# Weighted variational inequalities in non-pivot Hilbert spaces: existence and regularity results and applications

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# Outline

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# Part I

## Introduction

# Some contributions on variational and quasi-variational inequalities

- Fichera (1963–1964): a problem in elasticity with a unilateral boundary condition;
- Lions-Stampacchia (Commun. Pure Appl. Math., 1967): study of variational inequalities;
- Brezis (Comptes Rendus de l'Academie des Sciences, 1967): introduction of evolutionary variational inequalities;
- Kuratowski (1966) and Mosco (Adv. Math., 1969): study on convergence of convex sets and of solutions of variational inequalities;
- J-P. Aubin (1987): study on weighted Hilbert spaces.
- Daniele-Maugeri-Oettli (C.R. Acad. Sci. Paris, 1998): applications of evolutionary variational inequalities to dynamic equilibrium problems;

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# Preliminary concepts

## Proposition

*Let $\Omega \subset \mathbb{R}^p$ be an open subset of $\mathbb{R}^n$, $a : \Omega \rightarrow R^+ \setminus \{0\}$ a continuous and strictly positive function called "weight" and $s : \Omega \rightarrow R^+ \setminus \{0\}$ a continuous and strictly positive function called "real time density". The bilinear form defined on $\mathcal{C}_0(\Omega)$ (set of continuous functions with compact support on $\Omega$) by*

$$\mathcal{C}_0(\Omega) \times \mathcal{C}_0(\Omega) \rightarrow \mathbb{R}$$

$$\langle x, y \rangle_{a,s} = \int_\Omega x(\omega)y(\omega)a(\omega)s(\omega)d\omega$$

*is an inner product.*

# Preliminary definitions

If $a$ is a weight, $a^{-1} = 1/a$ is also a weight.

**Definition**

We call $L^2(\Omega, a, s)$ a completion of $\mathcal{C}_0(\Omega)$ for the inner product $< x, y >_{a,s}$

# Preliminary definitions

If we denote by $V_i = L^2(\Omega, \mathbb{R}, a_i, s_i)$ and $V_i^* = L^2(\Omega, \mathbb{R}, a_i^{-1}, s_i)$, the space

$$V = \prod_{i=1}^{m} V_i \tag{1}$$

is a non pivot Hilbert space for the inner product

$$\langle F, G \rangle_V = \langle F, G \rangle_{\mathbf{a},\mathbf{s}} = \sum_{i=1}^{m} \int_{\Omega} F_i(\omega) G_i(\omega) a_i(\omega) s_i(\omega) d\omega.$$

# Preliminary definitions

The space

$$V^* = \prod_{i=1}^{m} V_i^* \tag{2}$$

is clearly a non pivot Hilbert space for the following inner product

$$\langle F, G \rangle_{V^*} = \langle F, G \rangle_{\mathbf{a}^{-1},\mathbf{s}} = \sum_{i=1}^{m} \int_{\Omega} \frac{F_i(\omega) G_i(\omega) s_i(\omega)}{a_i(\omega)} d\omega$$

# Preliminary definitions

The bilinear form define a duality between $V$ and $V^*$:

$$V^* \times V \to \mathbb{R}$$

$$\langle f, x \rangle_{V^* \times V} = \langle f, x \rangle_{\mathbf{s}} = \sum_{i=1}^{m} \int_{\Omega} f_i(\omega) x_i(\omega) s_i(\omega) d\omega. \tag{3}$$

# Preliminary results

## Proposition

*The bilinear form (3) is defined over $V^* \times V$ and define a duality between $V^*$ and $V$. The duality mapping is given by*

$$J(F) = (a_1 F_1, \ldots, a_m F_m).$$

# Part II

## Existence results

# Some definitions

Let $V$ be the Hilbert spaces and let $S$ be a subset of $V$.

**Definition**

An operator $C : S \to V^*$ is said to be

- *monotone* on $S$ if
$$\langle C(x_1) - C(x_2), x_1 - x_2 \rangle_s \geq 0, \quad \forall x_1, x_2 \in S;$$
- *strictly monotone* on $S$ if
$$\langle C(x_1) - C(x_2), x_1 - x_2 \rangle_s > 0, \quad \forall x_1 \neq x_2;$$
- *strongly monotone* on $S$ if for some $\nu > 0$
$$\langle C(x_1) - C(x_2), x_1 - x_2 \rangle_s \geq \nu \|x_1 - x_2\|_V, \quad \forall x_1, x_2 \in S;$$

# Some definitions

## Definition

- *strongly pseudomonotone with degree $\alpha > 0$ on $K$ (strongly pseudo-monotone on $K$ if $\alpha = 2$)* , if and only if there exists $\nu > 0$ such that for all $x_1, x_2 \in S$

$$\langle C(x_2), x_1 - x_2 \rangle_s \geq 0 \;\Rightarrow\; \langle C(x_1), x_1 - x_2 \rangle_s \geq \nu \|x_1 - x_2\|_V^{\alpha},$$

- *stricly pseudomonotone* on $S$ if for all $x_1, x_2 \in S$

$$\langle C(x_1), x_1 - x_2 \rangle_s \geq 0 \Longrightarrow \langle C(x_2), x_1 - x_2 \rangle_s < 0.$$

- *pseudomonotone* on $S$ if for all $x_1, x_2 \in S$

$$\langle C(x_1), x_1 - x_2 \rangle_s \geq 0 \Longrightarrow \langle C(x_2), x_1 - x_2 \rangle_s \leq 0.$$

# Some definitions

Let $\mathbb{K}$ be a convex subset of $V$.

## Definition

An operator $C : \mathbb{K} \to V^*$ is said to be

- *hemicontinuous* if for any $x \in \mathbb{K}$, the function
$$\mathbb{K} \ni \xi \to \langle C(\xi), x - \xi \rangle_s$$
is upper semi-continuous on $\mathbb{K}$;
- *hemicontinuous along line segments* if and only if for any $x, y \in \mathbb{K}$, the function
$$\mathbb{K} \ni \xi \to \langle C(\xi), y - x \rangle_s$$
is upper semi-continuous on the line segment $[x, y]$.

# Evolutionary weighted variational inequality

## Definition

Let $\mathbb{K}$ be a nonempty, convex and closed subset of $V$ and let $C : \mathbb{K} \to V^*$ be a vector-function. The weighted variational inequality is the problem to find a vector $x \in \mathbb{K}$, such that

$$\langle C(x), y - x \rangle_s \geq 0, \quad \forall y \in \mathbb{K}. \tag{4}$$

# Existence theorem

**Theorem**

*Let $\mathbb{K}$ be a nonempty, convex and closed subset of $V$. Let $C : \mathbb{K} \to V^*$ such that $C$ is monotone and hemicontinous. Then there is a $u_0 \in \mathbb{K}$ such that*

$$\langle C(u_0), v - u_0 \rangle_s \geq 0, \ \forall v \in \mathbb{K}.$$

# Existence theorem

**Theorem**

*Let $\mathbb{K}$ be a nonempty, convex and closed subset of $V$. Let $C : \mathbb{K} \to V^*$ be monotone such that $C$ is continuous on finite dimensional subspaces of $\mathbb{K}$. Then there is a $u_0 \in \mathbb{K}$ such that*

$$\langle C(u_0), v - u_0 \rangle_s \geq 0, \ \forall v \in \mathbb{K}.$$

If $C$ is strictly monotone, $u_0$ is unique.

# Existence theorem

**Theorem**

*Let $\mathbb{K}$ be a nonempty, convex and closed subset of $V$. Let $C : \mathbb{K} \to V^*$ be a given function such that:*

(i) *there exist $A \subseteq \mathbb{K}$ nonempty, compact and $B \subseteq \mathbb{K}$ compact, convex such that, for every $y \in \mathbb{K} \setminus A$, there exists $\widehat{x} \in B$ with $\langle C(y), \widehat{x} - y \rangle_s < 0$;*

(ii) *$C$ is pseudomonotone and hemicontinuous along line segments.*

*Then, there exists $x \in A$ such that $\langle C(x), y - x \rangle_s \geq 0$, for all $y \in \mathbb{K}$.*

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**Theorem**

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(i) *there exist* $A \subseteq \mathbb{K}$ *nonempty, compact and* $B \subseteq K$ *compact, convex such that, for every* $y \in \mathbb{K} \setminus A$, *there exists* $\widehat{x} \in B$ *with* $\langle C(y), \widehat{x} - y \rangle_s < 0$;

(ii) $C$ *is hemicontinuous.*

*Then, there exists* $x \in A$ *such that* $\langle C(x), y - x \rangle_s \geq 0$, *for all* $y \in \mathbb{K}$.

# Part III

## Regularity results

# Sets convergence

Let $X$ be a nonempty set endowed with two topologies $\sigma \subseteq \tau$.
Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of $X$.

## Definition

We say that $K_n$ $(\sigma,\tau)$-converges to some subset $K \subseteq X$, and we briefly write $K_n \rightarrow^{(\sigma,\tau)} K$, if

- for any sequence $\{x_n\}_{n\in\mathbb{N}}$, with $x_n$ in $K_n$ $\forall n \in \mathbb{N}$, such that $x_n \rightarrow^{\sigma} x$ for some $x \in S$, then $x \in K$;
- for any $x \in K$ there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$, with $x_{k_n}$ in $K_{k_n}$ $\forall n \in \mathbb{N}$, such that $x_{k_n} \rightarrow^{\tau} x$.

# Sets convergence

## Definition

- Let $(X, d)$ be a metric space such that $\sigma = \tau = \tau_d$ is exactly the metric topology. In this case the $(\sigma, \tau)$-convergence is called Kuratowski convergence of sets; it will be denoted by $K_n \to^K K$.
- Let $X$ be a normed space, moreover let $\sigma$ and $\tau$ be respectively the weak and the strong topology on $X$. In this case the $(\sigma, \tau)$-convergence is called Mosco convergence of sets; it will be denoted by $K_n \to^M K$.

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# Regularity result for nonlinear strongly monotone evolutionary weighted variational inequalities

**Theorem**

*Let $V$ be the non-pivot Hilbert space, let $\Omega \subseteq \mathbb{R}^p$, let $t \in \Omega$ and $K(t)$ be a subset of $\mathbb{R}^m$ verifying Kuratowski's convergence assumptions, let $C : \Omega \times K \to V^*$ be a continuous function and $C(t, \cdot)$ strongly pseudo-monotone with degree $\alpha > 1$. Then the solution map $x : \Omega \ni t \to x(t) \in \mathbb{R}^m$ of the evolutionary weighted variational inequality is continuous on $\Omega$.*

# Regularity result for nonlinear strictly evolutionary weighted pseudomonotone evolutionary variational inequalities

For every $\varepsilon > 0$ and for any fixed $t \in \Omega$, let us consider the following perturbed variational inequality

$$\langle C(t, x(t)) + \varepsilon J_m(x(t)), y(t) - x(t) \rangle_{m,s(t)} \geq 0, \quad \forall y(t) \in K(t),$$

where $J_m$ is the duality mapping between $(\mathbb{R}^m, \|.\|_{m,a,s})$ and $(\mathbb{R}^m, \|.\|_{m,a^{-1},s})$.

# Regularity result for nonlinear strictly pseudomonotone evolutionary weighted variational inequalities

**Theorem**

*Let $V$ be the non-pivot Hilbert space, let $\Omega \subseteq \mathbb{R}^n$, let $K(t)$ e a nonempty closed convex and bounded (uniformly with respect to $t \in \Omega$) subset of $\mathbb{R}^m$, verifying the Kuratowski convergence. Let $C : \Omega \times K \to V^*$ be a continuous function so that $C(t, \cdot)$ is strictly pseudo-monotone. Then the solution map $x : \Omega \ni t \to x(t) \in \mathbb{R}^m$ of the evolutionary weighted variational inequality is continuous on $\Omega$.*

# Part IV

# Dynamic weighted traffic equilibrium problem

# Dynamic weighted traffic equilibrium problem

Let us introduce a network $\mathcal{N}$:

- $G = [N, L]$ is a graph;
- $\mathcal{W}$ is the set of Origin-Destination (O/D) pairs $w_j$, $j = 1, 2, \ldots, l$;
- $\mathcal{R}$ is the set of routes $R_r$, $r = 1, 2, \ldots, m$, which connect the pair $w \in \mathcal{W}$.

The set of all $r \in \mathcal{R}$ which link a given $w \in \mathcal{W}$ is denoted by $\mathcal{R}(w)$.

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# Dynamic weighted traffic equilibrium problem

Let be denote by

- $\Omega$ an open subset of $\mathbb{R}$,
- $\mathbf{a} = \{a_1, .., a_m\}$, $\mathbf{a}^{-1} = \{a_1^{-1}, .., a_m^{-1}\}$ and $\mathbf{s} = \{s_1, .., s_m\}$ three families of weights such that for each $1 \leq i \leq n$, $a_i, s_i \in \mathcal{C}(\Omega, \mathbb{R}^+ \setminus \{0\})$.

We use the framework of a non-pivot Hilbert space which is a multidimensional version of the weighted space $L^2(\Omega, \mathbb{R}, a, s)$, that we denote by $V$.

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# Dynamic traffic equilibrium problem

For a.e. $t \in \Omega = ]0, T[$ we consider vector flow $F(t) \in \mathbb{R}^m$. The feasible flows have to satisfy the time dependent capacity constraints and demand requirements, namely for all $r \in \mathcal{R}$, $w \in \mathcal{W}$ and for almost all $t \in \Omega$,

$$\lambda_r(t) \leq F_r(t) \leq \mu_r(t)$$

and

$$\sum_{r \in \mathcal{R}(w)} F_r(t) = \rho_w(t)$$

where $\lambda(t) \leq \mu(t)$ are given in $\mathbb{R}^m$, $\rho(t) \in \mathbb{R}^l$.

# Dynamic traffic equilibrium problem

If $\Phi = (\Phi_{w,r})$ is the pair route incidence matrix, with $w \in \mathcal{W}$ and $r \in \mathcal{R}$, that is

$$\Phi_{w,r} = \begin{cases} 1 \text{ if } w \in \mathcal{R}(r) \\ 0 \text{ otherwise,} \end{cases}$$

the demand requirements can be written in matrix-vector notation as

$$\Phi F(t) = \rho(t)$$

The set of all feasible flows is given by

$$\mathbf{K} := \{F \in V | \ \lambda(t) \leq F(t) \leq \mu(t), a.e. \ in \ \Omega,$$

$$\Phi F(t) = \rho(t), \ \ a.e \ in \ \Omega\}$$

Let us denote by

$$C : \Omega \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$$

the cost function.

# Time-dependent weighted variational inequality

**Definition**

$H \in V$ is an equilibrium flow if and only if

$$H \in \mathbf{K} : \ \langle C(t, H(t)), F(t) - H(t) \rangle_{\mathbf{s}} \geq 0, \ \forall F \in \mathbf{K}, \ a.e. \ in \ \Omega. \quad (5)$$

# Weighted Wardrop condition

Equivalence between condition (5) and what we will call the **weighted Wardrop condition**

**Theorem**

*$H \in \mathbf{K}$ is an equilibrium flow in the sense of (5) if and only if*

$$\forall w \in \mathcal{W},\ \forall q, m \in \mathcal{R}(w),\ a.e.\ in\ \Omega,$$

$$s_q(t)C_q(t, H(t)) < s_m(t)C_m(t, H(t)) \qquad (6)$$

$$\Rightarrow H_q(t) = \mu_q(t)\ or\ H_m(t) = \lambda_m(t).$$

# Pointed formulation

## Theorem

*The evolutionary variational inequality:*

$$H \in \mathbf{K} : \int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle dt \geq 0, \ \forall F \in \mathbf{K},$$

*is equivalent to*

$$H \in \mathbf{K} : \ \langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \ \forall F(t) \in \mathbf{K}(t), \ \text{a.e. in } \Omega,$$

*where*

$$\mathbf{K}(t) = \Big\{ F(t) \in \mathbb{R}^m) : \lambda(t) \leq F(t) \leq \mu(t), \ \Phi F(t) = \rho(t) \Big\}.$$

# Weighted Wardrop condition

We propose a way to define the Real Time Traffic Density (RTTD) for a route. This data will be the "weight" of the route considered and it will be use to define the duality pairing.
Using mobile phone connections data it is possible to establish the density of mobile phone connected over a monitored area.
The principle can be generalized to other wireless devices.
It is clear that to weight properly a link is really difficult and it is at least necessarily important to take into account network's geometry, which means for us the position of network's elements .

# Weighted Wardrop condition

We can suppose to have $I \subset \mathbb{R}^2$ closed and large enough to include the monitored area and a parametric continuous function $\gamma_t$ with $t \in \Omega$ such that:

$$\gamma_t : I \to \mathbb{R}^+$$

$$\gamma_t : (x, y) \to \gamma_t(x, y)$$

This function represent a normalized interpolation obtained using the communication data.

# Weighted Wardrop condition

For each route we construct a weight in the following way: let us fix $\vartheta \in \mathbb{R}^+ \setminus \{0\}$, a strict positive number called "**resolution**". We introduce the set $r^\vartheta = r \times \vartheta, \ r^\vartheta \subset \Omega$.
We propose now a definition of weight which not pretend to be exhaustive, all the contrary. We think that the weights should be calibrate case by case.
For examples one can decide to take into account very exceptional events that are not visible by mobile connection data adding to the definition given bellow, terms that will increase or decrease the weight.

# Weighted Wardrop condition

## Definition

Given $\vartheta$ a resolution and $\mathcal{N}$ a finite network, we call weight of the route $r$, the real positive number $s_r(t)$ such that

$$s_r(t) = \int_{r^\vartheta} \gamma_t(x,y)[\chi_{r^\vartheta \setminus (\bigcup_{p\neq r} p^\vartheta)}(x,y)$$

$$+ m_\vartheta(x,y,t) \sum_{p\neq r} \chi_{r^\vartheta \bigcap p^\vartheta}(x,y)]dxdy$$

where $m_\vartheta : \Omega \times [0, T] \to \mathbb{R}^+$ is continuous and called, proximity contribution weight and $\chi$ is the standard characteristic function.

# Weighted Wardrop condition

We assume that for each $r \in \mathcal{R}$, $s_r(t) \neq 0$ for a.e. $t \in \Omega$.

**Definition**

A given family of weights $\{s_r(t)\}_{r \in \mathcal{R}}$, is called Normalized Family of Weights if

$$\sum_{r \in \mathcal{R}} s_r(t) = 1, \forall\ t \ \in \Omega$$

It is clear that each family of weights can be normalized. To define the inner product $\langle \cdot, \cdot \rangle_{\mathbf{a},\mathbf{s}}$ we use a normalized family of weights $\mathbf{s}$.

# Set of feasible flows: set convergence in Mosco's sense

## Lemma

*Let* $\lambda, \mu \in C(\Omega, \mathbb{R}^m_+) \cap L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$, *let* $\rho \in C(\Omega, \mathbb{R}^l_+) \cap L^2(\Omega, \mathbb{R}^l, \mathbf{a}, \mathbf{s})$ *and let* $\{t_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $t_n \to t \in [0, T]$, *as* $n \to +\infty$.
*Then, the sequence of sets*

$$\mathbf{K}(t_n) = \Big\{F(t_n) \in \mathbb{R}^m) : \lambda(t_n) \leq F(t_n) \leq \mu(t_n),\ \Phi F(t_n) = \rho(t_n)\Big\},$$

$\forall n \in \mathbb{N}$, *converges to*

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m) : \lambda(t) \leq F(t) \leq \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

*as* $n \to +\infty$, *in Mosco's sense.*

# Regularity Weighted Wardrop condition

## Theorem

*Let* $\lambda, \mu \in C(\Omega, \mathbb{R}^m_+) \cap L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$, *let* $\rho \in C(\Omega, \mathbb{R}^l_+) \cap L^2(\Omega, \mathbb{R}^l, \mathbf{a}, \mathbf{s})$ *and let* $C : \Omega \times \mathbf{K} \to V^*$ *be a continuous function and* $C(t, \cdot)$ *strictly pseudo-monotone. Then the solution map of the dynamic weighted traffic equilibrium problem is continuous on* $\Omega$.

# Part V

## Computational method

# Discretization method

We consider the evolutionary variational inequality

$$H \in \mathbf{K} : \langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \ \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \quad (*)$$

and we suppose that the assumptions above established are satisfied and hence the solution

$$H \in C([0, T], \mathbb{R}^m_+).$$

As a consequence, $(*)$ holds for each $t \in [0, T]$, namely

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \ \forall t \in [0, T].$$

We consider a partition of $[0, T]$, such that

$$0 = t_0 < \ldots < t_i < \ldots < t_N = T.$$

# Discretization method

Then, for each value $t_i$, for $i = 0, \dots, N$, we obtain a static variational inequality

$$\langle C(t_i, H(t_i)), F(t_i) - H(t_i) \rangle \geq 0, \ \forall F(t_i) \in \mathbf{K}(t_i), \qquad (**)$$

where

$$\mathbf{K}(t_i) = \left\{ F(t_i) \in \mathbb{R}^m_+ : \ \lambda(t_i) \leq F(t_i) \leq \mu(t_i), \ \Phi F(t_i) = \rho(t_i) \right\}.$$

and we apply a projection method to solve it.
After the iterative procedure, we can construct an approximate equilibrium solution by linear interpolation.

# Solodov-Svaiter's method

If we denote by

$$r(H(t)) = H(t) - P_{\mathbf{K}}(H(t) - J_m^{-1} C(t, H(t)))$$

we can note that $r(H(t)) = 0 \Leftrightarrow H(t) \in SVI(C, K)$.

# Solodov-Svaiter's method

Choose $H^0(t_i) \in K$ and two parameters $\gamma \in ]0,1[$ and $\sigma \in ]0,1[$.
Having $H^k(t_i)$, compute $r(H^k(t_i))$. If $r(H^k(t_i)) = 0$ stop.
Otherwise, compute $G^k(t_i) = H^k(t_i) - \eta_i r(H^k(t_i))$, where $\eta_k = \gamma^{h_k}$, with $h_k$ the smallest nonnegative integer $h$ satisfying

$$\langle C(t_i, H^k(t_i) - \gamma^h r(H^k(t_i))), r(H^k(t_i))\rangle_{\mathbf{a},\mathbf{s}} \geq \sigma \|r(H^k(t_i))\|_{\mathbf{s}}^2 \quad (7)$$

Compute

where

$$\partial H^k(t_i) = \left\{ H(t_i) \in \mathbb{R}^m : \ \langle C(t_i, G^k(t_i)), H(t_i) - G^k(t_i)\rangle = 0 \right\}$$

# Discretization method

## Lemma

*Let $B$ be any nonempty closed convex subset of $V$, a not necessary pivot Hilbert space $V$. For any $x, y \in V$ and any $z \in V$ the following properties hold.*

- $(x - P_B(x), z - P_B(x))_V \leq 0.$
- $\|P_B(x) - P_B(y)\|_V^2 \leq \|x - y\|_V^2 - \|P_B(x) - x + y - P_B(y)\|_V^2$

*where $(.,.)_V$ and $\|.\|_V$ are respectively the inner product and the norm of $V$.*

# Discretization method

## Lemma

*Suppose that the linesearch procedure 7 of Algorithm is well defined. Then it holds that*

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)\cap\partial H^k(t_i)}(\bar{H}^k(t_i))$$

*where*

$$\bar{H}^k(t_i) = P_{\partial H^k(t_i)}(H^k(t_i)).$$

# Discretization method

## Lemma

*Let $X$ be strictly convex and smooth Banach space, if we denote $f$ an element of $X^* \setminus \{0\}$, by $\alpha$ a real number and by*

$$K_\alpha = \{x \in V | \langle f, x \rangle_{X^*,X} \leq \alpha\}$$

*We have*

$$P_{K_\alpha}(x) = x^i - max\{0, \frac{\langle f, x \rangle_{X^* \times X} - \alpha}{\|f\|^2_{X^*}}\} J^{-1}(f). \qquad (8)$$

# Discretization method

## Corollary

*For $H^k(t_i)$ construct as specified in Algorithm and $V$ a not necessarily pivot Hilbert space, if*

$$\partial H^k(t_i) = \left\{ H(t_i) \in \mathbb{R}^m : \ \langle C(t_i, G^k(t_i)), H(t_i) - G^k(t_i) \rangle = 0 \right\}$$

*then*

$$P_{\partial H^k(t_i)}(H^k(t_i)) = H^k(t_i) - \frac{\langle C(t_i, G^k(t_i)), H^k(t_i) \rangle_{V^* \times V} - \alpha}{\|C(t_i, G^k(t_i))\|^2_{V^*}} J^{-1}(C(t_i, G^k \tag{9}$$

# Discretization method

**Theorem**

*Let $C$ be continuous and monotone with respect to $\langle ., . \rangle_{V^*,V}$. Suppose $SVI(C, K)$ is nonempty. Then any sequence $\{H^k(t_i)\}_k$ generated by Algorithm converges to a solution of $VI(C, \mathbf{K})$*

# Discretization method

**Theorem**

*Assume that the conditions of Theorems above established are satisfied, then the approximate solution given by:*

$$u_k(t) = \begin{cases} 0 \text{ if } t \in ]0, \epsilon_n[ \\ \sum_{r=1}^{N_k} u(t_k^r)\chi_{[t_k^{r-1}, t_k^r[}(t) \text{ if } t \in [t_k^0, t_k^{N_k}[ \\ \\ 0 \text{ if } t \in ]T - \epsilon_k, T[ \end{cases}$$

*converges to* $u(t)$ *in* $L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ *sense*

# Extragradient method

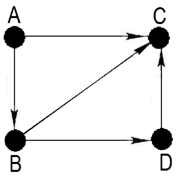
The algorithm, starting from any $H^0(t_i) \in \mathbf{K}(t_i)$ fixed, generates a sequence $\{H^k(t_i)\}_{k\in\mathbb{N}}$ such that

$$\overline{H}^k(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(H^k(t_i))),$$

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(\overline{H}^k(t_i))),$$

where $P_{\mathbf{K}(t_i)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_i)$ and $\alpha$ is constant for all iterations.
If $C$ is monotone and Lipschitz continuous on $\mathbf{K}$ (with Lipschitz constant $L$), and if $\alpha \in (0, 1/L)$, the extragradient method determines a sequence $\{H^k(t_i)\}_{k\in\mathbb{N}}$ convergent to the solution to the static variational inequality $(**)$.

# Example




- O/D pair of nodes $w = (A, C)$ and three paths;
- set of feasible flows $\mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^3_+, (2t, t, 10t), (2(t+1), 0.1t, 0.1(t+2))) : (t+1, 2t+1, t+2) \leq (F_1(t), F_2(t), F_3(t)) \leq (3(t+1), 4t+3, 3t+4), \quad F_1(t) + F_2(t) + F_3(t) = 7t+2, \quad \text{in } [0,2] \Big\}$.
- cost vector-function on the path

$$
\begin{aligned}
C_1(H(t)) &= \frac{1}{4} t H_1(t) + 3t + 1, \\
C_2(H(t)) &= 3t^2 H_2(t) + \sqrt{t^5} H_3^2(t) + t^2 + 2, \\
C_3(H(t)) &= t^2 \sqrt{H_1(t)} + \frac{7}{2} t^3 H_3(t) + \sqrt{t} + \frac{4}{3}.
\end{aligned}
$$

# Example

Monotonicity condition:

$$\langle C(H(t)) - C(F(t)), (H(t) - F(t))\rangle > 0, \quad \forall H \neq F, \ a.e. \ in ]0,2[$$

Lipschitz continuity:

$$\|C(H(t)) - C(F(t))\|_3^2 \leq 27168\|H(t) - F(t)\|_3^2,$$

# Example

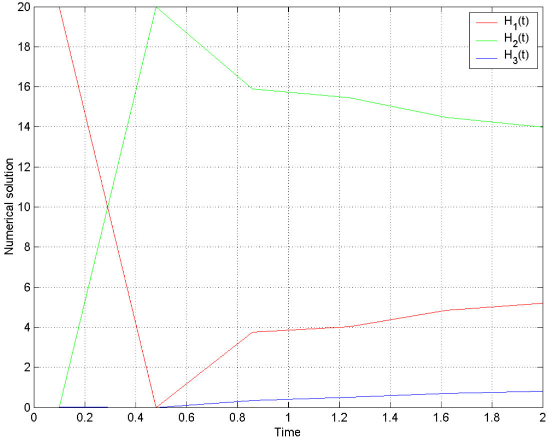
Then the extragradient method is convergent for $\alpha \in (0, 0.00003)$.

We can compute an approximate curve of equilibria, by selecting

$$t_i \in \left\{ \frac{k}{15} : \ k \in \{1, \ldots, 30\} \right\}.$$

# Example

## Curves of equilibria



# References

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