Modularity for community detection: history, perspectives and open issues

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Outline

Community Definition

- Studies of complex networks revealed some interesting properties which can be exploited in communication and information networks:
  - Small–world effect
  - Self–organisation and self–adaption
  - Arising of community structure
- Small–world effect and Self–* properties have been exploited to optimise routing strategies and to increase network robustness
- On the other hand, Community structures have been deeply studied by physicians and mathematicians, but are quite unknown to researchers in the ITC field.

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Communities in real world

- A community can be simply defined as a group of people which know each other and share interests and knowledge or collaborate to reach a given target
- Players of the same chess–club, classmates, members of a GNU/Linux User Group, students of engineering are examples of real–world communities....
- ....but also computers in the same LAN, people sharing rock music through eMule, programmers writing code for the same project are good examples of community structures
- Studying community structures in real–world networks can help to better understand and exploit community structures in information and communication networks
- Information about communities can be used to optimise message routing, to avoid waste of bandwidth and to speed–up collaboration and interaction in very large networks

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A community is....

- A sharp analytical definition of what a community is cannot be easily formulated. For this reason, many definitions of "community" have been proposed in the last few years
- One of the most widely used and accepted defines a community as "A group of nodes of a graph which are more strongly connected to each other than with other nodes in the same graph"
- Or as "A group of nodes of a graph which are more strongly connected to each other than expected in a corresponding random graph "
- Even if these definitions seem simple and clear, it is not easy to derive an algorithm to decide if a graph has communities and, in that case, to find which nodes belong to each community
- Note that the "community problem" has nothing in common with classical "clustering" problems. We're not looking for an optimal cut of a graph or for a min–flux decomposition....

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Finding communities: All of us agree.....

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Finding communities: algorithms

Many different methods have been proposed for community identification. Danon et al. identify the following classes:

- Link removal methods (centrality–based)
- Agglomerative methods (e.g. Hierarchical clustering)
- Modularity optimization methods (a lot of them!)
- Spectral methods
- Other methods

We'll focus on modularity–based methods

Modularity and community structure

- A possible measure for evaluation of community decomposition is the so–called "Modularity"
- Given an undirected graph $G(E, V)$, where each node is assigned to one of C possible communities, the modularity of the decomposition is defined as (Newman):

$$Q = \frac{1}{2m} \sum_{i,j \in V} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)$$

where:

- A_{ij} are the elements of the adjacency matrix of $G(E, V)$
- k_i is the out-degree of node i
- $m = |E|$
- $\delta(c_i, c_j)$ is equal to 1 if i and j belong to the same community, and is equal to 0 otherwise

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Modularity: what does it mean?

- The idea behind modularity definition is simple: a set of nodes form a community in the sense of modularity if the fraction of links inside the community is higher than expected in a network considered as “reference”(the so–called “null–model”)
- The term

$$\frac{1}{2m}\sum_{i,j\in V}[A_{ij}]\,\delta(c_i,c_j)$$

is the fraction of links connecting nodes which are in the same community, while

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Notes about modularity

- Q is always lesser than 1, and equal to 0 only if all nodes are put in the same community
- High values of Q indicate a strong community structure
- Modularity is defined only for undirected graphs
- This definition allows a node to be put in just one community at a time, while nodes in real networks usually belong to many communities

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Extending modularity for directed graphs

- It is intuitively simple to extend the modularity definition for directed graphs (Arenas et al. 2007, Leicht – Newman 2007, Nicosia, Mangioni et al. 2008)
- It is necessary to consider a null–model where the probability of having a link between nodes i and j is the product of the probability that a link starts at i and of the probability that a link ends at j.
- So the extension of modularity for directed graphs can be written as:

$$Q_d = \frac{1}{m} \sum_{i,j \in V} \left[A_{ij} - \frac{k_i^{out} k_j^{in}}{m} \right] \delta(c_i, c_j)$$

- Note that:
  - Adjacency matrix is not symmetric for directed networks
  - The real fraction of links is referred to the real number of links in the graph (each link is counted just once!)

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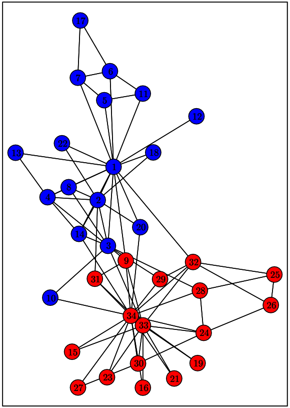
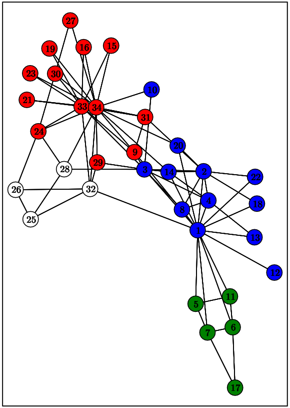
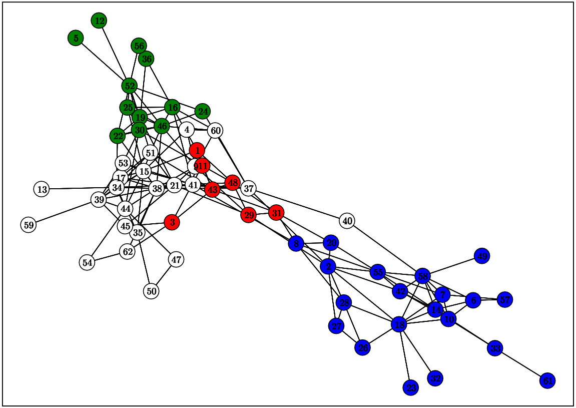
Modularity: The Zachary network (1)



Modularity: The Zachary network (2)



Modularity: The Dolphin network (1)



Drawbacks of modularity

Some drawbacks of modularity have been pointed out recently:

- It requires global knowledge of graph topology
- Many modularity optimization methods require a certain knowledge about the number of communities in the graph
- Modularity does not capture overlaps among communities in real networks (expecially in social and collaboration networks)
- Resolution limits (Fortunato, Barthelemy et al. 2006)

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Outline

Extension of modularity for overlapped communities

- Given a directed graph G(V,E) and a set of K overlapping communities of nodes of G, we can assing to each node i a vector of belonging coefficients $\alpha_{i,k}$
- $\alpha_{i,k}$ expresses how strongly node i belongs to community k, for each $k \in K$
- Without loss of generality, we can require that

$$0 \leq \alpha_{i,k} \leq 1 \forall i \in V, \quad \forall k \in K \tag{1}$$

and that

$$\sum_{k=1}^{|K|} \alpha_{i,k} = 1 \tag{2}$$

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- We assign a belonging coefficient to community k to each edge l_{ij} connecting nodes i and j. We call it β_{lk}
- We imagine that β_{lk} is a certain function of the belonging coefficients of the nodes i and j that are connected by l_{ij}

$$\beta_{l,k} = \mathcal{F}(\alpha_{i,k}, \alpha_{j,k})$$

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We can rewrite Newman's modularity distributing the factor $\delta(c_i, c_j)$ as follows:

$$Q_d = \frac{1}{m} \sum_{i,j \in V} \left[A_{ij} \delta(c_i, c_j) - \frac{k_i^{out} k_j^{in}}{m} \delta(c_i, c_j) \right] \tag{3}$$

In this case both the elements A_{ij} of the adjacency matrix and the probability of having a link between i and j in the null model are weighted by the belonging of i and j to the same community, i.e. $\delta(c_i, c_j)$

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We can simply reformulate modularity, where $\delta(c_i, c_j)$ is substituted, respectively, by two different coefficients c_{ij} and d_{ij}, obtaining:

$$Q_{ov} = \frac{1}{m} \sum_{i,j \in V} \left[c_{ij} A_{ij} - d_{ij} \frac{k_i^{out} k_j^{in}}{m} \right] \quad (4)$$

It is also possible to put in evidence the contribution to modularity given by each community, so that we can rewrite the modularity as:

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$$c_{ijk} = \beta_{l(i,j),k} = \beta_{l,k} = \mathcal{F}(\alpha_{i,k}, \alpha_{j,k}) \quad (6)$$

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$$Q_{ov} = \frac{1}{m} \sum_{k \in K} \sum_{i,j \in V} \left[c_{ijk} A_{ij} - d_{ijk} \frac{k_i^{out} k_j^{in}}{m} \right] \tag{5}$$

c_{ijk} is the conribute to the modularity of community k due to $l_{i,j}$. As said above, it corresponds to the belonging coefficient of link $l_{i,j}$ to community k:

$$c_{ijk} = \beta_{l(i,j),k} = \beta_{l,k} = \mathcal{F}(\alpha_{i,k}, \alpha_{j,k}) \tag{6}$$

A neat definition of d_{ijk} is a bit more complicated, and requires a clear definition of the *null–model* to be used as reference.

given a graph $G(E,V)$ we choose as null–model a random graph corresponding to $G(E,V)$ where each node has out–degree and in–degree as in the original graph, and where the probability that a node i belongs to a given community k with a belonging factor $\alpha_{i,k}$ does not depend upon the probability that any other node j in the network does belong to the same community with $\alpha_{j,k}$.

This is equivalent to say that the expected belonging coefficient of any possible link $l(i,j)$ starting from a node into community k is simply the average of all possible belonging coefficients of l to k, so that:

$$\beta^{out}_{l(i,j),k} = \frac{\sum_{j\in V}\mathcal{F}(\alpha_{i,k},\alpha_{j,k})}{|V|} \tag{7}$$

and conversely:

$$\beta^{in}_{l(i,j),k} = \frac{\sum_{i\in V}\mathcal{F}(\alpha_{i,k},\alpha_{j,k})}{|V|} \tag{8}$$

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As a consequence of reported considerations,

$$d_{ijk} = \beta^{out}_{l(i,j),k} \beta^{in}_{l(i,j),k}$$

and we can define modularity in the case of overlapped communities as:

$$Q_{ov} = \frac{1}{m} \sum_{k \in K} \sum_{i,j \in V} \left[\beta_{l(i,j),k} A_{ij} - \frac{\beta^{out}_{l(i,j),k} k^{out}_i \beta^{in}_{l(i,j),k} k^{in}_j}{m} \right] \tag{9}$$

We tested this definition with different $\mathcal{F}$s. Interesting results have been obtained using

$$\mathcal{F}(\alpha_{i,k}, \alpha_{j,k}) = \frac{1}{(1 + e^{-f(\alpha_{i,k})})(1 + e^{-f(\alpha_{j,k})})}$$

where f is a scaling function:

$$f(x) = 2kx - k \tag{10}$$

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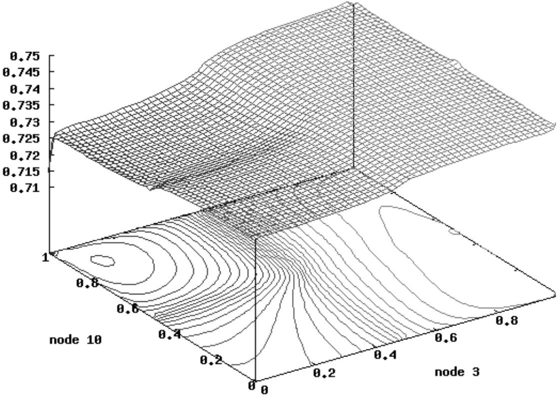
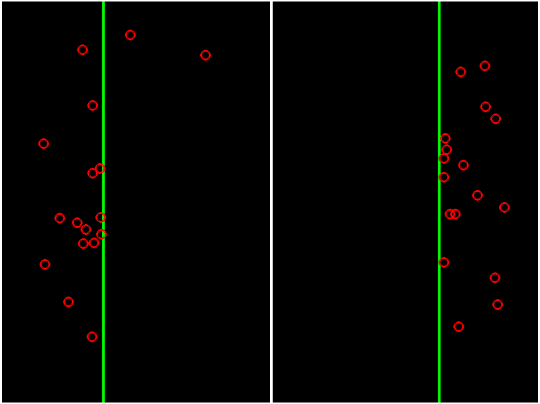
An example: the Zachary network



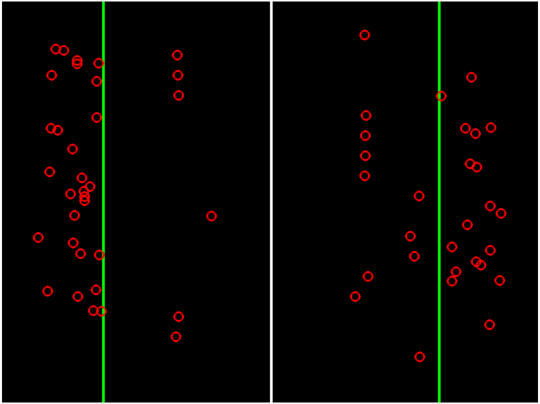
The highest value of modularity in the case of two communities is obtained when $\alpha_{3,1} = 0.81$ and $\alpha_{10,1} = 0.63$

Outline

Results: Zachary



Results: Engineering Students




"That's all Folks!"