Recent results and applications in Hilbert spaces for PDS and VI when duality appears.

Stéphane Nicolas Pia

Università di Catania, Italy
Join work with Monica Cojocaru University of Guelph - Ontario Canada

March 2008, Catania

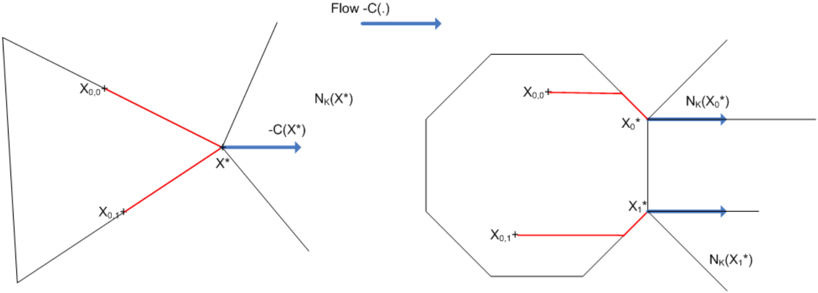
Outline

Outline

Just to fix some ideas

A lot of problems can be formulated in terms of Variational Inequalities (VI) and Projected Dynamical Systems (PDS) [Nagurney, Dong, 2002].
To fix the ideas we say that a solution of a VI gives the equilibrium point of a given problem and the solution of a PDS gives the trajectory that reaches the equilibrium (critical point for PDS) from an initial point [Cojocaru, Daniele, Nagurney, 2004].

Visio's approach



- X^* Equilibrium $\Leftrightarrow \langle C(X^*), y - X^* \rangle \geq 0, \ \forall y \in K$
- $X(t)$ gives the trajectory (red) and solves the PDS
  $\dot{X} = P_{T_K(X)}(-C(X)), X(0) = X_0 \in K$

Some relevant dates

- Variational Inequalities Theory introduced by Hartmann and Stampacchia in 1966.
- S.Dafermos recognized in 1980 that the traffic network equilibrium conditions as formulated by Smith in 1979 had a structure of variational inequality
- P.Dupuis and A.Nagurney introduced in 1993 the Projected Dynamical Systems theory to describe the states that preceed the VI equilibrium.
- P.Daniele, A.Maugeri and W.Oetlli in 1999 introduced the time dependent traffic equilibrium problem.
- M.Cojocaru, in 2002 extended the theory of PDS to Hilbert spaces.

To date, many problems are being formulated in terms of evolutionary variational inequalities and projected dynamical systems [Nagurney, Dong, 2002]. As usual, some questions appears...

Does it make sense to generalize?

Our questions:

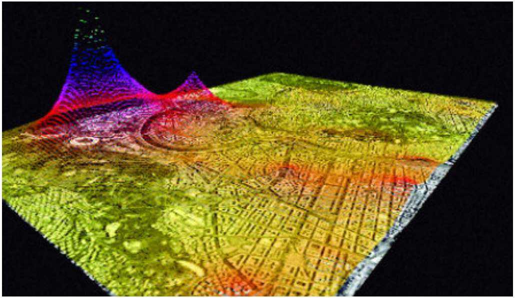
- How can we extend known results without Banach spaces? What are the possibilities in B-Spaces?
- Evolutionary problems are problems in functional spaces where the domain is 1 dimensional. What about n-dimensional domains?
- Do we have potential industrial applications?

Our trigger: Wireless Communications Applications and Traffic Problems

SENSEable City Laboratory at MIT developed the following smart idea:

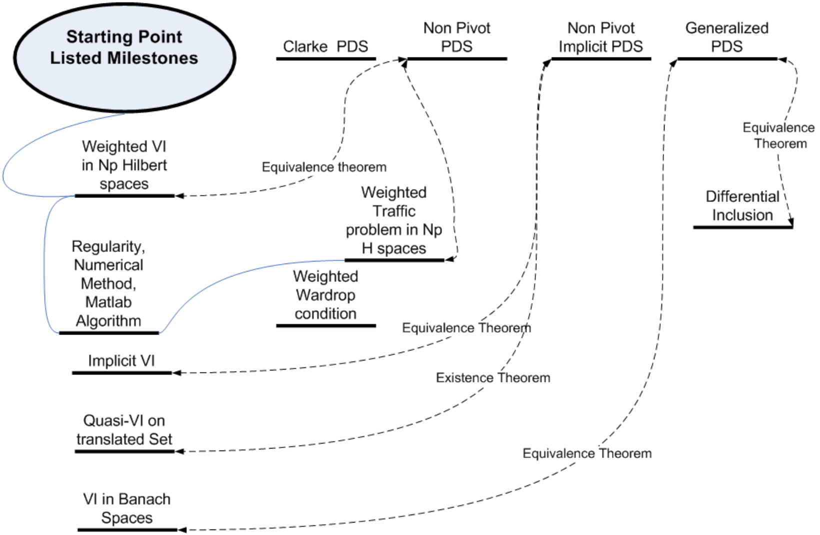
Use wireless devices to provide centralized localization data

$\Rightarrow$ Real time data and Low costs.

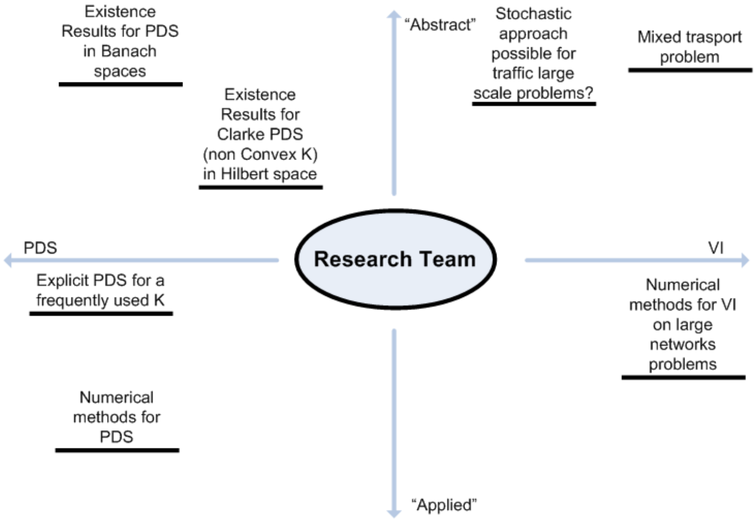


Trying to integrate wireless device information into the evolutionary traffic model developed in [Daniele, Maugeri, Oettli, 1999] we get in touch with interesting directions of research.

Basic summary of our achievements

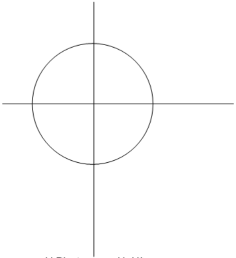


Directions of research

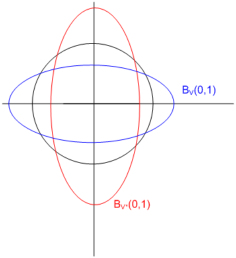


Outline

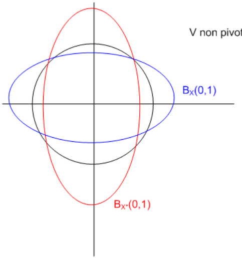
A question of Balls



H Pivot space, H=H*,
$B_H(0,1)=B_{H^*}(0,1)$



V non pivot space, $V \neq V^*$, $V^*=J(V)$, $B_{V^*}(0,1)=J(B_H(0,1))$,
J linear



$X \neq X^*$, $X^*=J(X)$, $B_{X^*}(0,1)=J(B_X(0,1))$, J non linear, X
reflexive strictly convex and smooth (SCS)

Dual Realization of a Hilbert Space - 1

First of all consider a pre-Hilbert space V for a inner-product (x, y) and its topological dual $V^* = \mathcal{L}(V, \mathbb{R})$, it is well known that V^* is a Banach Space for the classical dual norm ($\|f\|_* = \sup_{x \in V} \frac{|f(x)|}{\|x\|}$). It is also known that there exists an isometry $J : V \to V^*$ such that for all $x \in V$, $J(x) = grad(\frac{\|x\|^2}{2})$, J is linear and it is called the duality mapping.

Theorem ([Aubin, 1987])

Let V be a Hilbert space for the inner product (x, y) and $J \in \mathcal{L}(V, V^*)$ the duality mapping. Then J is a surjective isometry from V to V^*. The dual space V^* is a Hilbert space for the inner product:

$$((f, g))_* = ((J^{-1}f, J^{-1}g)) = f(J^{-1}g)$$

Even if V is a Hilbert space, V^* is an abstract space (no characterization of its elements). In order to have a concrete characterization for the element of the dual space we consider a concrete Hilbert space F. More precisely we define...

Dual Realization of a Hilbert Space - 2

Definition

Let V be a Hilbert space. We call $\{F, j\}$, where

- i) F is a Hilbert space
- ii) j is an isometry from F to V^*

a dual realization of V.
Then we set

$$< f, x >= j \circ f(x), \forall f \in F, \ \forall x \in V$$

$< f, x >$ is called the duality pairing for $F \times V$.

Convention: when we choose a dual realization $\{F, j\}$, we set $F = V^*$ and $j \circ f(x) =< f, x >$ we will say that the isometry $K : V \rightarrow V^*$, $K = j^{-1} \circ J$ is the duality operator associated to the inner product on V and to the duality pairing on $V^* \times V$ by the relation $(x, y) =< K(x), y >$

Definition

A Hilbert space H for the inner product (x, y) is called a pivot space, if we identify H^* with H (if we choose as dual realization of H, $\{H, J\}$). In that case

$$H^* = H, \ j = J, \ < x, y >= (x, y)$$

Dual Realization of a Hilbert Space - 3, Examples

- Sometimes it doesn't make sense to identify the space with its topological dual as the following example shows: Let's consider $V = L^2(\mathbb{R}, (1+|x|)) \subset L^2(\mathbb{R})$ (dense subspace of $L^2(\mathbb{R})$) endowed with the inner product;

$$(u, v)_V = \int_{\mathbb{R}} (1+|x|)u(x)v(x)dx$$

an element $\varphi \in L^2(\mathbb{R})^*$ is also an element of V^*. If we identify φ to an element $f \in L^2(\mathbb{R})$, this function doesn't define a linear form on V and the expression $\varphi(v) = \langle f, v\rangle_V$ has no meaning on V. In this situation it is necessarily to work in a non pivot Hilbert space. We have the following inclusions

$$V \subset L^2(\mathbb{R}) \subset V^*$$

$L^2(\mathbb{R})$ is the pivot space.

- More generally $V = L^2(\Omega, \mathbf{a})$, $V^* = L^2(\Omega, \mathbf{a}^{-1})$
- $H^m(\Omega) \subset L^2(\Omega)$, $H^m(\Omega) = \{x \in L^2(\Omega), D^p x \in L^2(\Omega), 1 \leq p \leq m\}$ endowed with the inner product $(x, y) = \sum_{k=0}^{m} \int_\Omega D^k x D^k y d\omega$

Properties of the Duality Mapping J

In a non pivot Hilbert space, the duality mapping J enjoys the following properties:

- J is strictly monotone and continuous.
- J linear $\Leftrightarrow$ X is a Hilbert space
- $J = Id_X \Leftrightarrow X$ is a pivot Hilbert space

Some example for J:

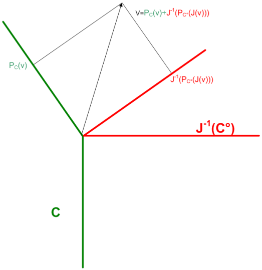
- If $V = L^2(\Omega, \mathbf{a})$, J is the multiplication operator by **a**
- If $V = L^p(\Omega)$, $J(x) = \|x\|^{2-p}|x|^{p-1}sgn(x)$ where $sgn(x) = \chi_{[x>0]} - \chi_{[x<0]}$

Variational Principle and decomposition results

- *The projection operator of V onto K, $P_K : V \to K$ given by* $||P_K(z) - z|| = \inf_{x \in K} ||x - z||$. Moreover we have the following characterization of $P_K(x)$;
$$\bar{x} = P_K(x) \Leftrightarrow \langle J(x - \bar{x}), y - \bar{x} \rangle \leq 0, \ \forall y \in K \tag{1}$$
- Let C be a nonempty closed convex cone of a non-pivot Hilbert space V. Then for all $x \in V$ and $f \in V^*$ the following decompositions hold:
$$x = P_C(x) + J^{-1} P_{C^0} J(x) \ \textit{and} \ \langle P_{C^0} J(x), P_C(x) \rangle = 0$$
$$f = P_{C^0}(f) + J P_C J^{-1}(f) \ \textit{and} \ \langle P_{C^0}(f), P_C J^{-1}(f) \rangle = 0$$
- The directional Gateaux derivative of the operator P_K is defined, for any $x \in K$ and any element $v \in V$, as the limit (the original proof is in [Zarantonello,1971], updated in [Cojocaru, Pia] for non pivot context):
$$\pi_K(x, v) := \lim_{\delta \to 0^+} \frac{P_K(x + \delta v) - x}{\delta}; \ \text{moreover} \ \pi_K(x, v) = P_{T_K(x)}(v).$$

.

Something more about cones



$\Rightarrow$ This intuitive result is true in an Infinite dimensional space (not necessarily Hilbert).

We remind that:
$T_K(x) = \overline{\bigcup_{\lambda>0} \lambda(K - x)}$
$N_K(x) = \{\xi \in X^*, <\xi, y - x> \leq 0, \forall y \in K\}$
$T_K(x) = N_K^o(x)$ and $T_K^o(x) = N_K(x)$.

Outline

PDS in non-pivot Hilbert spaces - Definitions

Definition

A non-pivot projected differential equation (NpPrDE) is a discontinuous ODE given by:

$$\frac{dx(t)}{dt} = \pi_K(x(t), -(J^{-1} \circ F)(x(t))) = P_{T_K(x(t))}(-(J^{-1} \circ F)(x(t))). \tag{2}$$

Consequently the associated **Cauchy problem** is given by:

$$\frac{dx(t)}{dt} = \pi_K(x(t), -(J^{-1} \circ F)(x(t))), \;\; x(0) = x_0 \in K. \tag{3}$$

Let's specify the notion of solution for the previous Cauchy problem

Definition

An absolutely continuous function $x : \mathcal{I} \subset \mathbb{R} \to X$, such that

$$x(t) \in K, \;\; x(0) = x_0 \in K, \, \forall t \in \mathcal{I}$$

$$\dot{x}(t) = \pi_K(x(t), -(J^{-1} \circ F)(x(t))), \;\; \text{a.e. on } \mathcal{I}$$

is called a solution for the initial value problem [3].

PDS in non-pivot Hilbert spaces - Existence Result

Theorem

Let V be a Hilbert space and V^* its topological dual and let $K \subset V$ be a non-empty, closed and convex subset. Let $F : K \to V^*$ be a Lipschitz continuous vector field. Let $x_0 \in K$. Then the initial value problem [3] has a unique solution on $\mathbb{R}_+$.

Sketches of proof [Cojocaru, Pia]:

step 1 we prove the equivalence between [3] and the differential inclusion

$$\begin{array}{l} x(t) \in K,\ x(0) = x_0 \in K,\ \forall t \in \mathcal{I} \\ \dot{x}(t) \in J^{-1}(-F(x) - \tilde{N}_K(x)),\ \text{ a.a. } t. \end{array}$$

($\tilde{N}_K(x)$ is a truncation of the normal cone)

step 2 we prove that the mapping $\mathcal{N}_p : K \cap B_V(x_0, L) \to \mathbb{R}$ given by $x \mapsto \langle F - \tilde{N}_K(x), p\rangle$ has for each $x \in K$ a closed graph.

step 3 we construct a sequence $\{x_k(.)\}$ of absolutely continuous fonction on an interval $\mathcal{I}$ such that $\forall k \geq k_0,\ (x_k(t), \dot{x}_k(t)) \in graph(J^{-1}(-F - \tilde{N}_K)) + \mathcal{M}$, $\mathcal{M}$ constant.

step 4 we prove the uniqueness and we extend the validity to $\mathbb{R}_+$.

Implicit PDS in non-pivot Hilbert spaces - Motivation

The motivation for the introduction of such an equation comes from the desire to study some aspects of a dynamics on a set K', not necessarily convex, but also to treat some problems on translated sets. We introduce the following definition:

Definition

A pair (g, K) such that $g : K' \to K \subset V$, with g continuous and strictly monotone, K convex and such that $F : K' \cup K \to V$ satisfying $(F \circ g)(y) = F(y)$, $\forall y \in K^{'}$ is called a **convexification pair of** $(F, K^{'})$.

Example. Here is an example of such a convexification pair in $\mathbb{R}^2$. Let $K^{'} = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ 0 \leq y \leq |x|\}$ and g be the map of $K^{'}$ into its **convex hull** $K = [-1, 1] \times [-1, 1]$, namely

$$g(x, y) = (x, \frac{2}{1 + |x|} y + \frac{1 - |x|}{1 + |x|})$$

We can easily check that g is continuous and monotone. Now take F to be $F(x, y) = (x, a)$, where a is an arbitrary constant in $\mathbb{R}$. Then we have $F \circ g(x, y) = (x, a) = F(x, y)$.

Implicit PDS in non-pivot Hilbert spaces - Definitions

Definition

A non-pivot implicit projected differential equation (NpImPrDE) is a discontinuous ODE given by:

$$\frac{dg(x(t))}{dt} = \pi_K(g(x(t)), -(J^{-1} \circ F)(x(t))) = P_{T_K(g(x(t)))}(-(J^{-1} \circ F)(x(t))). \quad (4)$$

Consequently the associated **Cauchy problem** is given by:

$$\frac{dg(x(t))}{dt} = \pi_K(g(x(t)), -(J^{-1} \circ F)(x(t))), \;\; g(x(0)) = g(x_0) \in K. \quad (5)$$

where (g, K) is a convexification pair of (F, K').

A solution of (5) is an absolutely continuous fonction x such that $x(t) \in K$ that satisfies (5).

PDS in non-pivot Hilbert spaces - Existence Result

Theorem

Let V be a not necessarily pivot Hilbert space, and let $K^{'}$ be a non-empty closed subset of V and (g, K) a convexification pair of $(F, K^{'})$. Then if $F : K^{'} \rightarrow V$ is a Lipschitz continuous vector field, then the initial value problem (5) , has a unique solution on the interval $\mathbb{R}_{+}$.

Outline

Equivalence theorem

Definition

We call g-variational inequality on a non necessarily convex set K' of a Hilbert space X the following variational inequality.

$$\textit{find } x \in K', \; < F(x), y - g(x) > \geq 0, \forall y \in K \tag{6}$$

where (g, K) is a convexification pair of $(F, K^{'})$.

Theorem

Let X be a not necessarily pivot Hilbert space and let $K \subset X$ be a non-empty, closed and convex subset. Let $F : X \to X^*$ be a vector field.
Then the solution set of the variational inequality (6) coincides with the set of critical points of the non pivot implicit dynamical system (5).

QVI

QVI

The following inequality is called a **(quasi-variational inequality)**.

$$\textit{find } x \in K(x), \ \langle F(x), y - x \rangle \geq 0, \ \forall y \in K(x) \tag{7}$$

Assuming $K(x)$ convex for all $x \in V$ not necessarily pivot Hilbert space and $F : H \to V^*$.
We can Introduce also the following projected dynamical system in order to study the disequilibrium behavior of (7).

Definition

We call Projected dynamical system associated to the quasi-variational inequality (7) the discontinuous right hand side differential equation given by

$$\frac{dx}{dt} = \lim_{\delta \to 0^+} \frac{P_{K(x)}(x - \delta J^{-1} F(x)) - x}{\delta} = P_{T_{K(x)}(x)}(-J^{-1} F(x)), \ x(0) = x_0 \in K(x_0) \tag{8}$$

Lipschitz like Assumption

In the general case we have no existence result for problem 8. An existence result for a class of PDS has been given in [Noor, 2003], assuming the following fact:

Assumption

For all $u, v, w \in \mathbb{H}$, $P_{K(u)}$ *satisfies the condition*

$$\|P_{K(u)}(w) - P_{K(v)}(w)\| \leq \lambda \|u - v\| \tag{9}$$

where $\lambda > 0$ *is a constant.*

which is not satisfied in trivial cases:
Take $\mathbb{H} = \mathbb{R}^2$, $C = [0, \epsilon]^2$, $u = (0, 0)$, $v = (\epsilon, \epsilon)$, $K(u) = T_C(u)$ and by $K(v) = T_C(v)$.
Complete proof: We denote by C a closed convex set and we take $u, v \in C$, we denote by $K(u) = T_C(u)$ and by $K(v) = T_C(v)$ the tangent cones of C at u and v.
In fact $w \in \mathbb{H}$ can only be chosen in one of the following four situations:

$$w \in K(u) \bigcap K(v)$$
$$w \in K(u) \setminus K(v)$$
$$w \in K(v) \setminus K(u)$$
$$w \in \mathbb{H} \setminus (K(u) \bigcup K(v))$$

Lipschitz like Assumption -2

Suppose now that we have $w \in K(u) \setminus K(v)$; then by Moreau's decomposition theorem we get

$$(9) \Leftrightarrow \|w - P_{K(v)}(w)\| = \|P_{N_C(v)}(w)\| \leq \lambda\|u - v\| \qquad (10)$$

where $N_C(v)$ is the normal cone of C at v. Consider now $\mathbb{H} = \mathbb{R}^2$, $C = [0,1]^2$, $u = (0,0)$ and $v = (\epsilon, \epsilon)$ with $\epsilon > 0$. It is clear that we have the following:

$$T_C(u) = \mathbb{R}^2_+$$
$$T_C(v) = \mathbb{R}^2_-$$
$$N_C(v) = \mathbb{R}^2_+ = T_C(u)$$

So for any $w \in N_C(v)$ we get

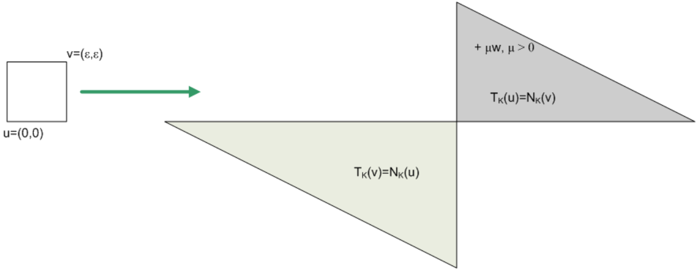
$$\|w\| \leq \lambda\|u - v\| = \sqrt{2}\epsilon\lambda$$

but by assumption, λ is a fixed positive constant so $\forall \mu > 0$,

$$\|\mu w\| \leq \lambda\|u - v\| = \sqrt{2}\epsilon\lambda$$

should be true. However this does not hold.

Lipschitz like Assumption -3



The assumption implies that

$$\|\mu w\| \leq \lambda \|u - v\| = \sqrt{2}\epsilon\lambda, \ \forall \mu > 0$$

Contradiction!
However in [Maugeri, Scrimali] the assumption is proven if K is the set of feasible solutions in network-based models.

Existence

If $K(x) = K + p(x)$ we can give the following equivalent formulation:

$$\frac{dx}{dt} = \lim_{\delta \to 0^+} \frac{P_{K+p(x)}(x - \delta J^{-1}F(x)) - P_{K+p(x)}(x)}{\delta} = P_{T_{K(x)}(x)}(-J^{-1}F(x)), \; x(0) = x_0 \in K \tag{11}$$

so we get

$$\frac{dx}{dt} = \lim_{\delta \to 0^+} \frac{P_K(x - p(x) - \delta J^{-1}F(x)) - P_K(x - p(x))}{\delta} = P_{T_K(g(x))}(-J^{-1}F(x)), \; x(0) = x_0 \in K \tag{12}$$

where $g(x) = x - p(x)$. We can observe that if $\frac{dp(x)}{dt} = 0$, then 12 is equal to the implicit projected differential equation (5), and therefore the theorem on ImNpPDS provide an existence solution without assuming any kind of Lipschitz condition of the projection operator.

Notion of decongestion and Interpretation

Definition

Suppose to have a traffic equilibrium problem (P) with constraints set $K \subset V_1$ without solution in K. If there exists $K' \in V_2 \supset V_1$ with $K' \cap V_1 = K$ such that (P) admits a solution in $K^{'}$ then $(V_2, K^{'})$ is called a **decongestion pair** for (P).

Non pivot spaces: application to Traffic problems

Introduce the notion of Non pivot Hilbert space on a traffic problem (P) permits to define **decongestion pairs** for (P).

And what about Wireless comunications? Info manage in the duality pairing.

Last

In the Next talk,"Weighted Variational Inequalities in Non-pivot Hilbert Spaces: Existence and Regularity Results and Applications." presented by Annamaria Barbagallo additional details will be provided regarding Traffic equilibrium problem in Np H-Spaces.

Thank you for your attention!

J-P. Aubin.*Analyse Fonctionelle appliquée.*Editions PUF, 1987.

M.G.Cojocaru, P.Daniele, A.Nagurney: *Projected Dynamical Systems and evolutionary (time-dependent) Variational Inequalities via Hilbert Spaces with Applications*, Journal of Optimization Theory and Applications, 2004,

M.G.Cojocaru, L.B. Jonker. *Existence of solutions to Projected Differential Equations in Hilbert Spaces*. Proceeding of the American Mathematical Society, 132, 2004, 183-193.

M.G.Cojocaru, S. Pia.*Projected Dynamical Systems in non-pivot Hilbert spaces.*Preprint.

P.Daniele, A.Maugeri, W.Oettli.*Time-Dependent Traffic Equilibria.*Journal of Optimization Theory and Applications, Vol 103, N.3, 1999,pages 543-555.

S.Giuffré, S.Pia.*Traffic Equilibrium problem in Hilbert spaces weighted by a real-time density*. Preprint.

A.Maugeri, L.Scrimali. *Global Lipschitz continuity of solutions to parameterized variational inequalities*, Bollettino dell'UMI, Sezione B, To appear.

A.Nagurney, J.Dong.*Supernetworks, Decision Making for the Information Age*. Edited by Edward Elgar, 2002.

Noor, M. A, *Implicit dynamical systems and quasi variational inequalities*. Applied Math. and Comput. 134 (2003), 69-81.

E.H. Zarantonello. *Projections on Convex sets in Hilbert space and spectral theory*, Contributions to Nonlinear Functional Analysis, Mathematics Research Center, Madison, April 12-14 1971 (E.H. Zarantonello ed.) Academic Press, 1971, pp 237-424.