multicriteria Network Equilibrium Conditions for the Elastic Demand Case

For each class i, for all O/D pairs $\omega \in \Omega$, and for all paths $p \in P_\omega$, the flow pattern $\tilde{x}^*$ is said to be *in equilibrium* if the following conditions hold:

$$C_p^i(\tilde{f}^*) \begin{cases} = \lambda_\omega^i(d^*), & \text{if} \quad x_p^{i*} > 0 \\ \geq \lambda_\omega^i(d^*), & \text{if} \quad x_p^{i*} = 0. \end{cases} \tag{51}$$

In other words, all utilized paths by a class connecting an O/D pair have equal and minimal generalized costs and the generalized cost on a used path by a class is equal to the inverse demand/ disutility for that class and the O/D pair that the path connects.

In the case of the fixed demand model, in which the demands in (40) are now assumed known and fixed, the multicriteria network equilibrium conditions now take the form:

Multiclass, Multicriteria Network Equilibrium Conditions for the Fixed Demand Case

For each class i, for all O/D pairs $\omega \in \Omega$, and for all paths $p \in P_\omega$, the flow pattern $\tilde{x}^*$ is said to be *in equilibrium* if the following conditions hold:

$$C_p^i(\tilde{f}^*) \begin{cases} = \lambda_\omega^i, & \text{if} \quad x_p^{i*} > 0 \\ \geq \lambda_\omega^i, & \text{if} \quad x_p^{i*} = 0, \end{cases} \tag{52}$$

where now the λ_ω^i denotes simply an indicator representing the minimal incurred generalized path cost for class i and O/D pair ω. Equilibrium conditions (52) state that all used paths by a class connecting an O/D pair have equal and minimal generalized costs.

We now present the variational inequality formulations of the equilibrium conditions governing the elastic demand and the fixed demand problems, respectively, given by (51) and (52).

Theorem 5: Variational Inequality Formulation of the Elastic Demand Model

The variational inequality formulation of the multicriteria network model with elastic demand *satisfying equilibrium conditions (51) is given by: Determine* $(\tilde{f}^*, d^*) \in \mathcal{K}^1$*, satisfying*

$$\sum_{i=1}^{k} \sum_{a \in \mathcal{L}} C_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) - \sum_{i=1}^{k} \sum_{\omega \in \Omega} \lambda_\omega^i(d^*) \times (d_\omega^i - d_\omega^{i*}) \geq 0, \quad \forall (\tilde{f}, d) \in \mathcal{K}^1, \tag{53a}$$

where $\mathcal{K}^1 \equiv \{(\tilde{f}, d) | \tilde{x} \geq 0$*, and (38), (39), and (40) hold*$\}$*; equivalently, in standard variational inequality form:*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{53b}$$

where $F \equiv (C, \lambda)$*,* $X \equiv (\tilde{f}, d)$*, and* $\mathcal{K} \equiv \mathcal{K}^1$*.*

Hence, a flow and demand pattern satisfies equilibrium conditions (14) if and only if it also satisfies the variational inequality problem (53a) or (53b).

In the case of fixed demands, we have the following:

Theorem 6: Variational Inequality Formulation of the Fixed Demand Model

The variational inequality formulation of the fixed demand multiciteria network equilibrium model satisfying equilibrium conditions (52) is given by: Determine $\tilde{f} \in \mathcal{K}^2$*, satisfying*

$$\sum_{i=1}^{k} \sum_{a \in \mathcal{L}} C_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) \geq 0, \qquad \forall \tilde{f} \in \mathcal{K}^2, \tag{54a}$$

where $\mathcal{K}^2 \equiv \{\tilde{f} | \exists \tilde{x} \geq 0,$ *and satisfying (38), (39), and (40), with d known*$\}$*; equivalently, in standard variational inequality form:*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{54b}$$

where $F \equiv C$*,* $X \equiv \tilde{f}$*, and* $\mathcal{K} \equiv \mathcal{K}^2$*.*

Therefore, a flow pattern satisfies equilibrium conditions (52) if and only if it satisfies variational inequality (54a) or (54b).

Note that both (53) and (54) are finite-dimensional variational inequality problems. Finite-dimensional variational inequality formulations were also obtained by Nagurney (2000d) for her bicriteria fixed demand traffic network equilibrium model in which the weights were fixed and only class-dependent. Nagurney and Dong (2000), in turn, formulated an elastic demand traffic network problem with two criteria and weights which were fixed but class- and link-dependent as a finite-dimensional variational inequality problem. The first use of a finite-dimensional variational inequality formulation of a multicriteria network equilibrium problem is due to Leurent (1993b), who, however, only allowed one of the two criteria to be flow-dependent. Moreover, although his model was an elastic demand model, the demand functions were separable and not class-dependent as are ours.

5.2 Applications

In this section, two applications of the multiclass, multicriteria network equilibrium framework are presented. In Section 5.2.1, the fixed demand multicriteria network equilibrium model is applied to telecommuting versus commuting, whereas in Section 5.2.2, the elastic demand model is applied to teleshopping versus shopping.

5.2.1 Modeling Telecommuting versus Commuting Decision-Making

In this subsection, the fixed demand model is applied to telecommuting versus commuting decision-making. According to Hu and Young (1996), person-trips and person-miles of commuting increased between 1990 and 1995, both in absolute terms and as a share of all personal travel. Constituting 18% of all person-trips and 22% of all person-miles in 1995, commuting is the single most common trip purpose. Furthermore, as argued by Mokhtarian (1998) (see also Mokhtarian (1991)), it is very likely that a greater proportion of commute trips rather than other types of trips will be amenable to substitution through telecommunications. Consequently, telecommuting most likely has the highest potential for travel reduction of any of the telecommunication applications. Therefore, the study of telecommuting and its impacts is a subject worthy of continued interest and research. Furthermore, recent legislation that allows federal employees to select telecommuting as an option (see United States (2000)), underscores the practical importance of this topic.

The decision-makers in the context of this application are travelers, who seek to determine their *optimal* routes of travel from their origins, which are residences, to their destinations, which are their places of work.

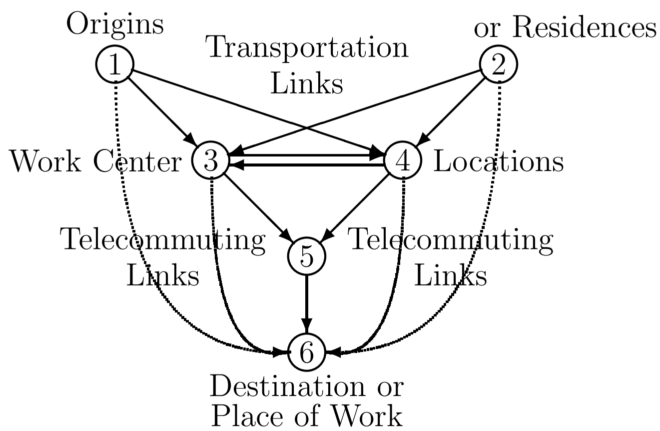

Figure 4: A Network Conceptualization of Commuting versus Telecommuting

Note that, in the supernetwork framework, a link may correspond to an actual physical link of transportation or an abstract or virtual link corresponding to a telecommuting link. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may also consist of a set of links corresponding to physical and virtual transportation choices such as would occur if a worker were to commute to a work center from which she could then telecommute. In Figure 4, a conceptualization of this idea is provided.

Observe that, in Figure 4, nodes 1 and 2 represent locations of residences, whereas node 6 denotes the place of work. Work centers from which workers can telecommute are located at nodes 3 and 4 which also serve as intermediate nodes for transportation routes to work. The links: $(1,6)$, $(3,6)$, $(4,6)$, and $(2,6)$ are telecommunication links depicting virtual transportation to work via telecommuting, whereas all other links are physical links associated with commuting. Hence, the paths $(1,6)$ and $(2,6)$ consisting, respectively, of the individual single links represent "going to work" virtually whereas the paths consisting of the links: $(1,3)$, $(3,6)$ and $(2,4)$, $(4,6)$ represent first commuting to the work centers located at nodes 3 and 4, from which the workers then telecommute. Finally, the remaining paths represent the commuting options for the residents at nodes 1 and 2. The conventional travel paths from node 1 to node 6 are as follows: (1,3), (3,5), (5,6); (1,3), (3,4), (4,5), (5,6); (1,4), (4,5), (5,6), and (1,4), (4,3), (3,5), (5,6). Note that there may be as many classes of users of this network as there are groups who perceive the tradeoffs among the criteria in a similar fashion.

Of course, the network depicted in Figure 4 is illustrative, and the actual network can be much more complex with numerous paths depicting the physical transportation choices from one's residence to one's work location. Similarly, one can further complexify the telecommunication link/path options. Also, we emphasize, that a *path* within this framework is sufficiently general to also capture a choice of mode, which, in the case of transportation, could correspond to busses, trains, or subways (that is, public transit) and, of course, to the use of cars (i.e., private vehicles). Similarly, the concept of path can be used to represent a distinct telecommunications option.

In the model, since the decision-makers are travelers, the path flows and link flows by class would correspond, respectively, to the number of travelers of the class selecting a particular path and link. Hence, the conservation of flow equations (38) and (39) would apply and since we have assumed a fixed demand model (of course, one could also consider an elastic demand version, which would have location choice implications), the expression (40) must also be satisfied, with the travel demand d^i_ω associated with class i traveling between origin/destination pair ω assumed known and given.

The Criteria

We now turn to a discussion of the criteria, which one can expect to be reasonable in the context of decision-making in this particular application. Recall that the first multicriteria traffic network models, due to Schneider (1968) and Quandt (1967), considered two criteria and these were travel time and travel cost. Of course, telecommuting was not truly an option in those days. Dafermos (1981), Leurent (1993a, b), Marcotte (1998), as well as Nagurney (2000d) also considered those two criteria. Nagurney, Dong, and Mokhtarian (2000), in turn, focused on the development of an integrated multicriteria network equilibrium model, which was the first to consider telecommuting versus commuting tradeoffs. They considered three criteria: travel time, travel cost, and an opportunity cost to trade-off the opportunity cost associated with not being physically able to interact with colleagues. Here, a fourth criterion is proposed, that of safety. Note, however, that the network equilibrium model with fixed demands in 5.1.1 can actually handle any number of criteria, provided that the number is finite.

Hence, consider the four criteria, given by (42) through (45), and representing, respectively, travel time, travel cost, the opportunity cost, and safety cost. Consider a generalized link cost for each class given by (46). Thus, the generalized cost on a path as perceived by a class of traveler is given by (49).

The behavioral assumption is that travelers of a particular class are assumed to choose the paths associated with their origin/destination pair so that the generalized cost on that path is minimal. An equilibrium is assumed to be reached when the multicriteria network equilibrium conditions (52) are satisfied. Hence, only those paths connecting an O/D pair are utilized such that the generalized costs on the paths, as perceived by a class, are equal and minimal. The governing variational inequality for this problem is given by (54a); equivalently, by (54b).

5.2.2 Modeling Teleshopping versus Shopping Decision-Making

In this subsection, a multicriteria network equilibrium model for teleshopping versus shopping is proposed. The model generalizes the model proposed in Nagurney, Dong, and Mokhtarian (2001) to the case of elastic demands. Furthermore, destinations, which will now correspond to locations where the product is received, need not necessarily correspond to the same origin at which the shopping experience was initiated. Moreover, the number of origins to be distinct from the number of destinations.

Although there is now a growing body of transportation literature on telecommuting (cf. Mokhtarian (1998)), the topic of teleshopping, which is a newer concept, has received less attention to date. In particular, shopping refers to a set of activities in which consumers seek and obtain information about products and/or services, conduct a transaction transferring ownership or right to use, and spatially relocate the product or service to the new owner (Mokhtarian and Salomon (1997)). Teleshopping, in turn, refers to a case in which one or more of those activities is conducted through the use of telecommunication technologies. Today, much attention is focused on the Internet as the technology of interest, and Internet-based shopping is, indeed, increasing. In this setting, teleshopping represents the consumer's role in B2C electronic commerce. Although the model is in the context of Internet-based shopping, the model can apply more broadly.

Note that outside the work of Nagurney, Dong, and Mokhtarian (2001), there has been essentially no study of the transportation impacts of teleshopping beyond speculation (e.g., Gould (1998), Mokhtarian and Salomon (1997)).

Assume that consumers are engaged in the purchase of a product which they do so in a repetitive fashion, say, on a weekly basis. The product may consist of a single good, such as a book, or a bundle of goods, such as food. Assume also that there are locations, both virtual and physical, where the consumers can obtain information about the product. The virtual locations are accessed through telecommunications via the Internet whereas the physical locations represent more classical shopping venues such as stores and require physical travel to reach.

The consumers may order/purchase the product, once they have selected the appropriate location, be it virtual or physical, with the former requiring shipment to the consumers' locations and the latter requiring, after the physical purchase, transportation of the consumer with the product to its final destination (which we expect, typically, to be his residence or, perhaps, place of work).

Refer to the network conceptualization of the problem given in Figure 5. We now identify the above concepts with the corresponding network component. The idea of such a shopping network was proposed in Nagurney, Dong, and Mokhtarian (2001). Here, several generalizations are given.

Observe that the network depicted in Figure 5 consists of four levels of nodes with the first (top) level and the last (bottom) level corresponding to the locations (destinations) of the consumers involved in the purchase of the product. There are a total of $m+2N+M$ nodes in the network with the number of consumer locations (origins) given by m and the number of information locations given by N where N also corresponds to the number of shopping sites. The number of consumer locations associated with the destinations is given by M. Denote the consumer location nodes (before the shopping experience) at the top level of nodes by: $1,\ldots,m$, with a typical such node denoted by j. We emphasize that each location may have

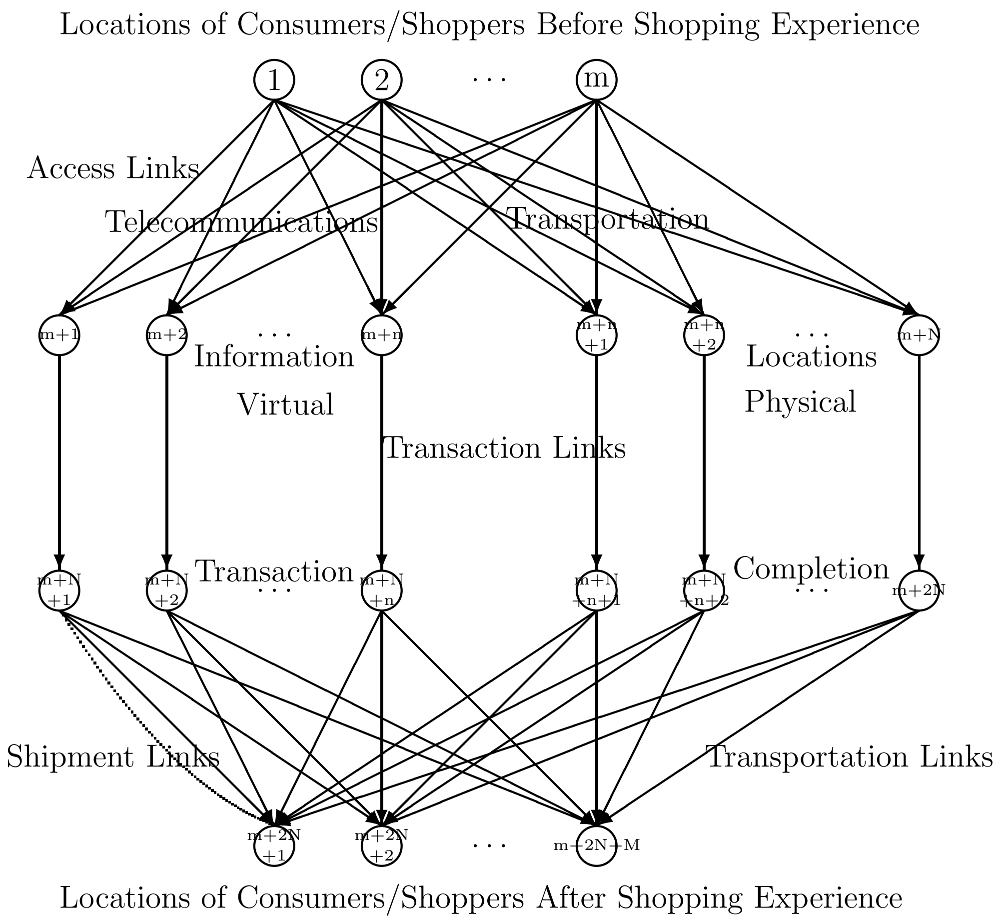

Figure 5: A Network Framework for Teleshopping versus Shopping

many consumers. The second level of nodes, in turn, corresponds to the information locations (and where the transactions also take place), with nodes: $m+1, \ldots, m+n$ representing the virtual or Internet-based locations and nodes: $m+n+1, \ldots, m+N$ denoting the physical locations of information corresponding to stores, for example. Such a typical node is denoted by κ. The third level of nodes corresponds to the completion of the transaction with nodes: $m+N+1, \ldots, m+N+n$ corresponding to Internet sites where the product could have been purchased (and where it has been assumed that information has also been made available in the previous level of nodes) and nodes: $m+N+n+1, \ldots, m+2N$ corresponding to the completion of the transaction at the physical stores. A typical such node is denoted by l. The bottom level of the nodes are enumerated as: $m+2N+1, \ldots m+2N+M$ and denote the locations of the consumers following the completion of the shopping experience. Note that we have, for flexibility purposes, let the number of nodes in the top level be distinct from the number at the bottom level.

We now discuss the links connecting the nodes in the network in Figure 5. There are four sets of links in the network. A typical link (j, κ) connecting a top level node (consumers' location) j to an information node κ at the second level corresponds to an *access* link for information. The links terminating in nodes: $m+1, \ldots, m+n$ of the second level correspond to telecommunication access links and the links terminating in nodes: $m+n+1, \ldots, m+N$ correspond to (aggregated) transportation links.

As can be seen from Figure 5, from each second tier node κ there emanates a link to a node l, which corresponds to a completion of a transaction node. The first mn such links correspond to virtual orders, whereas the subsequent links denote physical orders/purchases. Finally, there are links emanating from the transaction nodes to the consumers' (final) destination nodes, with the links emanating from transaction nodes: $m+N+1, \ldots, m+N+n$ denoting shipment links (since the product, once ordered, must be shipped to the consumer), and the links emanating from transaction nodes: $m+N+n+1, \ldots, m+2N+M$ representing physical transportation links to the consumers' destinations. Note that, in the case of the latter links, the consumers (after purchasing the product) transport it with themselves, whereas in the former case, the product is shipped to the consumers. Observe that in the supernetwork framework, we explicitly allow for alternative modes of shipping the product which is represented by an additional link (or links) connecting a virtual transaction node with the consumers' location.

The above network construction captures the *electronic dissemination* of goods (such as books or music, for example) in that an alternative shipment link in the bottom tier of links may correspond to the virtual or electronic shipment of the product.

Having fixed the above ideas we are now ready to present the notation which will allow us to clarify the costs, demands, and flows on the network. In addition, the behavior of the shoppers, who are assumed to be multicriteria decision-makers, is described. Recall that, as mentioned earlier, the shoppers can now shop from work and have their purchase delivered either to their work or to their home location.

An origin/destination pair in this network corresponds to a pair of nodes from the top tier in Figure 5 to the bottom tier. In the shopping network framework, a path consists of a sequence of choices made by a consumer. For example, the path consisting of the links: $(1, m+1), (m+1, m+N+1), (m+N+1, m+2N+1)$ would correspond to consumers located at location 1 accessing virtual location $m+1$ through telecommunications, placing an order at the site for the product, and having it shipped to them. The path consisting of the links: $(m, m+N)$, $(m+N, m+2N)$, and $(m+2N, m+2N+M)$, on the other hand, could reflect that consumers at location m (which could be a work location or home) drove to the store at location $m+N$, obtained the information there concerning the product, completed the transaction, and then drove to node M. Note that a path represents a sequence of possible options for the consumers. The flows, in turn, reflect *how many* consumers of a particular class actually select the particular paths and links, with a zero flow on a path corresponding to the situation that no consumer elects to choose that particular sequence of links.

The conservation of flow equations associated with the different classes of shoppers are given by (38), (39), and (40).

The Criteria

The criteria that, it is reasonable to assume, are relevant to decision-making in this application are: time, cost, opportunity cost, and safety or security risk, that is, (42) through (45), where, in contrast to the telecommuting application time need not be restricted simply to *travel* time and, depending on the associated link, may include transaction time. In addition, the cost is not exclusively a travel cost but depends on the associated link and can include the transaction cost as well as the product price, or shipment cost. Moreover, the opportunity cost now arises when shoppers on the Internet cannot have the physical experience of trying the good or the actual sociableness of the shopping experience itself. Finally, the safety or security risk cost now can reflect not only the danger of certain physical transportation links but also the potential of credit card fraud, etc.

For example, an article in *The Economist* (2001) notes that "websites are not much good for replicating the social functions of shopping" and that "consumers are often advised against giving their credit-card numbers freely over the Internet, and this remains one of the most-cited reasons for not buying things online."

Assuming weights for each class, link, and criterion, a generalized link cost for each class and link is given by (46). The generalized path cost for a class of consumer is given by (49).

Also, assume, as given, the inverse demand functions which reflect the "price" that the consumers of each class and O/D pair are willing to pay for the shopping experience as a functions of demand. Hence, assume inverse demand functions of the form (50).

The behavioral assumption is that consumers of a particular class are assumed to choose the paths associated with an O/D pair so that their generalized path costs are minimal. An equilibrium, hence, in the elastic demand model must satisfy conditions (51), which also require that if there is positive demand for a class and O/D pair, then the minimum generalized path cost is equal to the inverse demand for that class and O/D pair. The governing variational inequality is given by (53a); equivalently, (53b).

5.2.3 A Telecommuting versus Commuting Example

For illustrative purposes, we present a numerical example corresponds to the fixed demand model described in Section 5.2.1, which is governed by variational inequality (54a); equivalently, (54b). In order to compute the equilibrium flow pattern for the problem, the modifed projection method was applied. (See the book by Nagurney and Dong (2001) for complete details.)

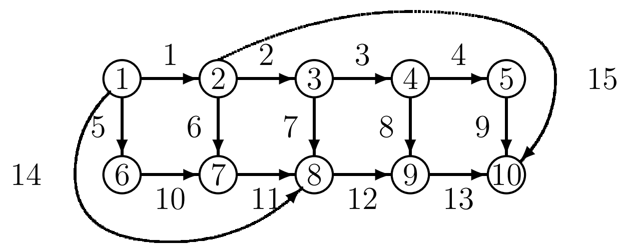

Figure 6: Network Topology for Telecommuting versus Commuting Example

The numerical example had the topology depicted in Figure 6. Links 1 through 13 are transportation links whereas links 14 and 15 are telecommunication links. The network consisted of ten nodes, fifteen links, and two O/D pairs where $\omega_1 = (1, 8)$ and $\omega_2 = (2, 10)$ with travel demands by class given by: $d^1_{\omega_1} = 10$, $d^1_{\omega_2} = 20$, $d^2_{\omega_1} = 10$, and $d^2_{\omega_2} = 30$. The paths connecting the O/D pairs were: for O/D pair ω_1: $p_1 = (1, 2, 7)$, $p_2 = (1, 6, 11)$, $p_3 = (5, 10, 11)$, $p_4 = (14)$, and for O/D pair ω_2: $p_5 = (2, 3, 4, 9)$, $p_6 = (2, 3, 8, 13)$, $p_7 = (2, 7, 12, 13)$, $p_8 = (6, 11, 12, 13)$, and $p_9 = (15)$. The weights were constructed as follows: For class 1, the weights were: $w^1_{1,1} = .25$, $w^1_{2,1} = .25$, $w^1_{3,1} = 1.$, $w^1_{1,2} = .25$, $w_{2,2} = .25$, $w^1_{3,2} = 1.$, $w^1_{1,3} = .4$, $w^1_{2,3} = .4$, $w^1_{3,3} = 1.$, $w^1_{1,4} = .5$, $w^1_{2,4} = .5$, $w^1_{3,4} = 2.$, $w^1_{1,5} = .4$, $w^1_{2,5} = .5$, $w^1_{3,5} = 1.$, $w^1_{1,6} = .5$, $w^1_{2,6} = .3$, $w^1_{3,6} = 2.$, $w^1_{1,7} = .2$, $w^1_{2,7} = .4$, $w^1_{3,7} = 1.$, $w^1_{1,8} = .3$, $w^1_{2,8} = .5$, $w^1_{3,8} = 1.$, $w^1_{1,9} = .6$, $w^1_{2,9} = .2$, $w^1_{3,9} = 2.$, $w^1_{1,10} = .3$, $w^1_{2,10} = .4$, $w^1_{3,10} = 1.$, $w^1_{1,11} = .2$, $w^1_{2,11} = .7$, $w^1_{3,11} = 1.$, $w^1_{1,12} = .3$, $w^1_{2,12} = .4$, $w^1_{3,12} = 1.$, $w^1_{1,13} = .2$, $w^1_{2,13} = .3$, $w^1_{3,13} = 2.$, $w^1_{1,14} = .5$, $w^1_{2,14} = .2$, $w^1_{3,14} = .1$, $w^1_{1,15} = .5$, $w^1_{2,15} = .3$, $w^1_{3,15} = .1$. All the weights $w^1_{4,a} = .2$ for all links a.

For class 2, the weights were: $w^2_{1,1} = .5$, $w^2_{2,1} = .5$, $w^2_{3,1} = .5$, $w^2_{1,2} = .5$, $w^2_{2,2} = .4$, $w^2_{3,2} = .4$, $w^2_{1,3} = .4$, $w^2_{2,3} = .3$, $w^2_{3,3} = .7$, $w^2_{1,4} = .3$, $w^2_{2,4} = .2$, $w^2_{3,4} = .6$, $w^2_{1,5} = .5$, $w^2_{2,5} = .4$, $w^2_{3,5} = .5$, $w^2_{1,6} = .7$, $w^2_{2,6} = .6$, $w^2_{3,6} = .7$, $w^2_{1,7} = .4$, $w^2_{2,7} = .3$, $w^2_{3,7} = .8$, $w^2_{1,8} = .3$, $w^2_{2,8} = .2$, $w^2_{3,8} = .6$, $w^2_{1,9} = .2$, $w^2_{2,9} = .3$, $w^2_{3,9} = .9$, $w^2_{1,10} = .1$, $w^2_{2,10} = .4$, $w^2_{3,10} = .8$, $w^2_{1,11} = .4$, $w^2_{2,11} = .5$, $w^2_{3,11} = .9$, $w^2_{1,12} = .5$, $w^2_{2,12} = .5$, $w^2_{3,12} = .7$, $w^2_{1,13} = .4$, $w^2_{2,13} = .6$, $w^2_{3,13} = .9$, $w^2_{1,14} = .3$, $w^2_{2,14} = .4$, $w^2_{3,14} = 1.$, $w^2_{1,15} = .2$, $w^2_{2,15} = .3$, $w^2_{3,15} = .2$. All the weights $w^2_{4,a} = .1$ for all links a.

Table 5: The Travel Time and Travel Cost Functions for the Links for the Telecommuting Example

Link a	$t_a(f)$	$c_a(f)$
1	$.00005f_1^4 + 4f_1 + 2f_3 + 2$	$.00005f_1^4 + 5f_1 + 1$
2	$.00003f_2^4 + 2f_2 + f_5 + 1$	$.00003f_2^4 + 4f_2 + 2f_3 + 2$
3	$.00005f_3^4 + f_3 + .5f_2 + 3$	$.00005f_3^4 + 3f_3 + f_1 + 1$
4	$.00003f_4^4 + 7f_4 + 3f_1 + 1$	$.00003f_4^4 + 6f_4 + 2f_6 + 4$
5	$5f_5 + 2$	$4f_5 + 8$
6	$.00007f_6^4 + 3f_6 + f_9 + 4$	$.00007f_6^4 + 7f_6 + 2f_2 + 6$
7	$4f_7 + 6$	$8f_7 + 7$
8	$.00001f_8^4 + 4f_8 + 2f_{10} + 1$	$.00001f_8^4 + 7f_8 + 3f_5 + 6$
9	$2f_9 + 8$	$8f_9 + 5$
10	$.00003f_{10}^4 + 4f_{10} + f_{12} + 7$	$.00003f_{10}^4 + 6f_{10} + 2f_8 + 3$
11	$.00004f_{11}^4 + 6f_{11} + 2f_{13} + 2$	$.00004f_{11}^4 + 4f_{11} + 3f_{10} + 4$
12	$.00002f_{12}^4 + 4f_{12} + 2f_5 + 1$	$.00002f_{12}^4 + 6f_{12} + 2f_9 + 5$
13	$.00003f_{13}^4 + 7f_{13} + 4f_{10} + 8$	$.00003f_{13}^4 + 9f_{13} + 3f_8 + 3$
14	$f_{14} + 2$	$.1f_{14} + 1$
15	$f_{15} + 1$	$.2f_{15} + 1$

The travel time functions and the travel cost functions for this example are reported in Table 5. The opportunity cost functions and the safety cost functions for the links for this example are reported in Table 6. The generalized link cost functions were constructed according to (46) using the weights given above.

Note that the opportunity costs associated with links 14 and 15 were high since these are telecommunication links and users by choosing these links forego the opportunities associated with working and associating with colleagues from a face to face perspective. Observe, however, that the weights for class 1 associated with the opportunity costs on the telecommunication links are low (relative to those of class 2). This has the interpretation that class 1 does not weight such opportunity costs highly and may, for example, prefer to be working from the home for a variety, including familial, reasons. Also, note that class 1 weights the travel time on the telecommunication links more highly than class 2 does. Furthermore, observe that class 1 weights the safety or security cost higher than class 2.

Table 6: The Opportunity Cost and Safety Cost Functions for the Links for the Example

Link a	$o_a(f)$	$s_a(f)$
1	$2f_1 + 4$	$f_1 + 1$
2	$3f_2 + 2$	$f_2 + 2$
3	$f_3 + 4$	$f_3 + 1$
4	$f_4 + 2$	$f_4 + 2$
5	$2f_5 + 1$	$2f_5 + 2$
6	$f_6 + 2$	$f_6 + 1$
7	$f_7 + 3$	$f_7 + 1$
8	$2f_8 + 1$	$2f_8 + 2$
9	$3f_9 + 2$	$3f_9 + 3$
10	$f_{10} + 1$	$f_{10} + 2$
11	$4f_{11} + 3$	$2f_{11} + 3$
12	$3f_{12} + 2$	$3f_{12} + 3$
13	$f_{13} + 1$	$f_{13} + 2$
14	$6f_{14} + 1$	$.5f_{14} + .1$
15	$7f_{15} + 4$	$.4f_{15} + .1$

The equilibrium multiclass link flow and total link flow patterns are reported in Table 7, which were induced by the equilibrium multiclass path flow pattern given in Table 8.

The generalized path costs were: for Class 1, O/D pair ω_1:

$$C^1_{p_1} = 13478.4365,\ C^1_{p_2} = 11001.0342,\ C^1_{p_3} = 8354.5420,\ C^1_{p_4} = 1025.4167,$$

for Class 1, O/D pair ω_2:

$$C^1_{p_5} = 45099.8047,\ C^1_{p_6} = 27941.5918,\ C^1_{p_7} = 25109.3223,\ C^1_{p_8} = 22631.9199,$$

$$C^1_{p_9} = 2314.7222;$$

for Class 2, O/D pair ω_1:

$$C^2_{p_1} = 15427.5996,\ C^2_{p_2} = 15427.2021,\ C^2_{p_3} = 8721.8945,\ C^2_{p_4} = 8721.3721,$$

Table 7: The Equilibrium Link Flows for the Example

Link a	Class 1 - f_a^{1*}	Class 2 - f_a^{2*}	Total flow - f_a^*
1	0.000	0.0000	0.0000
2	0.0000	24.0109	24.0109
3	0.0000	22.7600	22.7600
4	0.0000	17.3356	17.3356
5	0.0000	4.6901	4.6901
6	0.0000	5.9891	5.9891
7	0.0000	1.2509	1.2509
8	0.0000	5.4244	5.4244
9	0.0000	17.3556	17.3556
10	0.0000	4.6901	4.6901
11	0.0000	10.6792	10.6792
12	0.0000	7.2400	7.2400
13	0.0000	12.6644	12.6644
14	10.0000	5.3090	15.3099
15	20.0000	0.0000	20.0000

Table 8: The Equilibrium Path Flows for the Example

Path p	Class 1 - x_p^{1*}	Class 2 - x_p^{2*}
p_1	0.0000	0.0000
p_2	0.0000	0.0000
p_3	0.0000	4.6901
p_4	10.0000	5.3099
p_5	0.0000	17.3357
p_6	0.0000	5.4244
p_7	0.0000	1.2509
p_8	0.0000	5.9892
p_9	20.0000	0.0000

and for Class 2, O/D pair ω_2:

$$C^2_{p_5} = 34924.6602,\ C^2_{p_6} = 34924.6094,\ C^2_{p_7} = 34925.3789,\ C^2_{p_8} = 34924.9805,$$

$$C^2_{p_9} = 41574.2617.$$

It is interesting to see the separation by classes in the equilibrium solution. Note that all members of class 1, whether residing at node 1 or node 2, were telecommuters, whereas all members of class 2 chose to commute to work. This outcome is realistic, given the weight assignments of the two classes on the opportunity costs associated with the links (as well as the weight assignments associated with the travel times). Of course, different criteria functions, as well as their numerical forms and associated weights, will lead to different equilibrium patterns.

This example demonstrates the flexibility of the modeling approach. Moreover, it allows one to conduct a variety of "what if" simulations in that, one can modify the functions and the associated weights to reflect the particular telecommuting versus commuting scenario. For example, during a downturn in the economy, the opportunity costs associated with the telecommuting links may be high, and, also, different classes may weight this criteria on such links higher, resulting in a new solution. On the other hand, highly skilled employees who are in demand may have lower weights associated with such links in regards to the opportunity costs. This framework is, hence, sufficiently general to capture a variety of realistic situations while, at the same time, allowing decision-makers to identify their specific values and preferences.

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