Evolutionary Variational Inequalities and Applications to Complex Dynamic Multi-level Models

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Workshop on “Complex Networks - Equilibrium and Vulnerability Analysis with Applications”
Catania, March 10-12, 2008

Outline

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G. Stampacchia (1964): first theorem of existence and uniqueness of the solution of variational inequalities;
G. Fichera (1964, 1972): another founder of the variational inequality theory;
P. Hartman - G. Stampacchia (1966): study of partial differential equations;
J.L. Lions - G. Stampacchia (1967): second proof of the same theorem and introduction of evolutionary variational inequalities;
H. Brezis (1968): study of evolutionary variational inequalities.

Problem in $\mathbb{R}^n$:

$$\mathbb{K} \subseteq \mathbb{R}^n, \quad F : \mathbb{K} \to \mathbb{R}^n,$$

$$\text{Find } x \in \mathbb{K} \text{ such that } \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathbb{K},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

Problem in a Banach space E:

$$\mathbb{K} \subseteq E, \quad F : \mathbb{K} \to E^*,$$

$$\text{Find } x \in \mathbb{K} \text{ such that } \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathbb{K},$$

where $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is the duality pairing.

M.J. Smith (1979) and S. Dafermos (1980): formulation of a traffic network equilibrium problem in terms of a finite-dimensional variational inequality

Evolutionary Variational Inequalties

P. Daniele, A. Maugeri, W. Oettli (1998 - 1999): traffic network equilibrium problem with time-dependent capacity constraints and demands;
F. Raciti (2001): time-dependent traffic networks with delay;
L. Scrimali (2004): quasi-variational inequalities in transportation networks;
P. Daniele (2003): spatial price equilibrium problem with price and bounds depending on time;
P. Daniele (2003) and P. Daniele, S. Giuffre', S. Pia (2005): time-dependent financial network problem;

M.G. Cojocaru, P. Daniele, A. Nagurney (2005 - 2006 - 2007): connection between time-dependent variational inequalities projected dynamical systems;
A. Nagurney, Z. Liu, M.G. Cojocaru, P. Daniele (2007): dynamic electric power supply chains;
A. Nagurney, D. Parkes, P. Daniele (2007): Internet, evolutionary variational inequalities and time-dependent Braess paradox;
P. Daniele, S. Giuffre', G. Idone, A. Maugeri (2007): infinite-dimensional duality theory;
M.B. Donato, M. Milasi, C. Vitanza (2008): Walrasian price equilibrium problem with time-dependent data;
M.G. Cojocaru (2008): evolutionary variational inequality model of vaccination strategies games.

Time-dependent Variational Inequality:

$$\mathbb{K} \subseteq \mathcal{L} = L^p([0,T],\mathbb{R}^n), \quad F : \mathbb{K} \to \mathcal{L}^*,$$

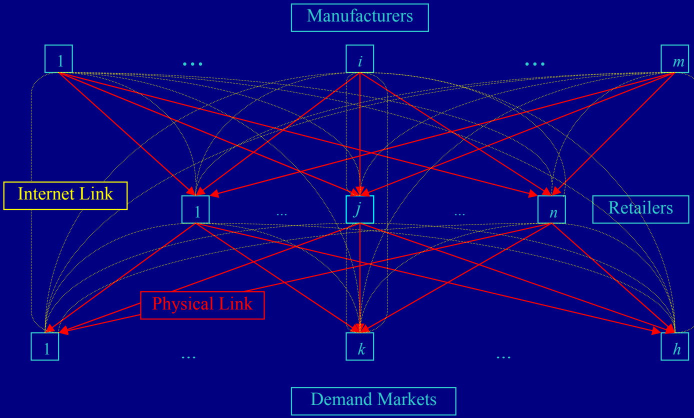
$$\text{Find } x \in \mathbb{K} \text{ such that } \ll F(x), y - x \gg \geq 0 \quad \forall y \in \mathbb{K},$$

where $\ll G, H \gg = \displaystyle\int_0^T \langle G(t), H(t) \rangle \, dt$ $G \in \mathcal{L}^*$, $H \in \mathcal{L}$.

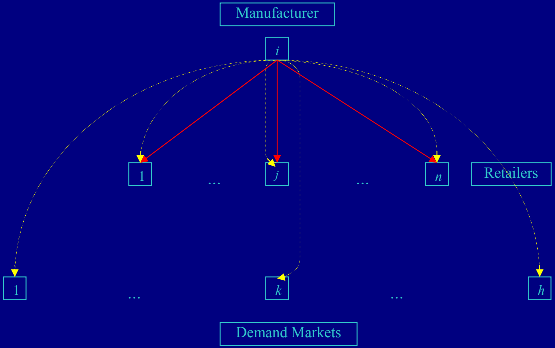
Supernetworks

A. Nagurney, J. Dong (2002): Supernetworks: Decision-Making for the Information Age;
A. Nagurney, Z. Liu (2007): application of an evolutionary variational inequality formulation to supply chain networks with time-varying demands;
J.M. Cruz (2008): dynamic framework for the modeling and analysis of supply chain networks with corporate social responsibility.

Dynamic Multilevel Supernetwork



Behavior of Manufacturers



$g_i(t)$: production of manufacturer i at $t \in [0, T] \Rightarrow$

$$g(t) \in L^2([0, T], \mathbb{R}^m_+);$$

$p_i(t)$: production cost function of manufacturer i at $t \Rightarrow$

$$p_i(t) = p_i(g(t)), \quad \forall i = 1, \ldots, m;$$

$g_{ijl}(t)$: product shipment between i, j via mode l at $t \in [0, T] \Rightarrow$

$$g^1(t) \in L^2([0, T], \mathbb{R}^{2mn}_+);$$

$c_{ijl}(t)$: transaction cost between i, j via mode l at $t \in [0, T] \Rightarrow$

$$c_{ijl}(t) = c_{ijl}(g_{ijl}(t)), \quad \forall i,\ j,\ l;$$

$g_{ik}(t)$: product shipment between i and k at $t \in [0, T] \Rightarrow$

$$g^2(t) \in L^2([0, T], \mathbb{R}_+^{mh});$$

$c_{ik}(t)$: transaction cost between i and j at $t \in [0, T] \Rightarrow$

$$c_{ik}(t) = c_{ik}(g_{ik}(t)), \quad \forall i, \ k;$$

$s_i(t)$: excess of production of i at $t \in [0, T] \Rightarrow$

$$s(t) \in L^2([0, T], \mathbb{R}_+^{m});$$

$c_i(t)$: storing cost of i at $t \in [0, T] \Rightarrow$

$$c_i(t) = c_i(s_i(t)) \quad \forall i;$$

Conservation of flow equation:

$$g_i(t) = \sum_{j=1}^{n} \sum_{l=1}^{2} g_{ijl}(t) + \sum_{k=1}^{h} g_{ik}(t) + s_i(t), \quad \forall i;$$

$\rho^*_{1ijl}(t)$: price charged by i to j via mode l at $t \in [0, T]$;
$\rho^*_{1ik}(t)$: price charged by i to k at time $t \in [0, T]$;
Set of feasible quantities:

$$\mathbb{K} = \mathbb{K}_1 \times \mathbb{K}_2 = \left\{ (g^1(t), g^2(t), s(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh+m}_+) \right\}$$

$$= \left\{ (g^1(t), g^2(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh}_+ \Big) \right\} \times \left\{ s(t) \in L^2([0, T], \mathbb{R}^m_+) \right\}$$

Revenue of i :

$$\begin{aligned} R_i(g^1(t), g^2(t), s^*(t)) &= \sum_{j=1}^{n} \sum_{l=1}^{2} \rho^*_{1ijl}(t)\, g_{ijl}(t) \\ &- p_i(g^1(t), g^2(t), s^*(t)) - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(g_{ijl}(t)) \\ &+ \sum_{k=1}^{h} \rho^*_{1ik}(t)\, g_{ik}(t) - \sum_{k=1}^{h} c_{ik}(g_{ik}(t)) \end{aligned}$$

Excesses regulation: $\forall i = 1, \ldots, m$, a.e. in $[0, T]$

$$\Big(R_i(g^{1*}(t), g^{2*}(t), s^*(t)) - c_i(s_i^*(t))\Big) \cdot s_i^*(t) = 0.$$

Optimality conditions for i: Find $(g^{1*}(t), g^{2*}(t), s^*(t)) \in \mathbb{K}$ such that:

$$\begin{cases} \max\limits_{(g^1(t), g^2(t)) \in \mathbb{K}_1} \Big(R_i(g^1(t), g^2(t), s^*(t)) - c_i(s_i^*(t))\Big) \\ \qquad = R_i(g^{1*}(t), g^{2*}(t), s^*(t)) - c_i(s_i^*(t)) \\ R_i(g^{1*}(t), g^{2*}(t), s^*(t)) - c_i(s_i^*(t)) \geq 0 \\ \Big(R_i(g^{1*}(t), g^{2*}(t), s^*(t)) - c_i(s_i^*(t))\Big) \cdot s_i^*(t) = 0 \end{cases} \tag{1}$$

Theorem

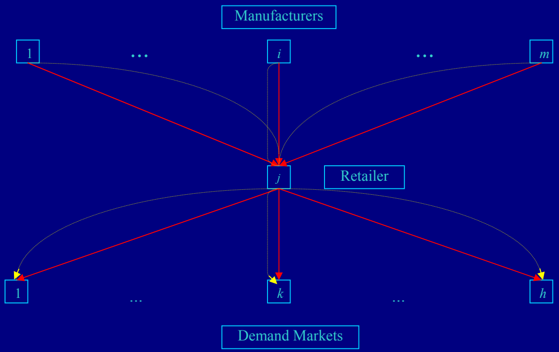
If $R_i(\cdot, s^*(t))$ is concave and differentiable for all $i = 1, \ldots, m$, then $(g^{1*}(t), g^{2*}(t), s^*(t)) \in \mathbb{K}$ is a solution to problem (1) if and only if it is a solution to the time-dependent variational inequality:

$$\text{Find } (g^{1*}(t), g^{2*}(t), s^*(t)) \in \mathbb{K} \text{ such that}$$

$$\int_0^T \langle -\, grad\, R(g^{1*}(t), g^{2*}(t), s^*(t)), (g^1(t), g^2(t)) - (g^{1*}(t), g^{2*}(t)) \rangle\, dt$$
$$+ \int_0^T \langle R(g^{1*}(t), g^{2*}(t), s^*(t)) - c(s^*(t)), s(t) - s^*(t) \rangle\, dt \geq 0$$
$$\forall (g^1(t), g^2(t), s(t)) \in \mathbb{K}, \quad (2)$$

where $c(s(t)) = (c_1(s_1(t)), \ldots, c_m(s_m(t)))$.

Behavior of Retailers



$g_{jkl}(t)$: product shipment between j and k via mode l at $t \Rightarrow$

$$g^3(t) \in L^2([0,T], \mathbb{R}^{2nh}_+);$$

$c_{jkl}(t)$: transaction cost between j and k via mode l at $t \Rightarrow$

$$c_{jkl}(t) = c_{jkl}(g_{jkl}(t)), \quad \forall j, \ \forall k, \ \forall l;$$

$\hat{c}_{ijl}(t)$: transaction cost between j and i via mode l at $t \Rightarrow$

$$\hat{c}_{ijl}(t) = \hat{c}_{ijl}(g_{ijl}(t)), \quad \forall i, \ \forall j, \ \forall l;$$

$c_j(t)$: handling cost associated to j at $t \in [0,T] \Rightarrow$

$$c_j(t) = c_j\left(g^1_j(t), g^3_j(t)\right), \quad \forall j = 1, \dots, n;$$

$\gamma^*_j(t)$: price imposed by j at $t \in [0,T]$.

Revenue of j

$$R_{1j}(g_j^1(t)) + R_{3j}(g_j^3(t)) = -c_j\left(g_j^1(t), g_j^3(t)\right)$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(g_{ijl}(t)) - \sum_{i=1}^{m}\sum_{l=1}^{2}\rho_{1ijl}^{*}\,g_{ijl}(t)$$

$$+\gamma_j^{*}\sum_{k=1}^{h}\sum_{l=1}^{2}g_{jkl}(t) - \sum_{k=1}^{h}\sum_{l=1}^{2}c_{jkl}(g_{jkl}(t))$$

Optimality Conditions

$$\max_{(g_j^1, g_j^3) \in \mathbb{K}_{1j} \times \mathbb{K}_{3j}} \left\{ R_{1j}(g_j^1(t)) + R_{3j}(g_j^3(t)) \right\} \tag{3}$$

where

$$\mathbb{K}_{1j} = \left\{ g_j^1(t) = (g_{ijl}(t)) \geq 0, \, \forall i, \, \forall l = 1, 2, \text{ a.e. in } [0, T] \right\},$$

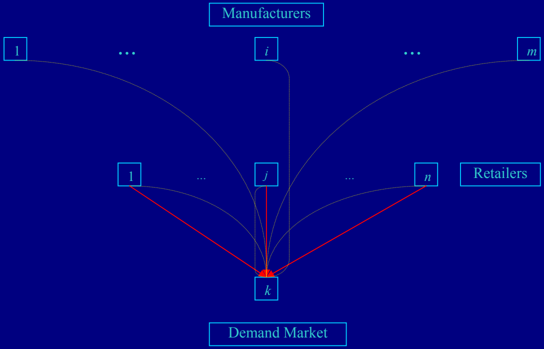
$$\mathbb{K}_{3j} = \left\{ g_j^3(t) = (g_{jkl}(t)) \geq 0, \, \forall k = 1, \ldots, h, \, \forall l = 1, 2, \text{ and } \sum_{k=1}^{h} \sum_{l=1}^{2} g_{jkl}(t) \leq \sum_{i=1}^{m} \sum_{l=1}^{2} g_{ijl}(t) \text{ a.e. in } [0, T] \right\}.$$

Theorem

If $R_{1j}(\cdot)$ and $R_{3j}(\cdot)$ are concave and differentiable for all j, then $(g^{1*}(t), g^{3*}(t)) \in \mathbb{K}_1 \times \mathbb{K}_3$ is a solution to problem (3) if and only if it is a solution to the time-dependent variational inequality:

$$\text{Find } (g^{1*}(t), g^{3*}(t)) \in \mathbb{K}_1 \times \mathbb{K}_3 \text{ such that}$$
$$\int_0^T \langle - \, grad \, R_1(g^{1*}(t)), g^1(t) - g^{1*}(t) \rangle \, dt$$
$$+ \int_0^T \langle - \, grad \, R_3(g^{3*}(t)), g^3(t) - g^{3*}(t) \rangle \, dt \geq 0$$
$$\forall (g^1(t), g^3(t)) \in \mathbb{K}_1 \times \mathbb{K}_3. \tag{4}$$

Behavior of Consumers



$\hat{c}_{jkl}(t)$: transaction cost between k and j via mode l at $t \in [0, T]$
$\Rightarrow$

$$\hat{c}_{jkl}(t) = \hat{c}_{jkl}(g^2(t), g^3(t)), \quad \forall j, \ \forall k, \ \forall l;$$

$\hat{c}_{ik}(t)$: transaction cost between k and i at $t \in [0, T]$ $\Rightarrow$

$$\hat{c}_{ik}(t) = \hat{c}_{ik}(g^2(t), g^3(t)), \quad \forall i, \ \forall k;$$

$\rho_{3k}(t)$: demand price at $t \in [0, T]$;
$f_k(t)$: demand at $t \in [0, T]$ $\Rightarrow$ $f_k(t) = f_k(\rho_3(t)), \quad \forall k;$
$\overline{\rho_{3k*}}(t)$: maximum price for consumers at k at time t;
$\tau_k(t)$: excess of demand at t $\Rightarrow$

$$f_k(t) = \sum_{i=1}^{m} g_{ik}(t) + \sum_{j=1}^{n} \sum_{l=1}^{2} g_{jkl}(t) + \tau_k(t), \quad \forall k.$$

Equilibrium Conditions

$$\gamma_j^*(t)+\hat{c}_{jkl}(g^{2*}(t),g^{3*}(t))\begin{cases} = \rho_{3k}^*(t), & \text{if} \quad g_{jkl}^*(t)>0 \\ \geq \rho_{3k}^*(t), & \text{if} \quad g_{jkl}^*(t)=0; \end{cases} \quad \forall j, \ \forall l;$$

$$\rho_{1ik}^*(t)+\hat{c}_{ik}(g^{2*}(t),g^{3*}(t))\begin{cases} = \rho_{3k}^*(t), & \text{if} \quad g_{ik}^*(t)>0 \\ \geq \rho_{3k}^*(t), & \text{if} \quad g_{ik}^*(t)=0; \end{cases} \quad \forall i;$$

$$\tau_k(t)\begin{cases} = 0, & \text{if} \quad 0 \leq \rho_{3k}^*(t) < \overline{\rho_{3k}^*(t)} \\ \geq 0, & \text{if} \quad \rho_{3k}^*(t) = \overline{\rho_{3k}^*(t)}. \end{cases} \quad \forall k.$$

Variational Formulation

Find $(g^{2*}(t), g^{3*}(t), \rho_3^*(t)) \in L^2([0,T], \mathbb{R}_+^{mh+2nh+h})$ such that

$$\int_0^T \Bigg\{ \sum_{j=1}^{n}\sum_{k=1}^{h}\sum_{l=1}^{2}\Big[\gamma_j^*(t) + \hat{c}_{jkl}(g^{2*}(t), g^{3*}(t)) - \rho_{3k}^*(t)\Big]\times\Big[g_{jkl}(t) - g_{jkl}^*(t)\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{h}\Big[\rho_{1ik}^*(t) + \hat{c}_{ik}(g^{2*}(t), g^{3*}(t)) - \rho_{3k}^*(t)\Big]\times[g_{ik}(t) - g_{ik}^*(t)]$$

$$+\sum_{k=1}^{h}\left[f_k(\rho_3^*(t)) - \sum_{j=1}^{n}\sum_{l=1}^{2} g_{jkl}^*(t) - \sum_{i=1}^{m} g_{ik}^*(t)\right]\times[\rho_{3k}(t) - \rho_{3k}^*(t)]\Bigg\}\,dt \geq 0,$$

$$\forall (g^2(t), g^3(t), \rho_3(t)) \in L^2([0,T], \mathbb{R}_+^{mh+2nh+h}).$$

Equilibrium Conditions for the Supply Chain Supernetwork

Theorem: Variational Formulation

$\left(g^{1*}(t), g^{2*}(t), g^{3*}(t), s^*(t), \rho_3^*(t)\right) \in L^2([0,T], \mathbb{R}_+^{2mn+mh+2nh+m+h})$ is an equilibrium for the supply chain supernetwork with electronic commerce and data depending on time if and only if $\forall (g^1(t), g^2(t), g^3(t), s(t), \rho_3(t)) \in L^2([0,T], \mathbb{R}_+^{2mn+mh+2nh+m+h})$ it satisfies the following variational inequality:

$$\int_0^T \Big\{ \langle - \text{grad } R(g^{1*}(t), g^{2*}(t), s^*(t)), (g_1(t), g_2(t)) - (g^{1*}(t), g^{2*}(t)) \rangle \, dt$$

$$+ \langle R(g^{1*}(t), g^{2*}(t), s^*(t)) - c(s^*(t)), s(t) - s^*(t) \rangle$$

$$+ \langle - \text{grad } R_1(g^{1*}(t)), g^1(t) - g^{1*}(t) \rangle$$

$$+ \langle - \text{grad } R_3(g^{3*}(t)), g^3(t) - g^{3*}(t) \rangle$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{h} \sum_{l=1}^{2} \left[\gamma_j^*(t) + \hat{c}_{jkl}(g^{2*}(t), g^{3*}(t)) - \rho_{3k}^*(t) \right] \times \left[g_{jkl}(t) - g_{jkl}^*(t) \right]$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{h} \left[\rho_{1ik}^*(t) + \hat{c}_{ik}(g^{2*}(t), g^{3*}(t)) - \rho_{3k}^*(t) \right] \times [g_{ik}(t) - g_{ik}^*(t)]$$

$$+ \sum_{k=1}^{h} \left[f_k(\rho_3^*(t)) - \sum_{j=1}^{n} \sum_{l=1}^{2} g_{jkl}^*(t) - \sum_{i=1}^{m} g_{ik}^*(t) \right] \times [\rho_{3k}(t) - \rho_{3k}^*(t)] \Big\} \, dt \geq 0.$$