Decision Making under Uncertainty
Worst-case Analysis & Expected Value Optimization

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Outline

- Macroeconomic policy example
- Properties of robustness
- Finance
- Engineering Design
- Defence
- Project scheduling
- Robust vs expected value optimality
- Conclusions
Macroeconomic policy example

\[
\min_U \left\{ \lambda J_{HMT}(Y_{HMT}(U), U) + (1 - \lambda) J_{NIESR}(Y_{NIESR}(U), U) \right\}
\]
General problem

\[ f(x^*, y^*) = \min_{x} \max_{y \in \mathbb{R}} f(x, y) \]
Robustness of mmx

\[ f(x^*, y^*) = \min_{x} \max_{y \in R} f(x, y) \]

\[ f(x^*, y^*) \geq f(x^*, y), \ \forall y \in R \]
Equivalent problems

- **Continuous mmx**

\[
\min_u \max_\lambda \left\{ \sum_i \lambda_i f_i(u) \mid \sum_i \lambda_i = 1, \lambda_i \geq 0, \forall i \right\}
\]

- **Discrete mmx**

\[
\min_u \max_i \left\{ f_i(u) \right\}
\]

- **NLP**

\[
\min_{u,v} \left\{ v \mid f_i(u) \leq v, \forall i \right\}
\]
SADDLE POINT

$f(x, y)$ convex in $x$ and concave in $y$

$f(x_*, y) \leq f(x_*, y_*) \leq f(x, y_*)$
MULTIPLE MAXIMA

\[ f(x, y) = x^2 + y^2 \]

\[ f(x^*, y_1^*) = 4 \]
\[ f(x^*, y_2^*) = 4 \]
\[ (x^*, y_1^*) = (0, -2) \]
\[ (x^*, y_2^*) = (0, 2) \]
RISK-RETURN FRONTIER: MARKOWITZ

\[
\max \alpha \begin{array}{c} \text{expected} \\ \text{return} \end{array} - (1 - \alpha) \begin{array}{c} \text{expected} \\ \text{risk} \end{array} \\
\max_{w \in \Omega} \left\{ \alpha (r^T w) - (1 - \alpha)(w - \bar{w})^T \Lambda (w - \bar{w}) \right\}
\]
MEAN-VARIANCE EFFICIENT FRONTIERS

\[
\max \left\{ \alpha (r^T w) - (1 - \alpha)(w - \overline{w})^T \Lambda (w - \overline{w}) \mid w \in \Omega \right\}
\]
WORST-CASE ANALYSIS: DISCRETE MINIMAX

- risk forecast $\Lambda$
- benchmark $w$
- rival return scenarios $r_i$, $i=1,...,I$
- risk level $\alpha$

$$\max_{w \in \Omega} \min_{i=1,...,I} \left\{ \alpha r_i^T (w - \overline{w}) - (1 - \alpha)(w - \overline{w})^T \Lambda (w - \overline{w}) \right\}$$
DISCRETE MINIMAX

\[
\max_{\mathbf{w} \in \Omega} \left\{ \alpha \left( \mathbf{r}_{DOOM}^T \mathbf{w} \right) - (1 - \alpha)(\mathbf{w} - \overline{\mathbf{w}})^T \Lambda (\mathbf{w} - \overline{\mathbf{w}}) \right\}
\]
Minimax strategy

Minimax Strategy
Prosperity Realised

Minimax Strategy
Core Realised

Minimax Strategy
Doom Realised

Minimax Strategy
Active Risk %/Annum

Absolute Return %/Annum

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Generalized discrete mmx

- return scenarios $r_i$ \quad $i=1,\ldots,I$
- risk scenarios $\Lambda_j$ \quad $j=1,\ldots,J$
- benchmarks $w_k$ \quad $k=1,\ldots,K$
- risk parameter $\alpha$
- current portfolio $p$, buy-sell costs $c_b$, $c_s$

$$\max_{w\in\Omega} \quad \min_{i=1,\ldots,I} \quad \min_{j=1,\ldots,J} \quad \min_{k=1,\ldots,K} \left\{ \begin{array}{c} \alpha [r_i^T(w - \bar{w}_k) - \tau(w, p, c_{b,s})] \\ - (1 - \alpha) [(w - \bar{w}_k)^T \Lambda_j (w - \bar{w}_k)] \end{array} \right\}$$
Nonlinear programming formulation

\[
\max_{w, b, s} \quad \alpha (\mu - \tau) - (1 - \alpha) \lambda \\
\text{subject to}
\]

\[
\begin{align*}
\mathbf{r}_i^T (\mathbf{w} - \overline{\mathbf{w}}_k) & \geq \mu, \quad \forall i, k \\
(\mathbf{w} - \overline{\mathbf{w}}_k)^T \Lambda_j (\mathbf{w} - \overline{\mathbf{w}}_k) & \leq \lambda, \quad \forall j, k \\
\mathbf{p} + \mathbf{b} - \mathbf{s} & = \mathbf{w} \\
\mathbf{c}_b^T \mathbf{b} + \mathbf{c}_s^T \mathbf{s} & = \tau \\
\sum_{i=1}^{n} w_i & = 1 \\
w, b, s & \geq 0
\end{align*}
\]

- \( \alpha \) = weighted return vs. risk
- \( \mu \) = worst-case return
- \( \lambda \) = worst-case risk
- \( \tau \) = transaction costs
- \( \sum_{i=1}^{n} w_i = 1 \) = scaled balance
## Application: fund management

### Input data: 11 assets, 8 return & 10 risk scenarios

<table>
<thead>
<tr>
<th>ASSETS</th>
<th>11</th>
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<tbody>
<tr>
<td>BCH</td>
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</tr>
<tr>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

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Application: multi-return & multi-risk mmx
Application: worst-case performance

rival return scenarios

| Scenario 0 | Low Growth - No War |
| Scenario 1 | Low Growth - Clean War |
| Scenario 2 | Low Growth - Messy War |
| Scenario 3 | Medium Growth - No War |
| Scenario 4 | Medium Growth - Clean War |
| Scenario 5 | Medium Growth - Messy War |
| Scenario 6 | High Growth - No War |
| Scenario 7 | High Growth - Clean War |
Application: robustness of mmx
Continuous mmx

- range of return forecasts: \( r^l \leq r \leq r^u \)
- rival risk scenarios: \( \Lambda_i \)
- m-v optimization problem as mmx

\[
\min_w \left\{ \alpha \max_{r^l \leq r \leq r^u} - \left[ (w - \bar{w})^T r + \tau \right] + (1 - \alpha) \max_i \left[ (w - \bar{w})^T \Lambda_i (w - \bar{w}) \right] \right\}
\]

- maximin problem

\[
\max_w \left\{ \alpha \min_{r^l \leq r \leq r^u} \left[ (w - \bar{w})^T r + \tau \right] - (1 - \alpha) \min_i \left[ (w - \bar{w})^T \Lambda_i (w - \bar{w}) \right] \right\}
\]
Continuous mmx: nonlinear programming

Define

\[ x^+ - x^- = w - \overline{w} \] 
so that \( x^+ x^- = 0 \) and \( x^+, x^- \geq 0 \)

\[
\begin{cases}
  w > \overline{w}, & x^+ > 0 \text{ and } x^- = 0 \\
  w < \overline{w}, & x^+ = 0 \text{ and } x^- > 0
\end{cases}
\]

\[ \text{mmx} \]

\[
\min_{w, x^+, x^-} \left\{ -\alpha [(x^-)^T r^u + (x^+)^T r^l] + \tau + (1 - \alpha) \max_i [(w - \overline{w})^T \Lambda_i (w - \overline{w})] \right\}
\]
Continuous mmx

\[ \min_{w, x^\pm} \alpha \left[ x^u r^u + x^+ r^+ \right] + \tau + (1 - \alpha) v + \gamma x^+ x^+ \]

\[ \begin{align*}
\text{st} & \quad 1^T w = 1 \\
& \quad p + b - s = w \\
& \quad c^T_b b + c^T_s s = \tau \\
& \quad x^+ - x^- = w - \bar{w} \\
& \quad (w - \bar{w})^T \Lambda_i (w - \bar{w}) \leq \nu, \quad i \in I \\
& \quad w, b, s, x^+, x^- \geq 0
\end{align*} \]
Multi-period m-v optimisation

- multi-period discrete time portfolio
- risky assets over investment horizon
- after initial investment
  - restructure portfolio at $t$ in terms of return & risk
  - redeem portfolio at $T$
- conflicting objectives: max wealth & min risk
- benchmark relative & transaction costs
Introduction to stochastic programming
economic factor scenarios

Inflation

GDP

Percentage increase in prices on a year earlier

Percentage increase in output on a year earlier

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Modelling the future

Scenario tree: discretised probabilistic model

N. Gulpinar, R. Settergren, B. Rustem, *Simulation and Optimization Approaches to Scenario Tree Generation*, JEDC, 2004
Capital allocation

- initial budget is normalised to $I$ & $p$ is current portfolio position:

\[ p + (1 - c_b) b_0 - (1 + c_s) s_0 = w_0 \]
\[ 1' b_0 - 1' s_0 = 1 - 1' p \]

Transaction constraint

- decision at time $t$ depends on $\rho^t$ and yields investment in asset $j$:

\[ w_t^j = r_t^j (\rho^t) w_{t-1}^j + (1 - c_b^j) b_t^j - (1 - c_s^j) s_t^j \quad \forall j = 1, \ldots, n \quad t=1,\ldots,T-1 \]
\[ w_T^j = r_T^j (\rho^T) w_{T-1}^j \]

Balance constraint

- subsequent transactions do not alter wealth:

\[ 1' b_t - 1' s_t = 0 \quad \text{for} \quad t = 1, \ldots, T - 1 \]
Expected wealth relative to benchmark

Given possible events $e \in N_t$ (discretisation of $p_t$)

$$W_t = \sum_{e \in N_t} P_e \left( \hat{r}_e (w_a(e) - \bar{w}_a(e)) \right)$$

Variance of wealth relative to benchmark

Risk - variance of portfolio return relative to benchmark

$$Var[r_t | \rho_t] = \sum_{e \in N_t} P_e \left( (w_a(e) - \bar{w}_a(e))^2 \right) \left( \Lambda + \hat{r}_e \hat{r}_e \right) \left( w_a(e) - \bar{w}_a(e) \right) - W_t^2$$

- uncertainty due to (continuous) variability of return at given realization
- uncertainty due to discrete stochastic realizations $\Rightarrow$ paths through tree
Multistage stochastic quadratic programming

\[
\min_{w,b,s} \sum_{t=1}^{T} \alpha_t \sum_{e \in N_t} P_e \left[ \left( w_{a(e)} - \bar{w}_{a(e)} \right)' \left( \Lambda + \hat{r}_e \hat{r}_e' \right) \left( w_{a(e)} - \bar{w}_{a(e)} \right) \right]
\]

Subject to
\[
p + (1-c_b)b_0 - (1+c_s)s_0 = w_0
\]
\[
1'b_0 - 1's_0 = 1 - 1'p
\]
\[
r_e \circ w_{a(e)} + (1-c_b) \circ b_e - (1-c_s) \circ s_e = w_e \quad e \in N_I
\]
\[
1'b_e - 1's_e = 0 \quad e \in N_I
\]
\[
\sum_{e \in N_T} P_e \left( \hat{r}_e' \left( w_{a(e)} - \bar{w}_{a(e)} \right) \right) \geq W
\]
\[
w^L_e \leq w_e \leq w^U_e \quad e \in N
\]
\[
0 \leq b_e \leq b^U_e \quad e \in N_I \cup 0
\]
\[
0 \leq s_e \leq s^U_e \quad e \in N_I \cup 0
\]
Worst-case optimal robust decisions

- extension of multi-period m-v to multiple return & risk scenarios
  - inherently inaccurate asset return forecasts & risk estimates
- proposed method based on min-max strategy
  - rival representations of future
  - computation of optimal portfolio simultaneously with worst case
  - take into account of rival risk & return scenarios
- min-max algorithm
Why worst-case analysis?

- guaranteed performance under worst-case conditions
- robustness of mmx strategy
- rival representations of future
  - rival returns for first period: investor wants to survive first period
  - risk scenarios: modelling future risk with all rival risk measures
m-v optimisation

\[
\max \{ \alpha \mathbb{E}[W_T] - (1 - \alpha) \text{var}[W_T] \}
\]

worst-case analysis

\[
\max \min_{i,j} \left\{ \alpha \mathbb{E}[W_T^i] - (1 - \alpha) \text{var}[W_T^j] \right\}
\]
Multi-period mmx model I

- \( i \) covariance matrices at each node of scenario tree & \( k \) rival return scenarios

\[
\min_w \left\{ \gamma \sum_{t=1}^{T} \sum_{e \in N_t} \max_i \left[ P_e(w_{a(e)} - \overline{w}_{a(e)})' \left( \Lambda_i + r_e' r_e \right) (w_{a(e)} - \overline{w}_{a(e)}) \right] \right\} - \min_k \left\{ \sum_{e \in N_T^k} P_e(w_{a(e)} - \overline{w}_{a(e)})' r_e \right\}
\]

\( i = 1, \ldots, I_e, \ k = 1, \ldots, K, \ t = 1, \ldots, T, \ e \in N_t \)

level of risk aversion \( \gamma = \begin{cases} 0; & \text{risk-seeking} \\ \infty; & \text{risk-averse} \end{cases} \)

worst-case return

\[
\sum_{e \in N_T^k} P_e(w_{a(e)} - \overline{w}_{a(e)})' r_e \geq \mu \quad e \in N_I, \ k = 1, \ldots, K
\]

worst-case risk at node \( e \) of scenario tree

\[
P_e(w_{a(e)} - \overline{w}_{a(e)})' \left( \Lambda_i + r_e' r_e \right) (w_{a(e)} - \overline{w}_{a(e)}) \leq v_e \quad e \in N_I, \ i = 1, \ldots, I_e
\]

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NLP model

$$\min_w \gamma \sum_{t=1}^T \alpha_t \sum_{e \in \mathcal{N}_t} v_e - \mu + \lambda \sum_{t=1}^T \sum_{e \in \mathcal{N}_t} b_e s_e$$

subject to

$$1'b_0 - 1's_0 = 1 - 1'p$$

$$p + (1-c_b)b_0 - (1+c_s)s_0 = w_0$$

$$r_e \circ w_{a(e)} + (1-c_b) \circ b_e - (1+c_s) \circ s_e = w_e \quad e \in \mathcal{N}_I$$

$$1'b_e - 1's_e = 0 \quad e \in \mathcal{N}_I$$

$$\sum_{e \in \mathcal{N}_I^k} P_e (w_{a(e)} - \overline{w}_{a(e)})' r_e \geq \mu \quad e \in \mathcal{N}_I, \ k = 1, \ldots, K$$

$$P_e (w_{a(e)} - \overline{w}_{a(e)})' \left( \Lambda_i + r_e r_e \right) (w_{a(e)} - \overline{w}_{a(e)}) \leq v_e \quad e \in \mathcal{N}_I, \ i = 1, \ldots, I_e$$

$$w_e^L \leq w_e \leq w_e^U \quad e \in \mathcal{N}$$

$$0 \leq b_e \leq b_e^U \quad e \in \mathcal{N}_I \cup 0$$

$$0 \leq s_e \leq s_e^U \quad e \in \mathcal{N}_I \cup 0$$

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Multi-period mmx model II

- covariance matrices at each time period & k rival return scenarios

\[
\min_w \left\{ \gamma \sum_{t=1}^{T} \alpha_t \max_i \left[ \sum_{e \in N_t} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right)' \left( \Lambda_i + r_e' r_e \right) \left( w_{a(e)} - \bar{w}_{a(e)} \right) \right] - \min_k \left[ \sum_{e \in N_T^k} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right)' r_e \right] \right\}
\]

\[i = 1, \ldots, I_t, \quad k = 1, \ldots, K, \quad t = 1, \ldots, T\]

worst-case return

\[
\sum_{e \in N_T^k} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right)' r_e \geq \mu
\]

worst-case risk at period \( t \)

\[
\sum_{e \in N_t} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right)' \left( \Lambda_i + r_e' r_e \right) \left( w_{a(e)} - \bar{w}_{a(e)} \right) \leq \nu_t
\]

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Multi-period mmx model III

- covariance matrices associated with each block & rival return scenarios

\[
\min_w \left\{ \gamma \sum_{j=1}^K \max_i \left[ \sum_{t=2}^T \sum_{e \in N_i^j} \alpha_i P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right) \right] \right\} 
\]

\[
\min_k \left[ \sum_{e \in N_T^k} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right) \right] \]

\[i=1,\ldots,I, \quad j,k=1,\ldots,K\]

worst-case return

\[
\sum_{e \in N_T^k} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right) \leq \mu \quad e \in N_I, \quad k=1,\ldots,K
\]

worst-case risk at block \( k \)

\[
\sum_{t=2}^T \sum_{e \in N_T^k} P_e \left( w_{a(e)} - \bar{w}_{a(e)} \right) \left( \Lambda_i + r'_e r_e \right) \left( w_{a(e)} - \bar{w}_{a(e)} \right) \leq v_k
\]

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Rival scenario specifications

- **return scenarios;** historical data – scenario tree

- **risk scenario specifications**
  - rival covariance matrices can be realised by observing the data during different past periods

- **mmx risk-return frontiers**

- **backtesting**
Backtesting – out of sample

- monthly history 1988-2000;
- 10 FTSE100 Assets
  - Barclays
  - BP Amoco
  - CGU
  - BG Group
  - Boots
  - Diageo
  - Bass
  - British Airways
  - Bank of Scotland
  - British Telecom

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Design a pump and pipeline system to minimize annual cost $C$, with uncertainties related to the efficiency of pump and fluid friction parameter
Two different types of parameters

• Uncertain Parameters
  • \( \eta \) - efficiency of pump
  • \( a \) - friction parameter

• Design Variables
  • \( W_0 \) - installed power of pump
  • \( W \) - required power of pump
  • \( D \) - diameter of pipe
Problem statement

\[ \min_{W_0, D} \max_{\eta, a} \quad C \]

s.t. \quad \begin{align*}
C &= 196850D + 0.84632W_0^{0.86} + 0.4251W \\
W &= \frac{994.018}{\eta} \left[ 381.024135 + \frac{5.66a}{0.04D^{4.84}} \right] \\
W &\leq W_0, \quad 0.508 \leq D \leq 0.9144 \\
0.3 &\leq \eta \leq 0.6, \quad 0.02 \leq a \leq 0.06.
\end{align*}
Experiment design

**PHYSICAL WORLD**
- apparatus configuration
- what is manipulated? (controls)
- what is measured? (responses)
- initial conditions, duration, no. and time of sample points

**MODELLING WORLD**
- proposed set of candidates
- identifiable?
- distinguishable?
- best model?
- utmost precision?

*reaction*

*time-varying system inputs*
Fermentation Model

\[
\frac{dy_1}{dt} = (r - u_1 - \theta_4)y_1
\]

\[
\frac{dy_2}{dt} = -\frac{ry_1}{\theta_3} + u_1(u_2 - y_2)
\]

\[
r = \frac{\theta_1 y_2}{\theta_2 + y_2}
\]

- Design an experiment to yield the best possible information of the parameters
- The only \textit{a priori} information:

\[
\theta_i \in [0.05, 0.98], \quad i = 1, 2, 3;
\]

\[
\theta_4 \in [0.01, 0.98];
\]
The Aim

How to adjust:
• Time-varying controls
• Initial conditions
• Duration of experiment,

For generating maximum amount of information for parameter identification in view of worst-case
Optimising Parameter Information

• Optimise parameter information in view of worst-case (wrt $\mathcal{D}$) sensitivity of predictions

• Large number of different sensitivity coefficients

\[
Q_r = \begin{bmatrix}
\frac{\partial y_r}{\partial \theta_1} & \frac{\partial y_r}{\partial \theta_2} & \ldots & \frac{\partial y_r}{\partial \theta_p} \\
\frac{\partial y_r}{\partial \theta_1} & \frac{\partial y_r}{\partial \theta_2} & \ldots & \frac{\partial y_r}{\partial \theta_p} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial y_r}{\partial \theta_1} & \frac{\partial y_r}{\partial \theta_2} & \ldots & \frac{\partial y_r}{\partial \theta_p}
\end{bmatrix}
\]

⇐ sampling point $t_{sp1}$

⇐ sampling point $t_{sp2}$

⇐ sampling point $t_{sp_{nsp}}$

• Experiments involve multiple response variables
Optimising Parameter Information

• *Step 1*: Combine sensitivities of all responses into a dynamic parameter information matrix:

\[ M_I(\theta, \phi) = \sum_{r=1}^{M} \sum_{s=1}^{M} \sigma_{rs} Q_r^T Q_s \]

• *Step 2*: Define scalar metric of \( M_I \) as “parameter information content” of experiment
Max-min formulation

\[ \phi_R = \max_{\phi \in \Phi} \{ \min_{\theta \in \Theta} \{ \det \{ M_I (\theta, \phi) \} \} \} \]

or

\[ \max_{\phi \in \Phi, \Psi \in R} \Psi \]

\[ s.t. \Psi \leq \phi_R, \forall \theta \in \Theta \]
# The Results

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>\textbf{max} 39.38954 violation -51.78269 at $\theta^{(1)} = [0.05,0.98,0.98,0.98]$</td>
</tr>
<tr>
<td>2</td>
<td>\textbf{max} 4.57472 violation -6.73966 at $\theta^{(2)} = [0.05,0.05,0.98,0.98]$</td>
</tr>
<tr>
<td>3</td>
<td>\textbf{max} 4.48416 violation -4.07338 at $\theta^{(3)} = [0.98,0.05,0.98,0.98]$</td>
</tr>
<tr>
<td>4</td>
<td>\textbf{max} 4.47273 violation -0.00001 at $\theta^{(4)} = [0.05,0.05,0.98,0.04]$</td>
</tr>
</tbody>
</table>
Task Assignment & Routing
Updating Decisions for Setup Changes (Repair)
Anticipating Opponent’s Strategy (Games)
Anticipating Opponent’s Strategy (Games)
Conclusions

- Expected value optimization leads to very large stochastic programming problems which can be avoided using robustness

- Expected value optima generally perform well in the mean but performance does deteriorate if worst-case scenario is realised

- Worst-case optima can only improve if worst-case not realised: performance is reasonably comparable with mean uncertainty realised. There is, however, a price to be paid for the insurance provided.