Complex Networks - Equilibrium and Vulnerability Analysis with Applications

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Monica Milasi

Existence of equilibrium for a Walrasian pure exchange economy with utility functions: duality and Lagrangean theory

Joint research with M.B. Donato and C. Vitanza
University of Messina

Competitive economic equilibrium

Definition

Let $\overline{p} \in P$ *and* $\overline{x} \in M(\overline{p}) = \prod_{a=1}^{n} M_a(\overline{p})$. *The pair* $(\overline{p}, \overline{x})$ *is a competitive equilibrium of a pure exchange economic market with utility function if and only if:*
for all $a = 1, ..., n$

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{1}$$

and for all $j = 1, 2, ..., l$:

$$z^j(\overline{p}) = \sum_{a=1}^{n} (\overline{x}^j_a(\overline{p}) - e^j_a) \leq 0. \tag{2}$$

Characterization Theorem

Theorem

The pair $(\overline{p},\overline{x})$ is a competitive equilibrium if and only if

$$\sum_{a=1}^{n}\langle -\nabla u_a(\overline{x}_a), x_a-\overline{x}_a\rangle + \langle \sum_{a=1}^{n}(e_a-\overline{x}_a), p-\overline{p}\rangle \geq 0$$

$$\forall\,(p,x)\in P\times\prod_{a=1}^{n}M_a(\overline{p}), \tag{3}$$

where:

$$P=\left\{p\in\mathbb{R}^l_+ : \sum_{j=1}^{l}p^j=1\right\},$$

$$M_a(\overline{p})=\left\{x_a\in\mathbb{R}^l:\ \ x_a^j\geq 0\ \ \forall j=1,\dots,l,\ \sum_{j=1}^{l}\overline{p}^j(x_a^j-e_a^j)\leq 0\right\}.$$

Existence result

Theorem

- *u_a is strictly concave and $u_a \in C^1(\mathbb{R}^l_+)$,*
- $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ *and*
  $\forall x_a \in \partial M_a(p) : \dfrac{\partial u_a(x_a)}{\partial x_a^s} > 0$, *when* $x_a^s = 0, \quad \forall p \in P$,
- $-\nabla u_a(x_a)$ *is strongly monotone:* $\exists \nu > 0$ *such that*
  $\langle -\nabla u_a(x_a) + \nabla u_a(y_a), x_a - y_a \rangle \geq \nu ||x_a - y_a||^2 \; \forall x_a, \; y_a \in M_a(p)$.

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Then there exists $(\overline{p}, \overline{x})$ *solution to quasi-variational inequality:*

$$\sum_{a=1}^{n} \langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle + \langle \sum_{a=1}^{n} (e_a - \overline{x}_a(\overline{p})), p - \overline{p} \rangle \geq 0 \,,$$

$$\forall \, (p, x) \in P \times M(\overline{p}).$$

Sketch of Proof

For all $p \in P$ and for all $a = 1, \dots, n$, we consider the variational inequality:

$$\langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \geq 0, \quad \forall x_a \in M_a(p)\,, \tag{4}$$

$$M_a(p) = \left\{ x_a \in \mathbb{R}^l : \quad x_a^j \geq 0 \;\; \forall j = 1, \dots, l, \sum_{j=1}^{l} p^j (x_a^j - e_a^j) \leq 0 \right\}.$$

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Since the operator is strongly monotone there exists a unique solution $\overline{x}_a$ to variational inequality (4).
We define the demand function:

$$p \longrightarrow \overline{x}_a\,.$$

Then we consider:

$$\langle \sum_{a=1}^{n} (e_a - \overline{x}_a(p)), p - \overline{p} \rangle \geq 0, \quad \forall p \in P\,. \tag{5}$$

Since P is a closed and bounded set, if we prove that, for all $a = 1, ..., n$, $\overline{x}_a(\cdot)$ is a continuous function on P, then variational inequality (5) admits at least a solution $\overline{p} \in P$.

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For all $n \in \mathbb{N}$, let us:

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The sequence $\{M_a(p_n)\}$ converges to $M_a(p)$ in Mosco's sense:

(M_1) for any $x_a \in M_a(p)$ there exists a sequence $\{(x_a)_n\}_{n\in\mathbb{N}}$ strongly converging to x_a such that $(x_a)_n \in M_a(p_n)$ for all n,

(M_2) for any $\{(x_a)_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to x_a, such that $(x_a)_{k_n} \in M_a(p_{k_n})$, then the weak limit x_a belongs to $M_a(p)$.

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Donato, M., Vitanza, "Quasi-variational inequality approach of a competitive economic equilibrium problem with utility function", *Mathematical Modelds and Methods in Applied Sciences*, 2008.

Second Step

$$\overline{x}_a(p_n) \longrightarrow \overline{x}_a(p),$$

where, for all $n \in \mathbb{N}$:

$$\langle -\nabla u_a(\overline{x}_a(p_n)), x_a - \overline{x}_a(p_n)\rangle \geq 0, \qquad \forall x_a \in M_a(p_n). \tag{6}$$

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From (M_1) there exists $\{(y_a)_n\}_{n\in\mathbb{N}}$ such that:

$$(y_a)_n \in M_a(p_n)\,, \qquad \lim_{n\to+\infty}(y_a)_n = \overline{x}_a(p). \tag{7}$$

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In (6) we replace x_a by $(y_a)_n$ and from strongly monotonicity of the operator of inequality (6) it follows:

$$||\overline{x}_a(p_n) - y_a(p_n)|| \leq \frac{||-\nabla u_a(y_a(p_n))||}{\nu}. \tag{8}$$

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Since $\{(-\nabla u_a(y_a))_n\}$ and $\{(y_a)_n\}$ are converging, by (8) there exist k and h such that:

$$||\overline{x}_a(p_n)|| \leq \frac{||-\nabla u_a(y_a(p_n))||}{\nu} + ||y_a(p_n)|| \leq \frac{h}{\nu} + k\,.$$

Hence there exists a subsequence $\{\overline{x}_a(p_{k_n})\}$ of $\{\overline{x}_a(p_n)\}$ such that:

$$\lim_{n\to+\infty} \overline{x}_a(p_{k_n}) = y_a\,.$$

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It results that y_a is a solution to variational inequality:

$$\langle -\nabla u_a(y_a), x_a - y_a \rangle \geq 0, \quad \forall x_a(p) \in M_a(p). \tag{9}$$

In fact: from condition (M_2) it follows that $y_a \in M_a(p)$.

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In fact: from condition (M_2) it follows that $y_a \in M_a(p)$.
By (M_1): for all $x_a \in M_a(p)$:

$$\exists\{(x_a)_n\} : \ (x_a)_n \in M_a(p_n) \ , \ \lim_{n\to+\infty}(x_a)_n = x_a \,.$$

For all $n \in \mathbb{N}$:

$$\langle -\nabla u_a(\overline{x}_a(p_{k_n})), (x_a)_{k_n} - \overline{x}_a(p_{k_n})\rangle \geq 0 \,, \tag{10}$$

by taking $n \to +\infty$ we get (9).
By the uniqueness of solution to (9) we have: $y_a = \overline{x}_a$.

Then every subsequence of $\{\overline{x}_a(p_n)\}$ converges to the same limit $\overline{x}_a(p)$. Hence

$$\lim_{n \to +\infty} \overline{x}_a(p_n) = \overline{x}_a(p).$$

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$$\lim_{n\to+\infty} \overline{x}_a(p_n) = \overline{x}_a(p).$$

The demand function $\overline{x}_a(p)$ is continuous on P, then there exists $\overline{p}$ solution to variational inequality:

$$\langle \sum_{a=1}^{n} (e_a - \overline{x}_a(p)), p - \overline{p} \rangle \geq 0, \quad \forall p \in P\,.$$

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So there exists $(\overline{p}, \overline{x}(\overline{p}))$ solution to quasi-variational inequality; then there exists a Walrasian competitive equilibrium.

Some Remarks on Basic Assumptions

Jofre A., Rockafellar R.T., and Wets R.J.-B., *A Variational Inequality Scheme for Determing an Economic Equilibrium of Classical or Extended Type*, Variational Analysis and Applications, 2005.

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- demand function:

$$p \longrightarrow \overline{x}_a(p)\,.$$

$$\Rightarrow \sum_{a=1}^{n} \langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle + \langle \sum_{a=1}^{n} (e_a - \overline{x}_a), p - \overline{p} \rangle \geq 0$$

$$\forall\ (p, x) \in P \times \prod_{a=1}^{n} M_a(\overline{p})\,.$$

- $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ and
  $\forall x_a \in \partial M_a(p) : \dfrac{\partial u_a(x_a)}{\partial x_a^s} > 0$, when $x_a^s = 0, \quad \forall p \in P$,

  $\Rightarrow$ Walras' law:
  $$\langle p, \overline{x}_a(p) - e_a \rangle = 0\,.$$

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  $\Rightarrow$ Walras' law:
  $$\langle p, \overline{x}_a(p) - e_a \rangle = 0 \,.$$

- Each agent is endowed with a positive quantity of at least one commodity:
  $$\forall a = 1, ..., n \quad \exists j : \quad e_a^j > 0.$$

Lagrangean theory

Theorem

$(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium if and only if for all $a = 1, ..., n$ there exist:

$$\overline{\alpha}_a \in \mathbb{R}^l_+, \quad \overline{\beta}_a \in \mathbb{R}_+ \setminus \{0\}, \quad \overline{\gamma} \in \mathbb{R}^l_+,$$

such that

i) $\langle \overline{\alpha}_a, \overline{x}_a \rangle = 0, \quad \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0, \quad \langle \overline{\gamma}, \overline{p} \rangle = 0;$

ii) $$\begin{cases} \dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}^j_a \qquad \forall j = 1, .., l; \\ \displaystyle\sum_{a=1}^{n} (e^j_a - \overline{x}^j_a) = \overline{\gamma}^j \qquad \forall j = 1, ..., l \\ \overline{\beta}_a = \displaystyle\sum_{j=1}^{l} \left(\dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} + \overline{\alpha}^j_a \right). \end{cases}$$

Sketch of Proof

Let $(\overline{p}, \overline{x})$ be a competitive equilibrium.

First step

$\overline{x}_a$ is a solution to $\quad \langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \geq 0 \qquad \forall x_a \in M_a(\overline{p})\,,$

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then $\overline{x}_a$ is a solution to optimization problem:

$$\min_{x_a \in M_a(\overline{p})} \Phi_{\overline{x}_a}(x_a) = \Phi_{\overline{x}_a}(\overline{x}_a) = 0$$

where $\quad \Phi_{\overline{x}_a}(x_a) = \langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \qquad \forall x_a \in \mathbb{R}^l$,

$$M_a(\overline{p}) = \left\{ x_a \in \mathbb{R}^l : \;\; x_a^j \geq 0 \;\; \forall j = 1, \ldots, l, \;\; \sum_{j=1}^{l} \overline{p}^j (x_a^j - e_a^j) \leq 0 \right\}.$$

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We associate the Lagrangean function:

$$\mathcal{L}_a^{(1)} : \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}$$

$$\mathcal{L}_a^{(1)}(x_a, \alpha_a, \beta_a) = \Phi_{\overline{x}_a}(x_a) - \langle \alpha_a, x_a \rangle - \beta_a \langle \overline{p}, e_a - x_a \rangle$$

Theorem

Let us $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be real normal spaces; $\widehat{S} \subseteq X$, $f : \widehat{S} \to \mathbb{R}$ $g : \widehat{S} \to Y$ such that (f, g) convex-like; let the ordering cone C of Y be closed. If $\overline{x} \in S$ is a minimal solution of the primal problem

$$\min_{x \in S} f(x). \tag{11}$$

where $S = \{x \in \widehat{S} : g(x) \in -C\} \neq \emptyset$
and the generalized Slater condition is satisfied, i.e. there is a vector $\widehat{x} \in \widehat{S}$ with $g(\widehat{x}) \in -intC$, then there exists $\overline{u} \in C^*$ such that $(\overline{x}, \overline{u})$ is a saddle point of the Lagrange functional associated to the problem (11):

$$\mathcal{L}(\overline{x}, u) \leq \mathcal{L}(\overline{x}, \overline{u}) \leq \mathcal{L}(x, \overline{u}), \quad \forall x \in \widehat{S},\ u \in C^*. \tag{12}$$

Jahn J., **Introduction to the theory of nonlinear optimization**, Springer-Verlag Berlin Heidelberg, New York, 1996.

By Lagrangean Theory there exist $(\overline{\alpha}_a, \overline{\beta}_a) \in \mathbb{R}^l_+ \times \mathbb{R}_+$ s. t.:

$$\mathcal{L}_a^{(1)}(\overline{x}_a, \alpha_a, \beta_a) \leq \mathcal{L}_a^{(1)}(\overline{x}_a, \overline{\alpha}_a, \overline{\beta}_a) \leq \mathcal{L}_a^{(1)}(x_a, \overline{\alpha}_a, \overline{\beta}_a)$$

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We deduce:

$$\langle \overline{\alpha}_a, \overline{x}_a \rangle = 0 \qquad \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0$$

$$\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}^j_a \qquad \forall j = 1, .., l;$$

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$$\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}^j_a \qquad \forall j = 1, .., l;$$

And, since $\sum_{j=1}^{l} \overline{p}^j = 1$, it results:

$$\overline{\beta}_a = \sum_{j=1}^{l} \left(\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} + \overline{\alpha}^j_a \right) \qquad \overline{\beta}_a \neq 0 \, .$$

Second step

$\overline{p} \in P$ is a solution to $$\langle \sum_{a=1}^{n} (e_a - \overline{x}_a(\overline{p})), p - \overline{p} \rangle \geq 0, \quad \forall p \in P,$$

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then $\overline{p}$ is a solution to optimization problem:

$$\min_{p \in P} \Psi_{\overline{p}}(p) = \Psi_{\overline{p}}(\overline{p}) = 0$$

where $\quad \Psi_{\overline{p}}(p) = \langle \sum_{a=1}^{n} (e_a - \overline{x}_a), p - \overline{p} \rangle \qquad \forall p \in \mathbb{R}^l$,

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$$P = \left\{ p \in \mathbb{R}^l_+ : \sum_{j=1}^{l} p^j = 1 \right\}.$$

We associate the Lagrangean function:

$$\mathcal{L}_a^{(2)} : \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}$$

$$\mathcal{L}^{(2)}(p, \gamma, \delta) = \Psi_{\overline{p}}(p) - \langle \gamma, p \rangle - \delta \Big(\sum_{j=1}^{l} p^j - 1 \Big)$$

By Lagrangean Theory there exist $(\overline{\gamma}, \overline{\delta}) \in \mathbb{R}^l_+ \times \mathbb{R}_+$ such that:

$$\mathcal{L}^{(2)}(\overline{p}, \gamma, \delta) \leq \mathcal{L}^{(2)}(\overline{p}, \overline{\gamma}, \overline{\delta}) \leq \mathcal{L}^{(2)}(p, \overline{\gamma}, \overline{\delta}).$$

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$$\mathcal{L}^{(2)}(\overline{p}, \gamma, \delta) \leq \mathcal{L}^{(2)}(\overline{p}, \overline{\gamma}, \overline{\delta}) \leq \mathcal{L}^{(2)}(p, \overline{\gamma}, \overline{\delta}).$$

Since $\overline{p} \in P$, we deduce:

$$\langle \gamma, p \rangle = 0 \qquad \overline{\gamma}^j = \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) - \overline{\delta}, \qquad \forall j = 1, ..., l.$$

By Lagrangean Theory there exist $(\overline{\gamma}, \overline{\delta}) \in \mathbb{R}^l_+ \times \mathbb{R}_+$ such that:

$$\mathcal{L}^{(2)}(\overline{p}, \gamma, \delta) \leq \mathcal{L}^{(2)}(\overline{p}, \overline{\gamma}, \overline{\delta}) \leq \mathcal{L}^{(2)}(p, \overline{\gamma}, \overline{\delta}) \,.$$

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Moreover, by Walras' law it follows:

$$0 = \langle \overline{\gamma}, \overline{p} \rangle = \sum_{a=1}^{n} \sum_{j=1}^{l} \overline{p}^j (e^j_a - \overline{x}^j_a) - \overline{\delta} \sum_{j=1}^{l} \overline{p}^j = \overline{\delta} \,,$$

then:

$$\overline{\gamma}^j = \sum_{a=1}^{n} (e^j_a - \overline{x}^j_a), \;\; \forall j = 1, ..., l, \qquad \overline{\delta} = 0 \,.$$

Moreover we get:

- for all $a = 1, ..., n$ there exists j^* such that $\overline{\alpha}_a^{j^*} = 0$
- for $j = 1, ..., l$ there exists a^* such that $\overline{\alpha}_{a^*}^{j} = 0$.

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Conversely, if $\overline{x}, \overline{p}, \overline{\alpha}_a$, $\overline{\beta}_a$, $\overline{\gamma}$, with $\overline{\alpha}_a \in \mathbb{R}^l_+$, $\overline{\beta}_a \in \mathbb{R}_+ \setminus \{0\}$, $\overline{\gamma} \in \mathbb{R}^l_+$, are such that:

i) $\langle \overline{\alpha}_a, \overline{x}_a \rangle = 0, \quad \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0, \quad \langle \overline{\gamma}, \overline{p} \rangle = 0;$

ii)
$$\begin{cases} \dfrac{\partial u_a(\overline{x}_a)}{\partial x_a^j} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}_a^j \qquad \forall j = 1, .., l; \\ \displaystyle\sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) = \overline{\gamma}^j \qquad \forall j = 1, ..., l \\ \overline{\beta}_a = \displaystyle\sum_{j=1}^{l} \left(\frac{\partial u_a(\overline{x}_a)}{\partial x_a^j} + \overline{\alpha}_a^j \right). \end{cases}$$

then $(\overline{x}, \overline{p})$ is a competitive equilibrium.

Remarks

The importance of the multipliers α^*, γ^*, derives from the fact that they are able to describe the behavior of the market:

if there exist a^* and j^* such that:

- $\overline{\alpha}_{a^*}^{j^*} > 0$ it follows: $\overline{x}_{a^*}^{j^*} = 0$,

$$\frac{\partial u_{a^*}(\overline{x}_{a^*})}{\partial x_{a^*}^{j^*}} > 0, \qquad \overline{p}^{j^*} > 0$$

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- $\overline{\gamma}^{j^*} > 0$ it follows: $\overline{p}^{j^*} = 0, \qquad \sum_{a=1}^{n} e^j_a > \sum_{a=1}^{n} \overline{x}^j_a$

  and for all $a = 1, ..., n$:

$$\overline{x}^{j^*}_a > 0, \qquad \frac{\partial u_a(\overline{x}_a)}{\partial x^{j^*}_a} = 0$$

Example

We consider a pure exchange economy with two agents ($a = 1, 2$) and two goods ($j = 1, 2$).

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Each agent has an utility function:

$$u_1(x_1) = -\frac{1}{2}(x_1^1)^2 - \frac{1}{2}(x_1^2)^2 + 4x_1^1 + 8x_1^2,$$

$$u_2(x_2) = -\frac{1}{2}(x_2^1)^2 - \frac{1}{2}(x_2^2)^2 + 12x_2^1 + 5x_2^2,$$

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and an endowment:

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Our aim is to find a competitive equilibrium $(\overline{x}, \overline{p})$ by using the characterization with the Lagrangean multipliers.

We find $\overline{p}$, $\overline{x}$, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ such that:

$$\overline{\alpha}_a \in \mathbb{R}^l_+, \quad \overline{\beta}_a \in \mathbb{R}_+ \setminus \{0\}, \quad \overline{\gamma} \in \mathbb{R}^l_+,$$

and

$$i) \ \langle \overline{\alpha}_a, \overline{x}_a \rangle = 0, \quad \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0, \quad \langle \overline{\gamma}, \overline{p} \rangle = 0;$$

$$ii) \ \begin{cases} \dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}^j_a & \forall j = 1, .., l; \\ \displaystyle\sum_{a=1}^{n} (e^j_a - \overline{x}^j_a) = \overline{\gamma}^j & \forall j = 1, ..., l \\ \overline{\beta}_a = \displaystyle\sum_{j=1}^{l} \Big(\dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} + \overline{\alpha}^j_a \Big). \end{cases}$$

From conditions:

$$\frac{\partial u_a(\overline{x}_a)}{\partial x_a^j} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}_a^j, \qquad \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) = \overline{\gamma}^j \tag{13}$$

it follows:

$$\begin{cases} (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^1 = \overline{\gamma}^1 + (\overline{\alpha}_1^1 + \overline{\alpha}_2^1) + 6 \\ (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^2 = \overline{\gamma}^2 + (\overline{\alpha}_1^2 + \overline{\alpha}_2^2) + 7 \end{cases} \tag{14}$$

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$$\begin{cases} (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^1 = \overline{\gamma}^1 + (\overline{\alpha}_1^1 + \overline{\alpha}_2^1) + 6 \\ \\ (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^2 = \overline{\gamma}^2 + (\overline{\alpha}_1^2 + \overline{\alpha}_2^2) + 7 \end{cases} \tag{14}$$

By (14) it results $\overline{\beta}_1 + \overline{\beta}_2 > 0,\ \overline{p}^1 > 0,\ \overline{p}^2 > 0$. Since $\langle \overline{\gamma}, \overline{p} \rangle = 0$ we have $\overline{\gamma}^1 = \overline{\gamma}^2 = 0$ and

$$\begin{cases} \overline{p}^1 = \dfrac{1}{(\overline{\beta}_1 + \overline{\beta}_2)} (\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) \\ \\ \overline{p}^2 = \dfrac{1}{(\overline{\beta}_1 + \overline{\beta}_2)} (\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) \end{cases} \tag{15}$$

From (15) and (13) we have:

$$\begin{cases} \overline{x}_1^1 = 4 - \dfrac{\overline{\beta}_1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) + \overline{\alpha}_1^1 \\ \overline{x}_1^2 = 8 - \dfrac{\overline{\beta}_1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) + \overline{\alpha}_1^2 \\ \overline{x}_2^1 = 12 - \dfrac{\overline{\beta}_2}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) + \overline{\alpha}_2^1 \\ \overline{x}_2^2 = 5 - \dfrac{\overline{\beta}_2}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) + \overline{\alpha}_2^2 \end{cases} \tag{16}$$

From (16) by considering the system

$$\overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0 \quad \forall a = 1, 2\,,$$

we have that $\overline{\alpha}_a^j = 0$ for all $a = 1, 2,\ j = 1, 2$ and $\overline{\beta}_1 = \dfrac{34}{51}\overline{\beta}_2$.

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Moreover from

$$\overline{\beta}_a = \sum_{j=1}^{l} \left(\frac{\partial u_a(\overline{x}_a)}{\partial x_a^j} + \overline{\alpha}_a^j \right)$$

we have:

$$\overline{\beta}_1 = \frac{26}{5},\ \overline{\beta}_2 = \frac{39}{5}\,.$$

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we have:

$$\overline{\beta}_1 = \frac{26}{5},\ \overline{\beta}_2 = \frac{39}{5}\,.$$

Hence the competitive equilibrium $(\overline{p}, \overline{x})$ is:

$$\overline{p} = \left(\frac{6}{13}, \frac{7}{13} \right),\ \overline{x}_1 = \left(\frac{9}{5}, \frac{26}{5} \right),\ \overline{x}_2 = \left(\frac{42}{5}, \frac{4}{5} \right).$$

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