Quasi-variational Inequalities and Applications to Complex Networks

Laura Scrimali

DMI, Università di Catania

Complex Network - Equilibrium and Vulnerability Analysis with Applications
Catania, March 10-12, 2008

Some of the related literature

P.T. Harker
Mixed Equilibrium Behaviors on Networks.
Transportation Science, 22 (1) (1988), 39-46.

B. W. Wie
A differential game approach to the dynamic mixed behavior traffic network equilibrium problem.
European Journal of Operational Research, 83 (1995), 117-136.

X. Zhang, H. Yang and W. Huang
Network equilibrium with mixed behavior.
Intelligent Transportation Systems, 2 (2003), 1499-1504.

Some of the related literature

P.T. Harker
Mixed Equilibrium Behaviors on Networks.
Transportation Science, 22 (1) (1988), 39-46.

B. W. Wie
A differential game approach to the dynamic mixed behavior traffic network equilibrium problem.
European Journal of Operational Research, 83 (1995), 117-136.

X. Zhang, H. Yang and W. Huang
Network equilibrium with mixed behavior.
Intelligent Transportation Systems, 2 (2003), 1499-1504.

Some of the related literature

P.T. Harker
Mixed Equilibrium Behaviors on Networks.
Transportation Science, 22 (1) (1988), 39-46.

B. W. Wie
A differential game approach to the dynamic mixed behavior traffic network equilibrium problem.
European Journal of Operational Research, 83 (1995), 117-136.

X. Zhang, H. Yang and W. Huang
Network equilibrium with mixed behavior.
Intelligent Transportation Systems, 2 (2003), 1499-1504.

Some of the related literature

P.T. Harker
Mixed Equilibrium Behaviors on Networks.
Transportation Science, 22 (1) (1988), 39-46.

B. W. Wie
A differential game approach to the dynamic mixed behavior traffic network equilibrium problem.
European Journal of Operational Research, 83 (1995), 117-136.

X. Zhang, H. Yang and W. Huang
Network equilibrium with mixed behavior.
Intelligent Transportation Systems, 2 (2003), 1499-1504.

My contribution

I present a mixed network equilibrium model in a time-dependent setting with implicit constraints formulated in terms of an infinite-dimensional quasi-variational inequality problem.

Outline

Outline

Typical network equilibria

Definition [User Equilibrium principle, Wardrop 1952]

The journey times of paths are equal and less than those which would be experienced by a single vehicle on any unused path.

Definition

For all $(s,d) \in D$ and $\forall p, q \in \mathcal{P}_{(s,d)}$, a feasible flow x^* is a User Equilibrium if it fulfills the following condition:

$$C_p^i(x^*) < C_q^i(x^*) \Rightarrow x_q^{*i} = 0.$$

Typical network equilibria

Definition [User Equilibrium principle, Wardrop 1952]

The journey times of paths are equal and less than those which would be experienced by a single vehicle on any unused path.

Definition

For all $(s, d) \in D$ and $\forall p, q \in \mathcal{P}_{(s,d)}$, a feasible flow x^* is a User Equilibrium if it fulfills the following condition:

$$C_p^i(x^*) < C_q^i(x^*) \Rightarrow x_q^{*i} = 0.$$

Definition [System Optimum principle, Wardrop 1952]

The average journey time is minimal.

Definition

A feasible flow x^* is a System Optimum Equilibrium if it satisfies the following minimization problem:

$$\min \sum_{a=1}^{n} f_a c_a(f_a)$$

$$\sum_{p \in \mathcal{P}_{(s,d)}} x_p = \delta_{(s,d)}, \quad \forall (s,d) \in D$$

$$f_a = \sum_{p \in \mathcal{P}} \gamma_{ap} x_p, \quad \forall a \in \mathcal{L}$$

$$x_p \geq 0, \quad \forall p \in \mathcal{P}.$$

Definition [System Optimum principle, Wardrop 1952]

The average journey time is minimal.

Definition

A feasible flow x^* is a System Optimum Equilibrium if it satisfies the following minimization problem:

$$\min \sum_{a=1}^{n} f_a c_a(f_a)$$

$$\sum_{p \in \mathcal{P}_{(s,d)}} x_p = \delta_{(s,d)}, \quad \forall (s,d) \in D$$

$$f_a = \sum_{p \in \mathcal{P}} \gamma_{ap} x_p, \quad \forall a \in \mathcal{L}$$

$$x_p \geq 0, \quad \forall p \in \mathcal{P}.$$

From network equilibria to game theory

One can assume that a finite number of players compete for the utilization of a common transportation network.

UE game-theoretic model

Infinitely many "infinitesimal" players aim to find the shortest path based on the choices of the other players. Users are fully competitive.

SO game-theoretic model

One player makes the route choices on behalf of all the other users and attempts to minimize the total cost of the system. Users are fully cooperative.

From network equilibria to game theory

One can assume that a finite number of players compete for the utilization of a common transportation network.

UE game-theoretic model

Infinitely many "infinitesimal" players aim to find the shortest path based on the choices of the other players. Users are fully competitive.

SO game-theoretic model

One player makes the route choices on behalf of all the other users and attempts to minimize the total cost of the system. Users are fully cooperative.

From network equilibria to game theory

One can assume that a finite number of players compete for the utilization of a common transportation network.

UE game-theoretic model

Infinitely many "infinitesimal" players aim to find the shortest path based on the choices of the other players. Users are fully competitive.

SO game-theoretic model

One player makes the route choices on behalf of all the other users and attempts to minimize the total cost of the system. Users are fully cooperative.

Typical game-theoretic equilibria

Definition [Nash equilibrium, Nash 1950]

The Nash game with N players consists in finding a tuple $x^* = (x^{*i})_{i=1}^N \in \mathbb{R}^n$, called Nash Equilibrium, such that for each $i = 1, \ldots, N$, x^{*i} is an optimal solution of the convex optimization problem in the variable x^i with x^{-i} fixed at x^{*-i}:

$$\begin{aligned} &\min u_i(x^{*-i}, x^i) \\ &x^i \in K^i, \end{aligned}$$

where $u_i : \mathbb{R}^n \to \mathbb{R}$ is the utility function, $K^i \subset \mathbb{R}^{n_i}$ is the set of strategies, $n_i \in \mathbb{N}$.

Definition [Generalized Nash equilibrium, Debreu 1952]

The generalized Nash game with N players consists in finding a tuple $x^* = (x^{*i})_{i=1}^N \in \mathbb{R}^n$, called Generalized Nash Equilibrium, such that for each $i = 1, \dots, N$, x^{*i} is an optimal solution of the convex optimization problem in the variable x^i with x^{-i} fixed at x^{*-i}:

$$\begin{array}{l} \min u_i(x^{*-i}, x^i) \\ x^i \in K^i(x^{*-i}), \end{array}$$

where $u_i : \mathbb{R}^n \to \mathbb{R}$ is the utility function, $K^i : R^{n_{-i}} \to 2^{\mathbb{R}^{n_i}}$ is a given set-valued map, $n_i \in \mathbb{N}$, $n = \sum_{i=1}^N n_i$, $n_{-i} = n - n_i$.

Common framework

All the equilibrium principles presented have a variational or quasi-variational inequality formulation.

Mixed network competition

In several equilibrium situations a mixed behavior is observed, namely there are both competition and cooperation among users over the network.

Examples

- In transportation networks carriers seek to minimize the total cost of transportation on their exclusive networks (SO model); smaller shippers look for the shortest path (UE model); larger shippers minimize their own costs (SO model).
- In telecommunication networks routing strategies to ship jobs are chosen either by a single decision maker (SO model) or they are determined individually by each user (UE model).

Mixed network competition

In several equilibrium situations a mixed behavior is observed, namely there are both competition and cooperation among users over the network.

Examples

- In transportation networks carriers seek to minimize the total cost of transportation on their exclusive networks (SO model); smaller shippers look for the shortest path (UE model); larger shippers minimize their own costs (SO model).
- In telecommunication networks routing strategies to ship jobs are chosen either by a single decision maker (SO model) or they are determined individually by each user (UE model).

Mixed network competition

In several equilibrium situations a mixed behavior is observed, namely there are both competition and cooperation among users over the network.

Examples

- In transportation networks carriers seek to minimize the total cost of transportation on their exclusive networks (SO model); smaller shippers look for the shortest path (UE model); larger shippers minimize their own costs (SO model).
- In telecommunication networks routing strategies to ship jobs are chosen either by a single decision maker (SO model) or they are determined individually by each user (UE model).

Outline

UE-GN players competition model

In our model we assume that there exist some classes of individual users and some GN players. Since there is a large number of individual users and a single driver has a negligible impact on the load of the network, we assume that for each class a single integrated UE player controls all the users who satisfy the UE principle. There are also some GN players whose objective is to minimize the cost of the users under their control. They have a significant impact on the load of the network and are responsible for the delay in which any other user may incur.

Model description

- $[0, T]$ time horizon
- $\mathcal{M}$ set of nodes
- $\mathcal{L}$ set of links
- $\mathcal{W}$ set of UE players
- $\mathcal{N}$ set of GN players
- $\mathcal{I} = \mathcal{N} \cup \mathcal{W}$
- a typical link
- p typical path

- $f_a^i(t)$ the flow sent by player i and circulating on link a at time t
- $(f^i(t))^T = (f_a^i(t))_{a\in\mathcal{L}}$, $(f(t))^T = (f^i(t)^T)_{i\in\mathcal{I}}$
- $x_p^i(t)$ the flow sent by player i and circulating on path p at time t
- $(x^i(t))^T = (x_p^i(t))_{p\in\mathcal{P}^i}$, $(x(t))^T = (x^i(t)^T)_{i\in\mathcal{I}}$
- $(x^*(t))^T = (x^{*i}(t)^T)_{i\in\mathcal{I}}$ optimal multiflow
- $c_a^i(f(t))$ link cost function
- $\delta^i_{(s,d)}(t,x) : [0,T]\times\mathbb{R}_+^{n+w}\to\mathbb{R}_+^m$, $i\in\mathcal{I}$ implicit load balancing for flows routed from s to d and leaving s at time t, such that
  a) Each component of $\delta_{(s,d)}(t,x)$ is measurable in t $\forall x\in\mathbb{R}_+^{n+w}$, continuous at x for a.e. $t\in[0,T]$, and

$$\exists\gamma_1\in L^2(0,T) : \|\delta_{(s,d)}(t,x)\|\le\gamma_1(t)+\|x\|.$$

Constraint set

For $i \in \mathcal{I}$, let E^i be a nonempty, convex, bounded and closed subset of $L^2(0,T;\mathbb{R}^{n+w})$ and let $K^i : \prod_{k \neq i} E^k \to E^i$ be the set-valued map which represents the set of the strategies of player i defined as

$$
\begin{aligned}
K^i(x^{-i}) \quad = \quad & \Big\{ x^i(t) \in E^i : \forall (s,d) \in D^i, \forall p \in \mathcal{P}^i_{(s,d)}, \\
& 0 \leq x^i_p(t) \leq \overline{x}^i_p(t), \sum_{p \in \mathcal{P}^i_{(s,d)}} x^i_p(t) = \delta^i_{(s,d)}(t, x^i(t)) \\
& \text{a.e. } t \in [0,T] \Big\},
\end{aligned}
$$

where x^{-i} is the vector of all the players' decision variables except those of player i, and $\overline{x}^i_p(t)$ is the upper bound on the flow sent through path p by user i at time t.

UE players' behavior

For all $i \in \mathcal{W}$, $p \in \mathcal{P}^{\rangle}$, we define the cost function

$$C_p^i(t,x) : [0,T] \times \mathbb{R}_+^{n+w} \to \mathbb{R}_+.$$

$$C(t,x) = (C_p^i(t,x))_{i \in \mathcal{W}, p \in \mathcal{P}^i}.$$

Assumptions

b) Each component of $C(t,x)$ is measurable in t $\forall x \in \mathbb{R}_+^{n+w}$, continuous at x for a.e. $t \in [0,T]$, and

$$\exists \gamma_2 \in L^2(0,T) : \|C(t,x)\| \leq \gamma_2(t) + \|x\|.$$

UE players' behavior

For all $i \in \mathcal{W}$, $p \in \mathcal{P}^{\rangle}$, we define the cost function

$$C_p^i(t, x) : [0, T] \times \mathbb{R}_+^{n+w} \to \mathbb{R}_+.$$

$$C(t, x) = (C_p^i(t, x))_{i \in \mathcal{W}, p \in \mathcal{P}^i}.$$

Assumptions

b) Each component of $C(t, x)$ is measurable in t $\forall x \in \mathbb{R}_+^{n+w}$, continuous at x for a.e. $t \in [0, T]$, and

$$\exists \gamma_2 \in L^2(0, T) : \|C(t, x)\| \leq \gamma_2(t) + \|x\|.$$

The aim of UE player $i \in \mathcal{W}$ is to choose the shortest path based on the choices of the other players, specifically to satisfy the following constrained Wardrop principle (Maugeri et al. 1997, Larsson et al. 1999).

Definition [Constrained Wardrop principle]

For all $i \in \mathcal{W}$, $\forall (s,d) \in D^i$ and $\forall p, q \in \mathcal{P}^i_{(s,d)}$, $x^*(t) \in K(x^{*-i})$ is a constrained User Equilibrium if it fulfills the following condition, a.e. on $[0,T]$

$$C^i_p(t, x^*(t)) < C^i_q(t, x^*(t)) \Rightarrow x^{*i}_p(t) = \overline{x}^{*i}_p(t) \text{ or } x^{*i}_q(t) = 0.$$

UE players' QVI

For each player $i \in \mathcal{W}$ the UE principle is equivalent with the quasi-variational inequality problem:

$$\int_0^T \sum_{p \in \mathcal{P}^i} C_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0, \forall x^i \in K(x^{*-i}). \quad (1)$$

GN players' behavior

For all $i \in \mathcal{N}$ we define the cost function

$$J^i(t,x) : [0,T] \times \mathbb{R}_+^{n+w} \to \mathbb{R}_+.$$

$J(t,x) = (J^i(t,x))_{i \in \mathcal{N}}$.

Assumptions

c) Each component of $J(t,x)$ is measurable in t $\forall x \in \mathbb{R}_+^{n+w}$, convex and continuously differentiable with respect to x for a.e. $t \in [0,T]$, and

$$\exists \gamma_3 \in L^1(0,T) : \|J(t,x)\| \leq \gamma_3(t) + \|x\|^2.$$

GN players' behavior

For all $i \in \mathcal{N}$ we define the cost function

$$J^i(t,x) : [0,T] \times \mathbb{R}_+^{n+w} \longrightarrow \mathbb{R}_+.$$

$J(t,x) = (J^i(t,x))_{i \in \mathcal{N}}$.

Assumptions

c) Each component of $J(t,x)$ is measurable in t $\forall x \in \mathbb{R}_+^{n+w}$, convex and continuously differentiable with respect to x for a.e. $t \in [0,T]$, and

$$\exists \gamma_3 \in L^1(0,T) : \|J(t,x)\| \leq \gamma_3(t) + \|x\|^2.$$

For $i \in \mathcal{N}$, let $\Psi^i_p(t,x)$ be the derivative of $J^i(t,x)$ with respect to x^i_p. Then it result $\Psi(t,x) = (\Psi^i_p(t,x))_{i\in\mathcal{N}, p\in\mathcal{P}^i} \in \mathbb{R}^n$.

Assumptions

d) Each component of $\Psi(t,x)$ is measurable in t $\forall x \in \mathbb{R}^{n+w}_+$, continuous at x for a.e. $t \in [0,T]$, and

$$\exists \gamma_4 \in L^2(0,T) : \|\Psi(t,x)\| \le \gamma_4(t) + \|x\|.$$

For $i \in \mathcal{N}$, let $\Psi_p^i(t,x)$ be the derivative of $J^i(t,x)$ with respect to x_p^i. Then it result $\Psi(t,x) = (\Psi_p^i(t,x))_{i\in\mathcal{N},p\in\mathcal{P}^i} \in \mathbb{R}^n$.

Assumptions

d) Each component of $\Psi(t,x)$ is measurable in t $\forall x \in \mathbb{R}_+^{n+w}$, continuous at x for a.e. $t \in [0,T]$, and

$$\exists \gamma_4 \in L^2(0,T) : \|\Psi(t,x)\| \le \gamma_4(t) + \|x\|.$$

Definition

The aim of GN player $i \in \mathcal{N}$, given the other players' strategies x^{*-i}, is to choose a Generalized Nash Equilibrium x^i that solves the minimization problem

$$i \in \mathcal{N}, \qquad \min_{x^i(t) \in K^i(x^{*-i})} \int_0^T J^i(t, x^{*-i}(t), x^i(t))dt. \tag{2}$$

Remark

The functional J^i is convex and weakly lower semi-continuous, $K^i(x^{*-i})$ is weakly compact, thus the minimization problem admit at least one solution.

Definition

The aim of GN player $i \in \mathcal{N}$, given the other players' strategies x^{*-i}, is to choose a Generalized Nash Equilibrium x^i that solves the minimization problem

$$i \in \mathcal{N}, \qquad \min_{x^i(t) \in K^i(x^{*-i})} \int_0^T J^i(t, x^{*-i}(t), x^i(t))dt. \qquad (2)$$

Remark

The functional J^i is convex and weakly lower semi-continuous, $K^i(x^{*-i})$ is weakly compact, thus the minimization problem admit at least one solution.

GN players' QVI

Under assumptions $c)$ and $d)$ the minimization problem of each GN player $i \in \mathcal{N}$, can be formulated as the quasi-variational inequality:

$$\int_0^T \sum_{p \in \mathcal{P}^i} \Psi_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0, \forall x^i \in K(x^{*-i}). \quad (3)$$

Remark

It is possible to consider a more complex model, where players' competition is explicitly taken into account at the level of costs. We introduce the perceived link costs $\overline{c}_a^i(f(t))$ of player $i \in \mathcal{I}$ given by

$$\overline{c}_a^i(f(t)) = \begin{cases} c_a^i(f(t)) & \text{if } i \in \mathcal{W}, \\ c_a^i(f(t)) + f_a^i(t) c_a^{i\prime}(f(t)) & \text{if } i \in \mathcal{N}. \end{cases}$$

We can choose

$$\begin{aligned} C_p^i(t, x(t)) &= \sum_{a \in \mathcal{L}} \gamma_{ap} \overline{c}_a^i(f(t)), \quad p \in \mathcal{P}^i, i \in \mathcal{W}, \\ \Psi_p^i(t, x(t)) &= \sum_{a \in \mathcal{L}} \gamma_{ap} \overline{c}_a^i(f(t)), \quad p \in \mathcal{P}^i, i \in \mathcal{N}. \end{aligned}$$

Mixed network equilibrium

Definition

If the quasi-variational inequality problems for each player

$$\int_0^T \sum_{p \in \mathcal{P}^i} C_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0, \forall x^i \in K(x^{*-i}),$$

$$\int_0^T \sum_{p \in \mathcal{P}^i} \Psi_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0, \forall x^i \in K(x^{*-i}),$$

are solved simultaneously, then the solution is called a Mixed Equilibrium.

QVI formulation

The ME can be formulated as a single quasi-variational inequality.

$$E = \prod_{i \in \mathcal{I}} E^i, \quad K(x) = \prod_{i \in \mathcal{I}} K^i(x^{-i}), \text{ with } x = (x^i) \in E.$$

Theorem

$x^*(t) \in K(x^*)$ is a ME iff $\forall x(t) \in K(x^*)$ the QVI holds:

$$\int_0^T \langle F(t, x^*(t)), x(t) - x^*(t) \rangle dt$$

$$= \sum_{i \in \mathcal{N}} \int_0^T \sum_{p \in \mathcal{P}^i} \Psi_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt$$

$$+ \sum_{i \in \mathcal{W}} \int_0^T \sum_{p \in \mathcal{P}^i} C_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0.$$

QVI formulation

The ME can be formulated as a single quasi-variational inequality.

$$E = \prod_{i \in \mathcal{I}} E^i, \quad K(x) = \prod_{i \in \mathcal{I}} K^i(x^{-i}), \text{ with } x = (x^i) \in E.$$

Theorem

$x^*(t) \in K(x^*)$ is a ME iff $\forall x(t) \in K(x^*)$ the QVI holds:

$$\int_0^T \langle F(t, x^*(t)), x(t) - x^*(t) \rangle dt$$

$$= \sum_{i \in \mathcal{N}} \int_0^T \sum_{p \in \mathcal{P}^i} \Psi_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt$$

$$+ \sum_{i \in \mathcal{W}} \int_0^T \sum_{p \in \mathcal{P}^i} C_p^i(t, x^*(t))(x_p^i(t) - x_p^{*i}(t))dt \geq 0.$$

Outline

Existence of solutions

Theorem

Let us assume that a), b), d) and the following assumptions hold

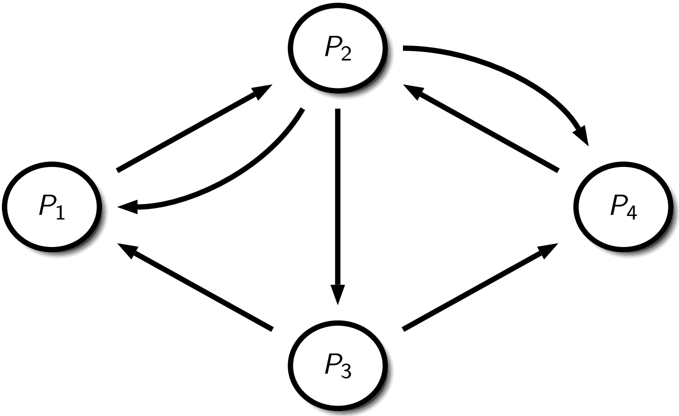
e) $\exists \nu \in L^2(0, T)$, $\nu(t) \geq 0$ for a.e. $t \in [0, T]$:

$$\|\delta(t, x_1) - \delta(t, x_2)\| \leq \nu(t)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^{n+w};$$

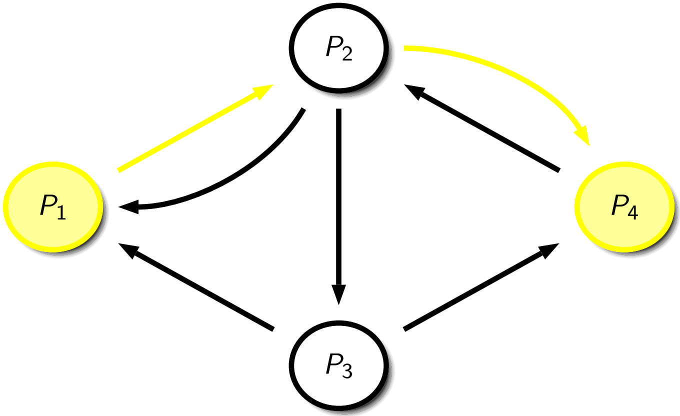
f) Each component of $\Psi(t, x)$, $C(t, x)$ and $\delta(t, x)$ is convex in x for a.e. $t \in [0, T]$ and upper semi-continuous with respect to the weak topology in $x \in E$ for a.e. $t \in [0, T]$.

Then quasi-variational inequality problem (4) admits a solution.

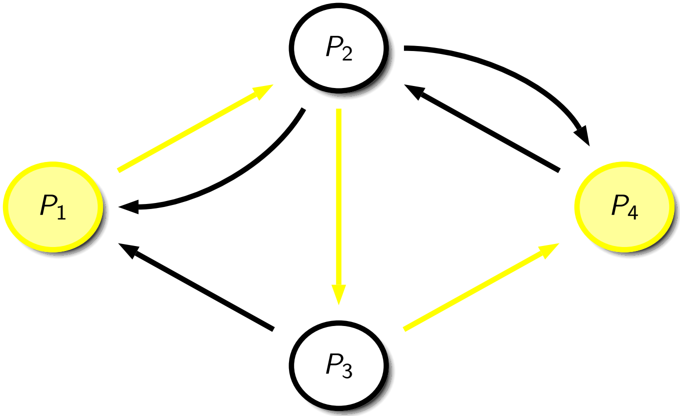
Numerical example



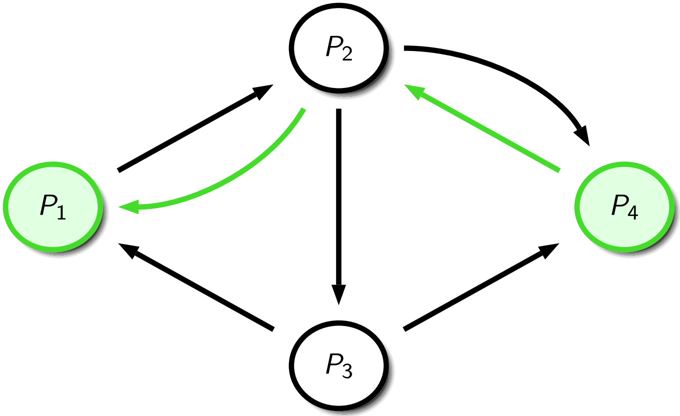
Numerical example



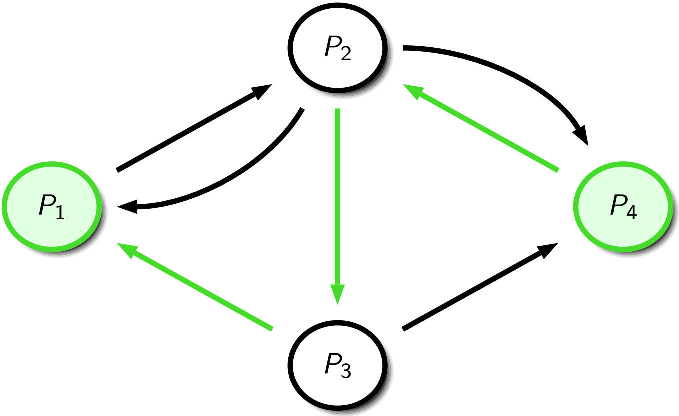
Numerical example



Numerical example



Numerical example



- Time horizon $[0, T] = [0, 1]$
- origin-destination pairs: (P_1, P_4) and (P_4, P_1)
- travel demands respectively

$$\begin{aligned} \delta_1(t, x^*(t)) &= 10(1-t) + \frac{2}{3}x_1^*(t) + 1, \\ \delta_2(t, x^*(t)) &= 4(1-t) + \frac{1}{2}x_4^*(t) + 3. \end{aligned}$$

O/D pair (P_1, P_4) is controlled by a GN player, O/D pair (P_4, P_1) is controlled by a UE player.

Cost functions

$$c_1(f(t)) = 20 + f_1(t), \quad c_2(f(t)) = 20 + f_2(t), \quad c_3(f(t)) = 5 + 2f_3(t),$$

$$c_4(f(t)) = 20 + f_4(t), \quad c_5(f(t)) = 5 + 2f_5(6), \quad c_6(f(t)) = 5 + 2f_6(t),$$

$$c_7(f(t)) = 20 + f_7(t).$$

$$J(x(t)) = \sum_{p=1}^{2} x_p(t) \sum_{a=1}^{7} \gamma_{ap} c_a(f(t)),$$

$$C_p(x(t)) = \sum_{a=1}^{7} \gamma_{ap} c_a(f(t)), \quad p = 3, 4.$$

QVI operator

$$
\begin{aligned}
F^{ME}(x(t)) &= \begin{pmatrix} \Psi(x(t)) \\ C(x(t)) \end{pmatrix}, \\
\Psi(x(t)) &= \begin{pmatrix} \sum_{a\in\mathcal{L}} \gamma_{a1}\left(c_a(f(t)) + x_a(t)\frac{\partial c_a(f(t))}{\partial f_a(t)}\right) \\ \sum_{a\in\mathcal{L}} \gamma_{a2}\left(c_a(f(t)) + x_a(t)\frac{\partial c_a(f(t))}{\partial f_a(t)}\right) \end{pmatrix}, \\
C(x(t)) &= \begin{pmatrix} C_3(x(t)) \\ C_4(x(t)) \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
F_1^{ME}(x(t)) &= 4x_1(t) + 2x_2(t) + 40, \\
F_2^{ME}(x(t)) &= 2x_1(t) + 10x_2(t) + 4x_4(t) + 30, \\
F_3^{ME}(x(t)) &= 2x_3(t) + x_4(t) + 40, \\
F_4^{ME}(x(t)) &= 2x_2(t) + x_3(t) + 5x_4(t) + 30.
\end{aligned}
$$

QVI formulation

Set of feasible flows:

$$K(x^*) = \Big\{x(t) \in L^2(0,1;\mathbb{R}^4) : x_i(t) \geq 0, i = 1, \ldots, 4; \\ x_1(t) + x_2(t) = \delta_1(t, x^*(t)), x_3(t) + x_4(t) = \delta_2(t, x^*(t)) \\ \text{a.e. in } [0, T]\Big\}.$$

The ME is a solution to the quasi-variational inequality

$$\int_0^1 \langle F^{ME}(x^*_{ME}(t)), x(t) - x^*_{ME}(t)\rangle dt \geq 0, \quad \forall x(t) \in K(x^*). \quad (4)$$

Solutions

ME Solution

$$x^*_{ME}(t) = \left(-\frac{888}{55}t + \frac{993}{55}, -\frac{254}{55}t + \frac{274}{55}, -\frac{252}{55}t + \frac{342}{55}, \frac{64}{55}t + \frac{86}{55}\right)$$

SO Solution

$$x^*_{SO}(t) = \left(-\frac{888}{55}t + \frac{933}{55}, -\frac{254}{55}t + \frac{294}{55}, -\frac{252}{55}t + \frac{377}{55}, \frac{64}{55}t + \frac{16}{55}\right)$$

Solutions

ME Solution

$$x^*_{ME}(t) = \left(-\frac{888}{55}t + \frac{993}{55}, -\frac{254}{55}t + \frac{274}{55}, -\frac{252}{55}t + \frac{342}{55}, \frac{64}{55}t + \frac{86}{55}\right)$$

SO Solution

$$x^*_{SO}(t) = \left(-\frac{888}{55}t + \frac{933}{55}, -\frac{254}{55}t + \frac{294}{55}, -\frac{252}{55}t + \frac{377}{55}, \frac{64}{55}t + \frac{16}{55}\right)$$

Solutions

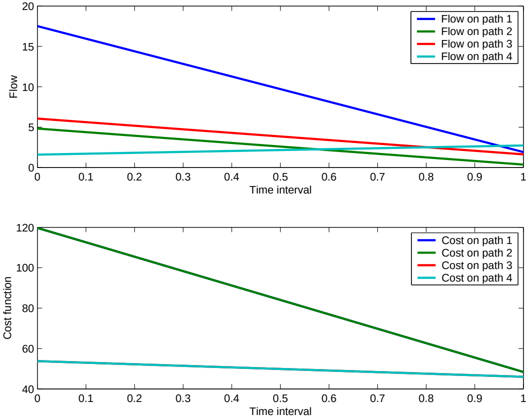
ME Solution

$$x^*_{ME}(t) = \left(-\frac{888}{55}t + \frac{993}{55}, -\frac{254}{55}t + \frac{274}{55}, -\frac{252}{55}t + \frac{342}{55}, \frac{64}{55}t + \frac{86}{55}\right)$$

SO Solution

$$x^*_{SO}(t) = \left(-\frac{888}{55}t + \frac{933}{55}, -\frac{254}{55}t + \frac{294}{55}, -\frac{252}{55}t + \frac{377}{55}, \frac{64}{55}t + \frac{16}{55}\right)$$

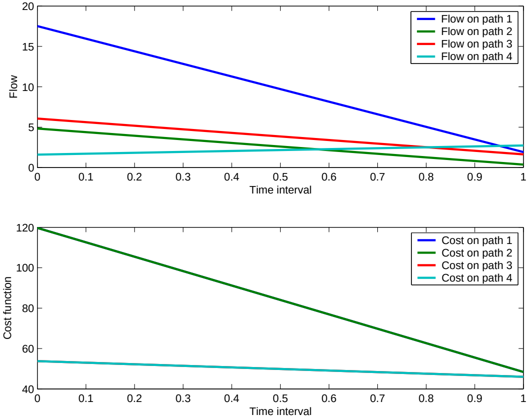
Graphical illustration



Remark

Paradox: $F_{(P_1,P_4)}(t,x(t)) > F_{(P_4,P_1)}(t,x(t))$

Graphical illustration



Remark

Paradox:
$F_{(P_1,P_4)}(t, x(t)) > F_{(P_4,P_1)}(t, x(t))$

Outline

Parametric perturbations

Original problem:

$$(QVI) \quad \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K(x^*),$$

Suppose that F is perturbed by means of a parameter μ, whereas the map δ is perturbed by a parameter λ. Given a pair (μ, λ) of parameters around the initial value $(\bar{\mu}, \bar{\lambda})$, the corresponding parametric quasi-variational inequality $QVI_{(\mu,\lambda)}$ is: Find $x^*(\mu, \lambda) \in K_\lambda(x^*(\mu, \lambda))$ such that

$$(QVI_{\mu,\lambda}) \quad \langle F(x^*(\mu, \lambda), \mu), x - x^*(\mu, \lambda) \rangle \geq 0, \quad \forall x \in K_\lambda(x^*(\mu, \lambda)).$$

The above problem can be considered as a perturbed form of the problem (QVI), so that $\bar{x^*} = x^*(\bar{\mu}, \bar{\lambda})$ is a solution to $(QVI_{\bar{\mu},\bar{\lambda}})$.

Assumptions

(h_0) δ is Hölder continuous, i.e., for some $L_1, L_2 > 0$ and $\xi, \xi' \in]0,1[$, and $\forall x^*, x \in X, \forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$

$$|\delta_\lambda(x^*) - \delta_{\lambda'}(x)| \leq L_1 \|\lambda - \lambda'\|^{\xi'} + L_2 |x^* - x|^{\xi};$$

(h_1) F is uniformly strongly monotone, i.e., for some $m > 0$,

$$\langle F(x^*, \mu) - F(x, \mu), x^* - x \rangle \geq m|x^* - x|^2, \ \forall x^*, x \in X, \ \forall \mu \in \mathcal{V}(\bar{\mu});$$

(h_2) for some $b_0 > 0$, for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $x^* \in X$ one has $|F(x^*, \mu)| \leq b_0$;

(h_3) for some $\gamma \in]0,1[$ and $c > 0$, $\mu \mapsto F(\cdot, \mu)$ is uniformly (in x^*) (γ, c)-Hölder, i.e., for all $x^* \in X$ and all $\mu, \mu' \in \mathcal{V}(\bar{\mu})$,

$$|F(x^*, \mu) - F(x^*, \mu')| \leq c\|\mu - \mu'\|^{\gamma}.$$

Analysis of constraints

Proposition

Assume that (h_0) holds. Then, there exist $k_1, k_2 > 0$ such that $\forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$, and $\forall x^*, x \in E$ one has:

$$K_\lambda(x^*) \subset K_{\lambda'}(x) + (k_1\|\lambda - \lambda'\|^{\xi'} + k_2|x^* - x|^{\xi})\bar{\mathbb{B}}_m, \tag{5}$$

where $\bar{\mathbb{B}}_m$ denotes the unit closed ball in $\mathbb{R}^m$.

Hölder continuity of solutions

Theorem [Ait Mansour and S. 2008]

Assume that $\bar{x^*} = x^*(\bar{\mu}, \bar{\lambda})$ is a solution to $(QVI) = (QVI_{\bar{\mu},\bar{\lambda}})$, conditions $(h_0) - (h_4)$ hold and $m > 2bk_2^{\beta}$, where $b = (\frac{2}{\delta_0})^{\beta-1} b_0$ and $\beta = \frac{2}{\xi}$. Then, the solution $x^*(\mu, \lambda)$ to $(QVI_{\mu,\lambda})$ is unique in a neighborhood X of the solution and verifies the following condition: there exist $c_1, c_2 > 0, d_1, d_2 \in]0, 1[$ such that

$$|x^*(\mu,\lambda) - x^*(\mu',\lambda')| \leq c_1\|\mu - \mu'\|^{d_1} + c_2\|\lambda - \lambda'\|^{d_2}, \qquad (6)$$

for all $\mu, \mu' \in \mathcal{V}(\bar{\mu}), \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$.