Evolutionary Variational Inequalities and Applications to Complex Dynamic Multi-level Models

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Outline

1. Variational and Evolutionary Variational Inequalities
2. Dynamic Multi-level Supernetwork
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1. Variational and Evolutionary Variational Inequalities

2. Dynamic Multi-level Supernetwork
G. Stampacchia (1964): first theorem of existence and uniqueness of the solution of variational inequalities; 
G. Fichera (1964, 1972): another founder of the variational inequality theory; 
J.L. Lions - G. Stampacchia (1967): second proof of the same theorem and introduction of evolutionary variational inequalities; 

Problem in $\mathbb{R}^n$:

\[ \mathbb{K} \subseteq \mathbb{R}^n, \quad F: \mathbb{K} \to \mathbb{R}^n, \]

Find $x \in \mathbb{K}$ such that $\langle F(x), y - x \rangle \geq 0 \quad \forall y \in \mathbb{K},$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. 
Problem in a Banach space $E$:

\[ K \subseteq E, \quad F : K \to E^*, \]

Find $x \in K$ such that $\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K,$

where $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is the duality pairing.

Evolutionary Variational Inequalities

F. Raciti (2001): time-dependent traffic networks with delay;
P. Daniele (2003): spatial price equilibrium problem with price and bounds depending on time;
A. Nagurney, Z. Liu, M.G. Cojocaru, P. Daniele (2007): dynamic electric power supply chains;
A. Nagurney, D. Parkes, P. Daniele (2007): Internet, evolutionary variational inequalities and time-dependent Braess paradox;
Time-dependent Variational Inequality:

\[ K \subseteq \mathcal{L} = L^p([0, T], \mathbb{R}^n), \quad F : K \rightarrow \mathcal{L}^*, \]

Find \( x \in K \) such that \( \ll F(x), y - x \gg \geq 0 \quad \forall y \in K, \)

where \( \ll G, H \gg = \int_0^T \langle G(t), H(t) \rangle \, dt \quad G \in \mathcal{L}^*, H \in \mathcal{L}. \)
Supernetworks

A. Nagurney, Z. Liu (2007): application of an evolutionary variational inequality formulation to supply chain networks with time-varying demands;
Dynamic Multilevel Supernetwork

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Evolutionary Variational Inequalities and Applications to Complex Dynamic Multi-level Models
Behavior of Manufacturers

- **Manufacturer**
  - $i$
  - 1
  - ... $j$
  - ... $n$
  - **Retailers**
    - $h$

- **Demand Markets**
  - 1
  - ... $k$
  - ... $h$

Variational and Evolutionary Variational Inequalities

Dynamic Multi-level Supernetwork

Evolutionary Variational Inequalities and Applications to Complex Dynamic Multi-level Models
$g_i(t)$: production of manufacturer $i$ at $t \in [0, T] \Rightarrow$

$$g(t) \in L^2([0, T], \mathbb{R}_+^m);$$

$p_i(t)$: production cost function of manufacturer $i$ at $t \Rightarrow$

$$p_i(t) = p_i(g(t)), \quad \forall i = 1, \ldots, m;$$

$g_{ijl}(t)$: product shipment between $i, j$ via mode $l$ at $t \in [0, T] \Rightarrow$

$$g^1(t) \in L^2([0, T], \mathbb{R}_+^{2mn});$$

$c_{ijl}(t)$: transaction cost between $i, j$ via mode $l$ at $t \in [0, T] \Rightarrow$

$$c_{ijl}(t) = c_{ijl}(g_{ijl}(t)), \quad \forall i, j, l;$$
$g_{ik}(t)$: product shipment between $i$ and $k$ at $t \in [0, T] \Rightarrow$

$$g^2(t) \in L^2([0, T], \mathbb{R}^{mh})$$

$c_{ik}(t)$: transaction cost between $i$ and $j$ at $t \in [0, T] \Rightarrow$

$$c_{ik}(t) = c_{ik}(g_{ik}(t)), \quad \forall i, k$$

$s_i(t)$: excess of production of $i$ at $t \in [0, T] \Rightarrow$

$$s(t) \in L^2([0, T], \mathbb{R}^m)$$

$c_i(t)$: storing cost of $i$ at $t \in [0, T] \Rightarrow$

$$c_i(t) = c_i(s_i(t)), \quad \forall i$$

Conservation of flow equation:

$$g_i(t) = \sum_{j=1}^{n} \sum_{l=1}^{2} g_{ijl}(t) + \sum_{k=1}^{h} g_{ik}(t) + s_i(t), \quad \forall i$$
\( \rho_{ijl}^*(t) : \) price charged by \( i \) to \( j \) via mode \( l \) at \( t \in [0, T] \);
\( \rho_{ik}^*(t) : \) price charged by \( i \) to \( k \) at time \( t \in [0, T] \);

Set of feasible quantities:

\[
K = K_1 \times K_2 = \left\{ (g^1(t), g^2(t), s(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh+m}_+) \right\}
\]

\[
= \left\{ (g^1(t), g^2(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh}_+) \right\} \times \left\{ s(t) \in L^2([0, T], \mathbb{R}^m_+) \right\}
\]

Revenue of \( i \):

\[
R_i(g^1(t), g^2(t), s^*(t)) = \sum_{j=1}^{n} \sum_{l=1}^{2} \rho_{ijl}^*(t) g_{ijl}(t)
\]

\[- p_i(g^1(t), g^2(t), s^*(t)) - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(g_{ijl}(t))
\]

\[+ \sum_{k=1}^{h} \rho_{ik}^*(t) g_{ik}(t) - \sum_{k=1}^{h} c_{ik}(g_{ik}(t))\]
Excesses regulation: \( \forall i = 1, \ldots, m \), a.e. in \([0, T]\)

\[
\left( R_i(g^1(t), g^2(t), s^*(t)) - c_i(s^*_i(t)) \right) \cdot s^*_i(t) = 0.
\]

Optimality conditions for \( i \): Find \((g^1(t), g^2(t), s^*(t)) \in K\) such that:

\[
\begin{cases}
\max_{(g^1(t), g^2(t)) \in \mathbb{K}_1} \left( R_i(g^1(t), g^2(t), s^*(t)) - c_i(s^*_i(t)) \right) = R_i(g^1(t), g^2(t), s^*(t)) - c_i(s^*_i(t)) \\
R_i(g^1(t), g^2(t), s^*(t)) - c_i(s^*_i(t)) \geq 0 \\
\left( R_i(g^1(t), g^2(t), s^*(t)) - c_i(s^*_i(t)) \right) \cdot s^*_i(t) = 0
\end{cases}
\] (1)
Theorem

If $R_i(\cdot, s^*(t))$ is concave and differentiable for all $i = 1, \ldots, m$, then $(g^1*(t), g^2*(t), s^*(t)) \in \mathbb{K}$ is a solution to problem (1) if and only if it is a solution to the time-dependent variational inequality:

\[
Find \ (g^1*(t), g^2*(t), s^*(t)) \in \mathbb{K} \text{ such that } \\
\int_0^T \left\langle - \nabla R(g^1*(t), g^2*(t), s^*(t)), (g^1(t), g^2(t)) - (g^1*(t), g^2*(t)) \right\rangle dt \\
+ \int_0^T \left\langle R(g^1*(t), g^2*(t), s^*(t)) - c(s^*(t)), s(t) - s^*(t) \right\rangle dt \geq 0 \\
\forall (g^1(t), g^2(t), s(t)) \in \mathbb{K},
\]

(2)

where $c(s(t)) = (c_1(s_1(t)), \ldots, c_m(s_m(t)))$. 
Behavior of Retailers
\( g_{jkl}(t) : \text{product shipment between } j \text{ and } k \text{ via mode } l \text{ at } t \Rightarrow \)

\[ g^3(t) \in L^2([0, T], \mathbb{R}^{2nh}) ; \]

\( c_{jkl}(t) : \text{transaction cost between } j \text{ and } k \text{ via mode } l \text{ at } t \Rightarrow \)

\[ c_{jkl}(t) = c_{jkl}(g_{jkl}(t)), \quad \forall j, \forall k, \forall l; \]

\( \hat{c}_{ijl}(t) : \text{transaction cost between } j \text{ and } i \text{ via mode } l \text{ at } t \Rightarrow \)

\[ \hat{c}_{ijl}(t) = \hat{c}_{ijl}(g_{ijl}(t)), \quad \forall i, \forall j, \forall l; \]

\( c_j(t) : \text{handling cost associated to } j \text{ at } t \in [0, T] \Rightarrow \)

\[ c_j(t) = c_j \left( g^1_j(t), g^3_j(t) \right), \quad \forall j = 1, \ldots, n; \]

\( \gamma^*_j(t) : \text{price imposed by } j \text{ at } t \in [0, T]. \)
Revenue of $j$

\[
R_{1j}(g^1_j(t)) + R_{3j}(g^3_j(t)) = -c_j \left( g^1_j(t), g^3_j(t) \right) \\
- \sum_{i=1}^{m} \sum_{l=1}^{2} \hat{c}_{ijl}(g_{ijl}(t)) - \sum_{i=1}^{m} \sum_{l=1}^{2} \rho_{ijl}^* g_{ijl}(t) \\
+ \gamma_j^* \sum_{k=1}^{h} \sum_{l=1}^{2} g_{jkl}(t) - \sum_{k=1}^{h} \sum_{l=1}^{2} c_{jkl}(g_{jkl}(t))
\]
Optimality Conditions

\[
\max_{(g_j^1, g_j^3) \in K_{1j} \times K_{3j}} \left\{ R_{1j}(g_j^1(t)) + R_{3j}(g_j^3(t)) \right\} \tag{3}
\]

where

\[
K_{1j} = \left\{ g_j^1(t) = (g_{ijl}(t)) \geq 0, \forall i, \forall l = 1, 2, \text{ a.e. in } [0, T] \right\},
\]

\[
K_{3j} = \left\{ g_j^3(t) = (g_{jkl}(t)) \geq 0, \forall k = 1, \ldots, h, \forall l = 1, 2, \text{ and } \sum_{k=1}^{h} \sum_{l=1}^{2} g_{jkl}(t) \leq \sum_{i=1}^{m} \sum_{l=1}^{2} g_{ijl}(t) \text{ a.e. in } [0, T] \right\}.
\]
**Theorem**

If $R_{1j}(\cdot)$ and $R_{3j}(\cdot)$ are concave and differentiable for all $j$, then $(g_1^*(t), g_3^*(t)) \in \mathbb{K}_1 \times \mathbb{K}_3$ is a solution to problem (3) if and only if it is a solution to the time-dependent variational inequality:

Find $(g_1^*(t), g_3^*(t)) \in \mathbb{K}_1 \times \mathbb{K}_3$ such that

$$
\int_0^T \langle - \nabla R_1(g_1^*(t)), g_1(t) - g_1^*(t) \rangle \, dt + \int_0^T \langle - \nabla R_3(g_3^*(t)), g_3(t) - g_3^*(t) \rangle \, dt \geq 0
$$

$\forall (g_1(t), g_3(t)) \in \mathbb{K}_1 \times \mathbb{K}_3.$

(4)
Behavior of Consumers

Variational and Evolutionary Variational Inequalities

Dynamic Multi-level Supernetwork

Manufacturers

Retailers

Demand Market
\[ \hat{c}_{jkl}(t) : \text{transaction cost between } k \text{ and } j \text{ via mode } l \text{ at } t \in [0, T] \]
\[ \Rightarrow \]
\[ \hat{c}_{jkl}(t) = \hat{c}_{jkl}(g^2(t), g^3(t)), \quad \forall j, \forall k, \forall l; \]
\[ \hat{c}_{ik}(t) : \text{transaction cost between } k \text{ and } i \text{ at } t \in [0, T] \Rightarrow \]
\[ \hat{c}_{ik}(t) = \hat{c}_{ik}(g^2(t), g^3(t)), \quad \forall i, \forall k; \]
\[ \rho_{3k}(t) : \text{demand price at } t \in [0, T]; \]
\[ f_k(t) : \text{demand at } t \in [0, T] \Rightarrow f_k(t) = f_k(\rho_{3k}(t)), \quad \forall k; \]
\[ \overline{\rho_{3k}*}(t) : \text{maximum price for consumers at } k \text{ at time } t; \]
\[ \tau_k(t) : \text{excess of demand at } t \Rightarrow \]
\[ f_k(t) = \sum_{i=1}^{m} g_{ik}(t) + \sum_{j=1}^{n} \sum_{l=1}^{2} g_{jkl}(t) + \tau_k(t), \quad \forall k. \]
Equilibrium Conditions

\[
\begin{align*}
\gamma_j^*(t) + \hat{c}_{jkl}(g^{2*}(t), g^{3*}(t)) & \begin{cases} 
= \rho_{3k}^*(t), & \text{if } g_{jkl}^*(t) > 0 \\
\geq \rho_{3k}^*(t), & \text{if } g_{jkl}^*(t) = 0;
\end{cases} & \quad \forall j, \forall l; \\
\rho_{1ik}^*(t) + \hat{c}_{ik}(g^{2*}(t), g^{3*}(t)) & \begin{cases} 
= \rho_{3k}^*(t), & \text{if } g_{ik}^*(t) > 0 \\
\geq \rho_{3k}^*(t), & \text{if } g_{ik}^*(t) = 0;
\end{cases} & \quad \forall i; \\
\tau_k(t) & \begin{cases} 
= 0, & \text{if } 0 \leq \rho_{3k}^*(t) < \rho_{3k}^*(t) \\
\geq 0, & \text{if } \rho_{3k}^*(t) = \rho_{3k}^*(t).
\end{cases} & \quad \forall k.
\end{align*}
\]

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Variational Formulation

Find \((g^2^*(t), g^3^*(t), \rho^*_{3k}(t)) \in L^2([0, T], \mathbb{R}_{+}^{mh+2nh+h})\) such that

\[
\int_{0}^{T} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{h} \sum_{l=1}^{2} \left[ \gamma^*_j(t) + \hat{c}_{jkl}(g^2^*(t), g^3^*(t)) - \rho^*_{3k}(t) \right] \times \left[ g_{jkl}(t) - g_{jkl}^*(t) \right] 
+ \sum_{i=1}^{m} \sum_{k=1}^{h} \left[ \rho^*_{1ik}(t) + \hat{c}_{ik}(g^2^*(t), g^3^*(t)) - \rho^*_{3k}(t) \right] \times \left[ g_{ik}(t) - g_{ik}^*(t) \right] 
+ \sum_{k=1}^{h} \left[ f_k(\rho^*_{3k}(t)) - \sum_{j=1}^{n} \sum_{l=1}^{2} g_{jkl}^*(t) - \sum_{i=1}^{m} g_{ik}^*(t) \right] \times \left[ \rho_{3k}(t) - \rho^*_{3k}(t) \right] \right\} \, dt \geq 0,
\]

\(\forall (g^2(t), g^3(t), \rho_{3k}(t)) \in L^2([0, T], \mathbb{R}_{+}^{mh+2nh+h}).\)
Equilibrium Conditions for the Supply Chain Supernetwork

Theorem: Variational Formulation

\[(g^1(t), g^2(t), g^3(t), s(t), \rho_3(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh+2nh+m+h})\]

is an equilibrium for the supply chain supernetwork with electronic commerce and data depending on time if and only if

\[\forall (g^1(t), g^2(t), g^3(t), s(t), \rho_3(t)) \in L^2([0, T], \mathbb{R}^{2mn+mh+2nh+m+h})\]

it satisfies the following variational inequality:
\[
\int_0^T \left\{ \langle -\nabla R(g_1^*(t), g_2^*(t), s^*(t)), (g_1(t), g_2(t)) - (g_1^*(t), g_2^*(t)) \rangle \right. \\
+ \langle R(g_1^*(t), g_2^*(t), s^*(t)) - c(s^*(t)), s(t) - s^*(t) \rangle \\
+ \langle -\nabla R_1(g_1^*(t)), g_1^*(t) - g_1(t) \rangle \\
+ \langle -\nabla R_3(g_3^*(t)), g_3^*(t) - g_3(t) \rangle \\
+ \sum_{j=1}^n \sum_{k=1}^h \sum_{l=1}^2 \left[ \gamma_j^*(t) + \hat{c}_{jkl}(g_2^*(t), g_3^*(t)) - \rho_{3k}^*(t) \right] \times \left[ g_{jkl}(t) - g_{jkl}^*(t) \right] \\
+ \sum_{i=1}^m \sum_{k=1}^h \left[ \rho_{1ik}^*(t) + \hat{c}_{ik}(g_2^*(t), g_3^*(t)) - \rho_{3k}^*(t) \right] \times \left[ g_{ik}(t) - g_{ik}^*(t) \right] \\
+ \sum_{k=1}^h \left[ f_k(\rho_3^*(t)) - \sum_{j=1}^n \sum_{l=1}^2 g_{jkl}^*(t) - \sum_{i=1}^m g_{ik}^*(t) \right] \times \left[ \rho_{3k}(t) - \rho_{3k}^*(t) \right] \left\} \, dt \geq 0. \right. 
\]