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A competitive equilibrium for a pure exchange Walrasian economy: a quasi-variational approach

joint research with M. Milasi and C. Vitanza

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The Walrasian pure exchange model

We consider a marketplace consisting of l different goods, indexed by $j = 1, .., l$ and n agents, indexed by $a = 1, ..., n$.
Each agent a has an initial endowment vector:

$$e_a = (e_a^1, e_a^2, ..., e_a^l) \in \mathbb{R}_+^l,$$

where e_a^j is the initial endowment relative to the commodity j and he chooses the consumption vector

$$x_a = (x_a^1, x_a^2, ..., x_a^l) \in \mathbb{R}_+^l,$$

where x_a^j is the consumption relative to the commodity j.

$$x \equiv (x_1, x_2, ..., x_n)^T \in \mathbb{R}_+^{nl}$$

represents the consumption of market.

Utility function

To each commodity $j = 1, 2, ..., l$ is associated a real positive number p^j representing its price.

$p = (p^1, p^2, ..., p^l) \in \mathbb{R}^l_+$ represents the price vector.

Utility function

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$$p = (p^1, p^2, ..., p^l) \in \mathbb{R}^l_+ \text{ represents the price vector.}$$

Consumptions vector x_a is rated by agent a according his utility, which is described by a function

$$u_a : \mathbb{R}^l_+ \to \mathbb{R}$$

$$x_a \to u_a(x_a).$$

For all $a = 1, ..., n$ and for all $p \in P$:

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(p)} u_a(x_a),$$

where

$$M_a(p) = \left\{ x_a \in \mathbb{R}^l : \;\; x_a^j \geq 0 \;\; \forall j = 1, ..., l, \;\; \sum_{j=1}^{l} p^j x_a^j \leq \sum_{j=1}^{l} p^j e_a^j \right\}$$

and

$$P = \left\{ p \in \mathbb{R}_+^l : \sum_{j=1}^{l} p^j = 1 \right\}.$$

$M_a(p)$ is a closed and convex set of $\mathbb{R}_+^l$ for each $a = 1, .., n$ and $p \in P$.

Basic assumptions

For all $a = 1, ..., n$

(U_1) u_a is strictly concave;

(U_2) $u_a \in C^1(\mathbb{R}^l_+)$;

(U_3) $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ and

$$\forall x_a \in \partial M_a(p) : \frac{\partial u_a(x_a)}{\partial x_a^s} > 0, \text{ when } x_a^s = 0, \quad \forall p \in P;$$

(U_4) $lim_{\substack{||x_a|| \to +\infty \\ x_a \in M_a(p)}} u_a(x_a) = -\infty, \quad \forall p \in P$;

(U_5) each agent is endowed with a positive quantity of at least one commodity:

$$\forall a = 1, ..., n \quad \exists j : \quad e_a^j > 0.$$

Demand function

Under basic assumptions, for all $a = 1, ..., n$, the maximization problem:

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(p)} u_a(x_a)$$

has a unique solution for each $p \in P$, then it arises a function:

$$\overline{x}_a : P \longrightarrow \mathbb{R}^l_+$$
$$p \longrightarrow \overline{x}_a(p)$$

called demand function.

Excess demand function

We define, now, a particular aggregate excess demand function:

$$z^j : \mathbb{R}^l_+ \to \mathbb{R}, \quad j = 1, 2, .., l$$

$$p \to z^j(p) = \sum_{a=1}^{n} (\overline{x}^j_a(p) - e^j_a)$$

where $\overline{x}^j_a(p) - e^j_a$ is the individual excess demand of the agent a relative to the commodity j.
Grouping this components in a vector we introduce:

$$z(p) = (z^1(p), z^2(p), ..., z^l(p)) \in \mathbb{R}^l.$$

Walras' law

For all $a = 1, ..., n$ and for all $p \in P$, let $\overline{x}_a$ be a maximal point of utility function $u_a(x_a)$ in $M_a(p)$. By assumption:

(U_3) $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ and

$$\forall x_a \in \partial M_a(p) : \frac{\partial u_a(x_a)}{\partial x_a^s} > 0, \text{ when } x_a^s = 0, \quad \forall p \in P,$$

it results

$$\overline{x}_a \in \Gamma_a(p) = \left\{ x_a \in \mathbb{R}^l : x_a^j \geq 0 \, \forall j = 1, ..., l, \sum_{j=1}^{l} p^j (x_a^j - e_a^j) = 0 \right\}.$$

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it results

$$\overline{x}_a \in \Gamma_a(p) = \left\{ x_a \in \mathbb{R}^l : x_a^j \geq 0 \, \forall j = 1, ..., l, \sum_{j=1}^{l} p^j (x_a^j - e_a^j) = 0 \right\}.$$

Proof. Assumption (U_3) shows that there exists $x_a' \in \mathbb{R}_+^l$ s. t.:

$$u_a(x_a') > u_a(\overline{x}_a). \quad (1)$$

We suppose, ab absurdum, that $\sum_{j=1}^{l} p^j (\overline{x}_a^j(p) - e_a^j) < 0.$

By condition (1) and being $u_a(x_a)$ strictly concave, we have:

$$u_a[(tx'_a) + (1-t)\overline{x}_a] > u_a(\overline{x}_a) \quad \forall t \in [0,1]. \tag{2}$$

It follows that, for all $\delta > 0$, there exists $t \in [0,1]$ such that:

$$tx'_a + (1-t)\overline{x}_a$$

belongs to the neighborhood of $\overline{x}_a$ and radius δ.

Moreover, there exists $\delta < -\sum_{j=1}^{l} p^j(\overline{x}_a^j - e_a^j)$ such that:

$$\text{for all } \; t < \frac{\delta}{||x'_a - \overline{x}_a||}, \; tx'_a + (1-t)\overline{x}_a \in M_a(p). \tag{3}$$

But the conditions (2) and (3) contradict that $\overline{x}_a$ is a maximal point for utility function in $M_a(p)$. Hence, we can deduce that, for all a and for all $p \in P$ the maximal point $\overline{x}_a$ satisfies:

$$\sum_{j=1}^{l} p^j(\overline{x}_a^j(p) - e_a^j) = 0 \quad \textbf{(Walras'law)}.$$

Competitive equilibrium conditions

Definition

Let $\overline{p} \in P$ *and* $\overline{x} \in M(\overline{p}) = \prod_{a=1}^{n} M_a(\overline{p})$. *The pair* $(\overline{p}, \overline{x})$ *is a competitive equilibrium of a pure exchange economic market with utility function if and only if:*

for all $a = 1, ..., n$

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{4}$$

and for all j=1,2,...,l

$$z^j(\overline{p}) = \sum_{a=1}^{n} (\overline{x}_a^j(\overline{p}) - e_a^j) \leq 0. \tag{5}$$

For the sake of brevity in the sequel we will write $\overline{x}$ instead of $\overline{x}(\overline{p})$.

For the sake of brevity in the sequel we will write $\overline{x}$ instead of $\overline{x}(\overline{p})$. We can observe that conditions

i)

$$z^j(\overline{p}) = \sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \leq 0 \quad \forall j = 1, ..., l \tag{6}$$

and

ii)

$$\sum_{j=1}^{l} p^j(\overline{x}_a^j(p) - e_a^j) = 0 \quad \forall p \in P, \ \forall a = 1, ..., n \tag{7}$$

are equivalent to

$$\sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \begin{cases} \leq 0 & \textit{if} \quad \overline{p}^j = 0 \\ = 0 & \textit{if} \quad \overline{p}^j > 0 \end{cases} \quad \forall j = 1, ..., l. \tag{8}$$

Definition

A competitive equilibrium of a pure exchange economic market with utility function consists of a competitive equilibrium price vector $\overline{p} \in P$ *and a consumption vector* $\overline{x} \in \mathbb{R}_+^{nl}$ *such that:*

a) *for all* $a = 1, ..., n$, $\overline{x}_a$ *is a solution to*

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a) \tag{9}$$

and

$$\sum_{j=1}^{l} \overline{p}^j (\overline{x}_a^j - e_a^j) = 0 \qquad \textit{(Walras' law)}. \tag{10}$$

b) *For all* $j = 1, ..., l$:

$$\sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \begin{cases} \leq 0 & \text{if} \quad \overline{p}^j = 0 \\ = 0 & \text{if} \quad \overline{p}^j > 0. \end{cases} \tag{11}$$

Quasi-variational formulation

We are able to characterize a competitive equilibrium of a pure exchange economic market as a solution to the following quasi-variational inequality:

$$\text{"}Find\ (\overline{p}, \overline{x}) \in P \times M(\overline{p})\ such\ that:$$

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \rangle + \sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \leq 0$$

$$\forall\ (p, x) \in P \times M(\overline{p})\text{"}.$$

Theorem

The pair $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium if and only if is a solution to quasi-variational inequality

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \rangle + \sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \leq 0$$

$$\forall \, (p, x) \in P \times M(\overline{p}).$$

We observe that $(\overline{p}, \overline{x})$ is a solution to the quasi-variational inequality if and only if $\overline{p}$ is a solution to:

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \rangle \leq 0 \qquad \forall p \in P. \tag{12}$$

and, for all $a = 1, ..., n$, $\overline{x}_a$ is a solution to:

$$\langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \leq 0 \qquad \forall x_a \in M_a(\overline{p}). \tag{13}$$

Proof. First, we prove that if $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ satisfies equilibrium conditions then $(\overline{p}, \overline{x})$ is a solution to quasi-variational inequality.
By Walras'law the variational inequality (12) is equivalent to

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p \rangle \leq 0 \qquad \forall p \in P. \tag{14}$$

Since $\overline{p}$ verifies $z^j(\overline{p}) = \sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \leq 0, \ \forall j = 1, ..., l$ and $p \in P$, then variational inequality (12) holds true.
Taking into account that the maximization problem

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \ \forall a = 1, ..., n$$

is equivalent to variational inequality (13), $\overline{x}_a$ is a solution to variational inequality (13).
Hence, $(\overline{p}, \overline{x})$ is a solution to quasi-variational inequality.

Conversely, we suppose that $\overline{p}$ is a solution to variational inequality (12) and $\overline{x}_a$ is solution to variational inequality (13).
Analogously if $\overline{x}_a$ is a solution to variational inequality (13), it is a maximal point of utility function in $M_a(p)$. In variational inequality

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p \rangle \leq 0 \qquad \forall p \in P,$$

selecting $p = (0, ..., 0, 1, 0, ..., 0)$ with 1 at the j-th position, we have:

$$z^j(\overline{p}) = \sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \leq 0 \quad \forall j = 1, ..., l.$$

Then, $(\overline{p}, \overline{x})$ is a competitive equilibrium of a pure exchange economic market.

A dynamic competitive equilibrium problem

During a period of time $[0, T]$, $T > 0$, we consider a marketplace consisting of l different goods indexed by $j = 1, .., l$ and n agents indexed by $a = 1, ..., n$.
At the time $t \in [0, T]$ each agent a has an initial endowment:

$$e_a(t) = (e_a^1(t), e_a^2(t), ..., e_a^l(t)),$$

where $e_a^j(t)$ represents the endowment, at the time t, relative to the commodity j.
We assume that each agent a is endowed with a positive quantity of at least one commodity:

$$\forall a = 1, ..., n, \quad \exists j : e_a^j(t) > 0 \;\; a.\,e.\; in\, [0, T].$$

At the time t each agent a will choose a consumption vector:

$$x_a(t) = (x_a^1(t), x_a^2(t), ..., x_a^l(t)),$$

where $x_a^j(t)$ represents the consumption, at the time t, relative to the commodity j, and

$$x(t) \equiv (x_1(t), x_2(t), ..., x_n(t))^T$$

represents the consumption of market at the time t.
At the time t, the price of goods j is denoted by $p^j(t)$ and the market is governed by a price vector

$$p(t) = (p^1(t), p^2(t), ..., p^l(t)).$$

We assume that vectors:

$$e_a, x_a, p \in L^2([0, T], \mathbb{R}^l) = L$$

$$x \in L^2([0, T], \mathbb{R}^{nl})$$

Consumption vector x_a is rated by agent a, at the instant t, according his utility which is described by a function $u_a(t, x_a(t))$.

For all $a = 1, ..., n$ and for all $p \in \overline{P}$:

$$\mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(p)} \int_0^T u_a(t, x_a(t))\, dt,$$

where

$$M_a(p) = \left\{ x_a \in L : \quad x_a^j(t) \geq 0 \ \ \forall j \ \ a.\,e.\, in\, [0, T], \right.$$

$$\left. \int_0^T \langle p, x_a \rangle_l dt \leq \int_0^T \langle p, e_a \rangle_l dt \right\}$$

and

$$p \in \overline{P} = \Bigg\{ p \in L : p^j(t) \geq 0 \ \ \forall j, \ \ \sum_{j=1}^{l} p^j(t) = 1 \, a.\, e.\, in \, [0, T],$$

$$\lim_{|h| \to 0} \int_0^T |p(t+h) - p(t)|^2 dt = 0 \ \ uniformly \ \ in \ \ p \Bigg\}.$$

The regularity condition

$$\lim_{|h| \to 0} \int_0^T |p(t+h) - p(t)|^2 dt = 0 \ \ uniformly \ \ in \ \ p \qquad (15)$$

can be interpreted as the uniform integral continuity of prices and it is assumed in order to describe and analyze the dynamic model.

Assumptions on utility function

For each agent $a = 1, ..., n$:

(U_1) $u_a(t, x_a(t))$ is strictly concave a. e. in $[0, T]$;

(U_2) $u_a(t, \cdot) \in C^1(\mathbb{R}^l_+)$ for almost all $t \in [0, T]$;

(U_3) $\forall p \in P \ \ \forall x_a \in M_a(p)$, $\dfrac{\partial u_a(t, x_a(t))}{\partial x_a^j} \neq 0$ a. e. in $[0, T]$, $\forall j = 1, ..., l$ and $\forall x_a \in \partial M_a(p)$ $\dfrac{\partial u_a(t, x_a(t))}{\partial x_a^s} > 0$, a. e. in $[0, T]$ when $x_a^s(t) = 0$ a. e. in $[0, T]$;

(U_4) $lim_{\substack{||x_a||_L \to +\infty \\ x_a \in M_a(p)}} u_a(x_a) = -\infty$.

In our assumptions, for all $a = 1, \dots, n$, and for each $p \in P$ the maximization problem:

$$\mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(p)} \int_0^T u_a(t, x_a(t))\, dt$$

has a unique solution $\overline{x}_a \in M_a(p)$, Then it arises the demand function:

$$\overline{x}_a : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l_+$$

$$(t, p(t)) \to \overline{x}_a(t, p(t)).$$

We define, for all $j = 1, \dots, l$, the aggregate excess demand function:

$$z^j(t, p(t)) = \sum_{a=1}^{n} (\overline{x}^j_a(t, p(t)) - e^j_a(t)).$$

Walras' law

Assumption (U_3): $\forall p \in P$, $\forall x_a \in M_a(p)$ $\dfrac{\partial u_a(t, x_a(t))}{\partial x_a^j} \neq 0$ a. e. in $[0, T]$, $\forall j = 1, ..., l$ and $\forall x_a \in \partial M_a(p)$ $\dfrac{\partial u_a(t, x_a(t))}{\partial x_a^s} > 0$, a. e. in $[0, T]$ when $x_a^s(t) = 0$ a. e. in $[0, T]$;

$$\Downarrow$$

there exists $x_a' \in C = \{x_a \in L : x_a(t) \geq 0 \, a.\, e.\, in \, [0, T]\}$ such that:

$$\int_0^T u_a(t, x_a'(t))dt > \int_0^T u_a(t, \overline{x}_a(t))dt.$$

$$\Downarrow$$

$$\int_0^T \langle p(t), \overline{x}_a(t) - e_a(t)\rangle_l dt = 0 \quad \forall p \in P \quad (\textbf{Walras' law}).$$

Definition

Let $\overline{p} \in P$ and $\overline{x}(\overline{p}) \in M(\overline{p}) = \Pi_{a=1}^{n} M_a(\overline{p})$.
The pair $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ is a dynamic competitive equilibrium if and only if:

for all $a = 1, ..., n$

$$\mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(p)} \int_0^T u_a(t, x_a(t))dt, \tag{16}$$

and for all j=1,2,...,l and a. e. in $[0, T]$:

$$\sum_{a=1}^{n} (\overline{x}_a^j(t, \overline{p}(t)) - e_a^j(t)) \leq 0. \tag{17}$$

We characterize a dynamic competitive equilibrium as a solution to an evolutionary quasi-variational inequality:

Theorem

The pair $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a dynamic competitive equilibrium of a pure exchange economic market with utility function if and only if is a solution to evolutionary quasi-variational inequality:

$$\int_0^T \langle \sum_{a=1}^{n} (\overline{x}_a(t, \overline{p}(t)) - e_a(t)), (p(t) - \overline{p}(t)) \rangle dt$$

$$+ \sum_{a=1}^{n} \int_0^T \langle \nabla u_a(t, \overline{x}_a(t)), x_a(t) - \overline{x}_a(t)) \rangle dt \leq 0$$

$$\forall\, (p, x) \in P \times M(\overline{p}).$$

Competitive equilibrium model

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Walrasian price equilibrium model

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