Vulnerability Analysis of Complex Networks from Transportation Networks to the Internet and Electric Power Supply Chains

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- Background
- The Transportation Network Equilibrium Problem and Methodological Tools
- The Braess Paradox
- Transportation and Critical Infrastructure Networks
- A New Network Performance/Efficiency Measure with Applications to Critical Infrastructure Networks
- A New Approach to Transportation Network Robustness
- What About Dynamic Networks?
- Evolutionary Variational Inequalities, the Internet, and the Time-Dependent (Demand-Varying) Braess Paradox
- Extension of the Efficiency Measure to Dynamic Networks
- Where Are We Now? An Empirical Case Study to Real-World Electric Power Supply Chains
Background
We Are in a New Era of Decision-Making Characterized by:

- complex interactions among decision-makers in organizations;
- alternative and at times conflicting criteria used in decision-making;
- constraints on resources: natural, human, financial, time, etc.;
- global reach of many decisions;
- high impact of many decisions;
- increasing risk and uncertainty, and
- the importance of dynamics and realizing a fast and sound response to evolving events.
Network problems are their own class of problems and they come in various forms and formulations, i.e., as optimization (linear or nonlinear) problems or as equilibrium problems and even dynamic network problems.

Complex network problems, with a focus on critical infrastructure systems and emphasis on transportation, will be the focus of this talk.
Transportation, Communication, and Energy Networks

Bus Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks

British Electricity Grid

Rail Network
## Components of Common Physical Networks

<table>
<thead>
<tr>
<th>Network System</th>
<th>Nodes</th>
<th>Links</th>
<th>Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transportation</strong></td>
<td>Intersections, Homes, Workplaces, Airports, Railyards</td>
<td>Roads, Airline Routes, Railroad Track</td>
<td>Automobiles, Trains, and Planes,</td>
</tr>
<tr>
<td><strong>Manufacturing and logistics</strong></td>
<td>Workstations, Distribution Points</td>
<td>Processing, Shipment</td>
<td>Components, Finished Goods</td>
</tr>
<tr>
<td><strong>Communication</strong></td>
<td>Computers, Satellites, Telephone Exchanges</td>
<td>Fiber Optic Cables, Radio Links</td>
<td>Voice, Data, Video</td>
</tr>
<tr>
<td><strong>Energy</strong></td>
<td>Pumping Stations, Plants</td>
<td>Pipelines, Transmission Lines</td>
<td>Water, Gas, Oil, Electricity</td>
</tr>
</tbody>
</table>
US Railroad Freight Flows

Natural Gas Pipeline Network in the US
World Oil Trading Network
The study of the efficient operation on transportation networks dates to *ancient Rome* with a classical example being the publicly provided Roman road network and the *time of day chariot policy*, whereby chariots were banned from the ancient city of Rome at particular times of day.
Characteristics of Networks Today

• *large-scale nature* and complexity of network topology;

• *congestion*;

• the *interactions among networks* themselves such as in transportation versus telecommunications;

• *policies* surrounding networks today may have a *major impact* not only economically but also environmentally, *socially, politically, and security-wise*. 
• alternative behaviors of the users of the network

  – system-optimized versus

  – user-optimized (network equilibrium),

which may lead to

paradoxical phenomena.
The Transportation Network Equilibrium Problem and Methodological Tools
Transportation science has historically been the discipline that has pushed the frontiers in terms of methodological developments for such problems (which are often large-scale) beginning with the book, *Studies in the Economics of Transportation*, by Beckmann, McGuire, and Winsten (1956).
Dafermos (1980) showed that the transportation network equilibrium (also referred to as user-optimization) conditions as formulated by Smith (1979) were a finite-dimensional variational inequality. In 1981, Dafermos proposed a multicriteria transportation network equilibrium model in which the costs were flow-dependent.

In 1993, Dupuis and Nagurney proved that the set of solutions to a variational inequality problem coincided with the set of solutions to a projected dynamical system (PDS) in $\mathbb{R}^n$.

In 1996, Nagurney and Zhang published *Projected Dynamical Systems and Variational Inequalities*.

Daniele, Maugeri, and Oettli (1998, 1999) introduced evolutionary variational inequalities for time-dependent (dynamic) traffic network equilibrium problems.

In 2002, Nagurney and Dong published *Supernetworks: Decision-Making for the Information Age*.
The Transportation Network Equilibrium (TNE) Problem

Consider a general network $G = [N, L]$, where $N$ denotes the set of nodes, and $L$ the set of directed links. Let $a$ denote a link of the network connecting a pair of nodes, and let $p$ denote a path consisting of a sequence of links connecting an $O/D$ pair. $P_w$ denotes the set of paths, assumed to be acyclic, connecting the $O/D$ pair of nodes $w$ and $P$ the set of all paths.

Let $x_p$ represent the flow on path $p$ and $f_a$ the flow on link $a$. The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link $a$ is contained in path $p$, and 0, otherwise. This expression states that the load on a link $a$ is equal to the sum of all the path flows on paths $p$ that contain (traverse) link $a$. 
Moreover, if we let $d_w$ denote the demand associated with O/D pair $w$, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair $w$ must be equal to the given demand $d_w$.

Let $c_a$ denote the user cost associated with traversing link $a$, which is assumed to be continuous, and $C_p$ the user cost associated with traversing the path $p$. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$
Transportation Network Equilibrium

The network equilibrium conditions are then given by:
For each path \( p \in P_w \) and every O/D pair \( w \):

\[
C_p \begin{cases} 
= \lambda_w, & \text{if } x^*_p > 0 \\
\geq \lambda_w, & \text{if } x^*_p = 0
\end{cases}
\]

where \( \lambda_w \) is an indicator, whose value is not known a priori. These equilibrium conditions state that the user costs on all used paths connecting a given O/D pair will be minimal and equalized.
As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that \( \frac{\partial c_b}{\partial c_a} = \frac{\partial c_a}{\partial c_b} \) for all links \( a, b \) in the network then the solution to the above network equilibrium problem can be reformulated as the solution to an associated optimization problem. For example, if we have that \( c_a = c_a(f_a), \forall a \in L \), then the solution can be obtained by solving:

\[
\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y) \, dy
\]

subject to:

\[
d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,
\]

\[
f_a = \sum_{p \in P} x_p, \quad \forall a \in L,
\]

\[
x_p \geq 0, \quad \forall p \in P.
\]
**The Braess (1968) Paradox**

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: \( p_1 = (a,c) \) and \( p_2 = (b,d) \).

For a travel demand of 6, the equilibrium path flows are \( x_{p_1}^* = x_{p_2}^* = 3 \) and

The equilibrium path travel cost is \( C_{p_1} = C_{p_2} = 83 \).

\[
\begin{align*}
&c_a(f_a) = 10 f_a \\
&c_b(f_b) = f_b + 50 \\
&c_c(f_c) = f_c + 50 \\
&c_d(f_d) = 10 f_d
\end{align*}
\]
Adding a Link Increases Travel Cost for All!

Adding a new link creates a new path \( p_3 = (a, e, d) \).
The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path \( p_3 \), \( C_{p_3} = 70 \).
The new equilibrium flow pattern network is

\[ x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2. \]

The equilibrium path travel costs:

\[ C_{p_1} = C_{p_2} = C_{p_3} = 92. \]
The 1968 Braess article has been translated from German to English and appears as

**On a Paradox of Traffic Planning**

by Braess, Nagurney, Wakolbinger

in the November 2005 issue of *Transportation Science*. 
If no such symmetry assumption holds for the user link costs functions, then the equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a variational inequality problem!
VI Formulation of TNE
Dafermos (1980), Smith (1979)

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequality problem: determine \( x^* \in K \), such that

\[
\sum_{p} C_p(x^*) \times (x_p - x^*_p) \geq 0, \quad \forall x \in K.
\]

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine \( x^* \in K \subset \mathbb{R}^n \) such that

\[
\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,
\]

where \( \langle \cdot, \cdot \rangle \) denoted the inner product in \( \mathbb{R}^n \) and \( K \) is closed and convex.
A Geometric Interpretation of a Variational Inequality and a Projected Dynamical System

Dupuis and Nagurney (1993)

Nagurney and Zhang (1996)
The variational inequality problem, contains, as special cases, such classical problems as:

- systems of equations
- optimization problems
- complementarity problems

and is also closely related to fixed point problems.

Hence, it is a unifying mathematical formulation for a variety of mathematical programming problems.
Transportation and Critical Infrastructure Systems
The TNE Paradigm is the Unifying Paradigm for Critical Infrastructure Problems:

- Transportation Networks
- The Internet
- Financial Networks
- Electric Power Supply Chains.
The TNE Paradigm can also capture multicriteria decision-making. Decision-makers (manufacturers, retailers, and/or consumers) in multitiered networks may seek to:

- maximize profits
- minimize pollution (emissions/waste)
- minimize risk

with individual weights associated with the different criteria.
The Equivalence of Supply Chains and Transportation Networks

Supply Chain - Transportation Supernetwork Representation

Transaction cost information

Demand or order information

Travel time information

Unexpected issues information

Real-Time Information System

Two-way information exchanges between specific decision-makers

The fifth chapter of Beckmann, McGuire, and Winsten’s book, *Studies in the Economics of Transportation* (1956) describes some unsolved problems including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

Specifically, they asked whether electric power generation and distribution networks can be reformulated as transportation network equilibrium problems.
Electric Power Supply Chains
The Electric Power Supply Chain Network

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks

Electric Power Supply Chain Network with Fuel Suppliers

In 1952, Copeland wondered whether money flows like water or electricity.
The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation

We have shown that money as well as electricity flow like transportation and have answered questions posed fifty years ago by Copeland and by Beckmann, McGuire, and Winsten!
Recent disasters have demonstrated the importance and the vulnerability of network systems.

Examples:

• 9/11 Terrorist Attacks, September 11, 2001;
• The biggest blackout in North America, August 14, 2003;
• Two significant power outages in September 2003 -- one in the UK and the other in Italy and Switzerland;
• Hurricane Katrina, August 23, 2005;
• The Minneapolis I35 Bridge Collapse, August 1, 2007.
Communication Network Disasters

www.tx.mb21.co.uk

www.w5jgv.com

www.wirelessestimator.com
Recent Literature on Network Vulnerability

- Holme, Kim, Yoon and Han (2002)
- Taylor and D’este (2004)
- Chassin and Posse (2005)
- Barrat, Barthélemy and Vespignani (2005)
- Sheffi (2005)
- Dall’Asta, Barrat, Barthélemy and Vespignani (2006)
- Jenelius, Petersen and Mattson (2006)
- Taylor and D’Este (2007)
Our Research on Network Efficiency, Vulnerability, and Robustness


Robustness of Transportation Networks Subject to Degradable Links, Nagurney and Qiang, *Europhysics Letters*, 80, December (2007).

A New Network Performance/Efficiency Measure with Applications to Critical Infrastructure Networks
The network performance/efficiency measure $\varepsilon(G,d)$, for a given network topology $G$ and fixed demand vector $d$, is defined as

$$\varepsilon(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_w},$$

where $n_w$ is the number of O/D pairs in the network and $\lambda_w$ is the equilibrium disutility for O/D pair $w$.

**Importance of a Network Component**

**Definition: Importance of a Network Component**

The importance, $I(g)$, of a network component $g \in G$ is measured by the relative network efficiency drop after $g$ is removed from the network:

$$I(g) = \frac{\Delta \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, d) - \mathcal{E}(G - g, d)}{\mathcal{E}(G, d)},$$

where $G-g$ is the resulting network after component $g$ is removed.
**The Latora and Marchiori (L-M) Network Efficiency Measure**

**Definition: The L-M Measure**

The network performance/efficiency measure, $E(G)$ for a given network topology, $G$, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where $n$ is the number of nodes in the network and $d_{ij}$ is the shortest path length between node $i$ and node $j$. 
The L-M Measure vs. the N-Q Measure

Theorem:

If positive demands exist for all pairs of nodes in the network, $G$, and each of demands is equal to 1, and if $d_{ij}$ is set equal to $\lambda_w$, where $w=(i,j)$, for all $w \in W$, then the N-Q and L-M network efficiency measures are one and the same.
The Approach to Study the Importance of Network Components

The elimination of a link is treated in the N-Q network efficiency measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks.

The measure generalizes the Latora and Marchiori network measure for complex networks.
Example 1

Assume a network with two O/D pairs: \( w_1 = (1,2) \) and \( w_2 = (1,3) \) with demands: \( d_{w_1} = 100 \) and \( d_{w_2} = 20 \).

The paths are:
for \( w_1 \), \( p_1 = a \); for \( w_2 \), \( p_2 = b \).

The equilibrium path flows are:
\( x_{p_1}^* = 100 \), \( x_{p_2}^* = 20 \).

The equilibrium path travel costs are:
\( c_{p_1} = c_{p_2} = 20 \).

\[ c_a(f_a) = 0.01f_a + 19 \]
\[ c_b(f_b) = 0.05f_b + 19 \]
### Importance and Ranking of Links and Nodes

<table>
<thead>
<tr>
<th>Link</th>
<th>Importance Value from Our Measure</th>
<th>Importance Ranking from Our Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.8333</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>0.1667</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Node</th>
<th>Importance Value from Our Measure</th>
<th>Importance Ranking from Our Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.8333</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0.1667</td>
<td>3</td>
</tr>
</tbody>
</table>
Example 2

The network is given by:

\[ w_1 = (1,20) \quad w_2 = (1,19) \]

\[ d_{w_1} = 100 \quad d_{w_2} = 100 \]

From: Nagurney,

*Transportation Research B* (1984)
**Example 2: Link Cost Functions**

<table>
<thead>
<tr>
<th>Link ( a )</th>
<th>Link Cost Function ( c_a(f_a) )</th>
<th>Link ( a )</th>
<th>Link Cost Function ( c_a(f_a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.00005 f_1^4 + 5f_1 + 500)</td>
<td>15</td>
<td>(0.00003 f_{15}^4 + 9f_{15} + 200)</td>
</tr>
<tr>
<td>2</td>
<td>(0.00003 f_2^4 + 4f_2 + 200)</td>
<td>16</td>
<td>(8f_{16} + 300)</td>
</tr>
<tr>
<td>3</td>
<td>(0.00005 f_3^4 + 3f_3 + 350)</td>
<td>17</td>
<td>(0.00003 f_{17}^4 + 7f_{17} + 450)</td>
</tr>
<tr>
<td>4</td>
<td>(0.00003 f_4^4 + 6f_4 + 400)</td>
<td>18</td>
<td>(5f_{18} + 300)</td>
</tr>
<tr>
<td>5</td>
<td>(0.00006 f_5^4 + 6f_5 + 600)</td>
<td>19</td>
<td>(8f_{19} + 600)</td>
</tr>
<tr>
<td>6</td>
<td>(7f_6 + 500)</td>
<td>20</td>
<td>(0.00003 f_{20}^4 + 6f_{20} + 300)</td>
</tr>
<tr>
<td>7</td>
<td>(0.00008 f_7^4 + 8f_7 + 400)</td>
<td>21</td>
<td>(0.00004 f_{21}^4 + 4f_{21} + 400)</td>
</tr>
<tr>
<td>8</td>
<td>(0.00004 f_8^4 + 5f_8 + 650)</td>
<td>22</td>
<td>(0.00002 f_{22}^4 + 6f_{22} + 500)</td>
</tr>
<tr>
<td>9</td>
<td>(0.00001 f_9^4 + 6f_9 + 700)</td>
<td>23</td>
<td>(0.00003 f_{23}^4 + 9f_{23} + 350)</td>
</tr>
<tr>
<td>10</td>
<td>(4f_{10} + 800)</td>
<td>24</td>
<td>(0.00002 f_{24}^4 + 8f_{24} + 400)</td>
</tr>
<tr>
<td>11</td>
<td>(0.00007 f_{11}^4 + 7f_{11} + 650)</td>
<td>25</td>
<td>(0.00003 f_{25}^4 + 9f_{25} + 450)</td>
</tr>
<tr>
<td>12</td>
<td>(8f_{12} + 700)</td>
<td>26</td>
<td>(0.00006 f_{26} + 7f_{26} + 300)</td>
</tr>
<tr>
<td>13</td>
<td>(0.00001 f_{13}^4 + 7f_{13} + 600)</td>
<td>27</td>
<td>(0.00003 f_{27}^4 + 8f_{27} + 500)</td>
</tr>
<tr>
<td>14</td>
<td>(8f_{14} + 500)</td>
<td>28</td>
<td>(0.00003 f_{28}^4 + 7f_{28} + 650)</td>
</tr>
</tbody>
</table>
Algorithms for Solution

The projection method (cf. Dafermos (1980) and Nagurney (1999)) embedded with the equilibration algorithm of Dafermos and Sparrow (1969) was used for the computations.

In addition, the column generation method of Leventhal, Nemhauser, and Trotter (1973) was implemented to generate paths, as needed, in the case of the large-scale Sioux Falls network example.
Example 2: Importance and Ranking of Links

<table>
<thead>
<tr>
<th>Link α</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9086</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.8984</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.8791</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>0.8672</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>0.8430</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>0.8226</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>0.7750</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>0.5483</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>0.0362</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>0.6641</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>12</td>
<td>0.0006</td>
<td>20</td>
</tr>
<tr>
<td>13</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>14</td>
<td>0.0000</td>
<td>22</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>Link α</th>
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</thead>
<tbody>
<tr>
<td>15</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>16</td>
<td>0.0001</td>
<td>21</td>
</tr>
<tr>
<td>17</td>
<td>0.0000</td>
<td>22</td>
</tr>
<tr>
<td>18</td>
<td>0.0175</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>0.0362</td>
<td>17</td>
</tr>
<tr>
<td>20</td>
<td>0.6641</td>
<td>14</td>
</tr>
<tr>
<td>21</td>
<td>0.7537</td>
<td>13</td>
</tr>
<tr>
<td>22</td>
<td>0.8333</td>
<td>10</td>
</tr>
<tr>
<td>23</td>
<td>0.8598</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>0.8939</td>
<td>5</td>
</tr>
<tr>
<td>25</td>
<td>0.4162</td>
<td>16</td>
</tr>
<tr>
<td>26</td>
<td>0.9203</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>0.9213</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>0.0155</td>
<td>19</td>
</tr>
</tbody>
</table>
Example 2: Link Importance Rankings
Example 3 - Sioux Falls Network

The network data are from LeBlanc, Morlok, and Pierskalla (1975).

The network has 528 O/D pairs, 24 nodes, and 76 links.

The user link cost functions are of Bureau of Public Roads (BPR) form.
The Bureau of Public Roads (BPR) link cost functional form is:

\[ c_a(f_a) = t_a^0 \left[ 1 + k \left( \frac{f_a}{u_a} \right)^\beta \right] \quad \forall a \in L \]

where \( k \) and \( \beta \) are greater than zero and the \( u \)’s are the practical capacities on the links.
Example 3 - Sioux Falls Network Link Importance Rankings
Example 4: An Electric Power Supply Chain Network

Nagurney and Liu (2006) and Nagurney, Liu, Cojocaru and Daniele (2007) have shown that an electric power supply chain network can be transformed into an equivalent transportation network problem.
Supernetwork Transformation

Figure 3: Electric Power Supply Chain Network and the Corresponding Supernetwork

Five Demand Ranges

- Demand Range I: $d_w \in [0, 1]$
- Demand Range II: $d_w \in (1, \frac{4}{3}]$
- Demand Range III: $d_w \in (\frac{4}{3}, \frac{7}{3}]$
- Demand Range IV: $d_w \in (\frac{7}{3}, \frac{11}{3}]$
- Demand Range V: $d_w \in (\frac{11}{3}, \infty )$
Importance Ranking of Links in the Electric Power Supply Chain Network
Importance Ranking of Nodes in the Electric Power Supply Chain Network

Importance Ranking of Nodes in Demand Range I
Importance Ranking of Nodes in Demand Range II
Importance Ranking of Nodes in Demand Range III
Importance Ranking of Nodes in Demand Range IV
Importance Ranking of Nodes in Demand Range V

Node

Ranking
The Advantages of the N-Q Network Efficiency Measure

• The measure captures demands, flows, costs, and behavior of users, in addition to network topology;
• The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
• It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
• It can be applied to assess the efficiency/performance of a wide range of network systems.
• It is applicable also to elastic demand networks (Qiang and Nagurney, *Optimization Letters* (2008)).
• It has been extended to dynamic networks (Nagurney and Qiang, *Netnomics*, in press).
Motivation for Research on Transportation Network Robustness

According to the American Society of Civil Engineering:

Poor maintenance, natural disasters, deterioration over time, as well as unforeseen attacks now lead to estimates of $94 billion in the US in terms of needed repairs for roads alone.

Poor road conditions in the United States cost US motorists $54 billion in repairs and operating costs annually.
The focus of the robustness of networks (and complex networks) has been on the impact of different network measures when facing the removal of nodes on networks.

We focus on the *degradation of links through reductions in their capacities* and the effects on the induced travel costs in the presence of known travel demands and different functional forms for the links.
Global Annual Mean Temperature Trend 1950-1999

(http://www.epa.gov/globalwarming/climate/trends/temperature.html)

Source: Global Historical Climate Network,
National Oceanic and Atmospheric Administration
Impacts of Climate Change on Transportation Infrastructure

Examples from Alaska (Smith and Lavasseur)
According to the European Environment Agency (2004), since 1990 the *annual number of extreme weather and climate related events has doubled*, in comparison to the previous decade. These events account for approximately 80% of all economic losses caused by catastrophic events. In the course of climate change, catastrophic events are projected to occur more frequently (see Schulz (2007)).

Schulz (2007) applied the Nagurney and Qiang (2007) network efficiency measure to a German highway system in order to identify the critical road elements and found that this measure provided more reasonable results than the measure of Taylor and D’Este (2007).
Robustness in Engineering and Computer Science

IEEE (1990) defined robustness as *the degree to which a system of component can function correctly in the presence of invalid inputs or stressful environmental conditions.*

Gribble (2001) defined system robustness as *the ability of a system to continue to operate correctly across a wide range of operational conditions, and to fail gracefully outside of that range.*

Schilllo et al. (2001) argued that robustness has to be studied *in relation to some definition of the performance measure.*
“Robustness” in Transportation

Sakakibara et al. (2004) proposed a topological index. The authors considered a transportation network to be robust if it is “dispersed” in terms of the number of links connected to each node.

Scott et al. (2005) examined transportation network robustness by analyzing the increase in the total network cost after removal of certain network components.
A New Approach to Transportation Network Robustness
The robustness measure $\mathcal{R}^\gamma$ for a transportation network $G$ with the vector of demands $d$, the vector of user link cost functions $c$, and the vector of link capacities $u$ is defined as the relative performance retained under a given uniform capacity retention ratio $\gamma$ ($\gamma \in (0, 1]$) so that the new capacities are given by $\gamma u$. Its mathematical definition is given as:

$$\mathcal{R}^\gamma = \mathcal{R}(G, c, d, \gamma, u) = \frac{\mathcal{E}^\gamma}{\mathcal{E}} \times 100\%$$

where $\mathcal{E}$ and $\mathcal{E}^\gamma$ are the network performance measures with the original capacities and the remaining capacities, respectively.

We utilize BPR functions user link cost functions $c$ for the robustness analysis.
Simple Example

Assume a network with one O/D pair: \( w_1 = (1,2) \) with demand given by \( d_{w_1} = 10 \).
The paths are: \( p_1 = a \) and \( p_2 = b \).
In the BPR link cost function, \( k = 1 \) and \( \beta = 4 \); \( t_a^0 = 10 \) and \( t_a^0 = 1 \).
Assume that there are two sets of capacities:
Capacity Set A, where \( u_a = u_b = 50 \);
Capacity Set B, where \( u_a = 50 \) and \( u_b = 10 \).
Robustness of the Simple Network
Instead of using the original cost functions, we construct a set of BPR functions as below under which the Braess Paradox still occurs. The new demand is 110.

\[
\begin{align*}
  c_a(f_a) &= 1 + \left(\frac{f_a}{20}\right)^{\beta}, \\
  c_b(f_b) &= 50(1 + \left(\frac{f_b}{50}\right)^{\beta}), \\
  c_c(f_c) &= 50(1 + \left(\frac{f_b}{50}\right)^{\beta}), \\
  c_d(f_d) &= 1 + \left(\frac{f_d}{20}\right)^{\beta}, \\
  c_e(f_e) &= 10(1 + \left(\frac{f_e}{100}\right)^{\beta}).
\end{align*}
\]
Network Robustness for the Braess Network Example

\( \beta = 2 \)
Network Robustness for the Braess Network Example

\[ \beta = 3 \]
\[ \beta = 4 \]
Some Theoretical Results

Theorem
Consider a network consisting of two nodes 1 and 2, which are connected by a single link a and with a single O/D pair \( w_1 = (1,2) \). Assume that the user link cost function associated with link a is of the BPR form. Then the network robustness given by the expression is given by the explicit formula:

\[
\mathcal{R} \gamma = \frac{\gamma^\beta [w_1^\beta + kd_{w_1}^\beta]}{[\gamma^\beta w_1^\beta + kd_{w_1}^\beta]} \times 100\%,
\]

where \( d_{w_1} \) is the given demand for O/D pair \( w_1 = (1,2) \).
Moreover, the network robustness \( \mathcal{R} \) is bounded from below by \( \gamma^\beta \times 100\% \).
**Theorem**

Consider a network consisting of two nodes 1 and 2 as in the figure below, which are connected by a set of parallel links. Assume that the associated BPR link cost functions have $\beta = 1$. Furthermore, let’s assume that there are positive flows on all the links at both the original and partially degraded capacity levels. Then the network robustness given by the expression is given by the explicit formula:

$$R^\gamma = \frac{\gamma U + k \gamma d_{w_1}}{\gamma U + k d_{w_1}} \times 100\%,$$

where $d_{w_1}$ is the given demand for O/D pair $w_1 = (1, 2)$ and $U \equiv u_a + u_b + \cdots + u_n$.

Moreover, the network robustness $R^\gamma$ is bounded from below by $\gamma \times 100\%$. 
What About Dynamic Networks?
We are using evolutionary variational inequalities to model dynamic networks with:

- dynamic (time-dependent) supplies and demands
- dynamic (time-dependent) capacities
- structural changes in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).
Evolutionary variational inequalities, which are infinite dimensional, were originally introduced by Lions and Stampacchia (1967) and by Brezis (1967) in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems.

Steinbach (1998) studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity.

Daniele, Maugeri, and Oettli (1998, 1999), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others. See also Ran and Boyce (1996).
2005-2006 Radcliffe Institute for Advanced Study Fellowship Year at Harvard Collaboration with Professor David Parkes of Harvard University and Professor Patrizia Daniele of the University of Catania
A network like the Internet is volatile. Its traffic patterns can change quickly and dramatically... The assumption of a static model is therefore particularly suspect in such networks. (page 10 of Roughgarden’s (2005) book, *Selfish Routing and the Price of Anarchy*).

A Dynamic Model of the Internet

We now define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0, T], R^{Kn_p})$ (where $[0, T]$ denotes the time interval under consideration) given by

$$
\mathcal{K} = \left\{ x \in L^2([0, T], R^{Kn_p}) : 0 \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T]; \\
\sum_{p \in P_w} x^k_w(t) = d^k_w(t), \forall w, \forall k \text{ a.e. in } [0, T] \right\}.
$$

We assume that the capacities $\mu^k_r(t)$, for all $r$ and $k$, are in $\mathcal{L}$, and that the demands, $d^k_w \geq 0$, for all $w$ and $k$, are also in $\mathcal{L}$. Further, we assume that

$$
0 \leq d(t) \leq \Phi \mu(t), \text{a.e. on } [0, T],
$$

where $\Phi$ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element $(kw, kr)$ equal to 1 if route $r$ is contained in $P_w$, and 0, otherwise. The feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$
\langle \langle G, x \rangle \rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.
$$
Furthermore, the cost mapping $C : \mathcal{K} \rightarrow \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

The conditions below are a generalization of the Wardrop’s (1952) first principle of traffic behavior.

**Definition: Dynamic Multiclass Network Equilibrium**

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop’s first principle) if, for every O/D pair $w \in W$, every route $r \in P_w$, every class $k$; $k = 1, \ldots, K$, and a.e. on $[0,T]$:

$$C^k_r(x^*(t)) = \lambda^k_w(t) \begin{cases} 
\leq 0, & \text{if } x^k_r(t) = \mu^k_r(t), \\
= 0, & \text{if } 0 < x^k_r(t) < \mu^k_r(t), \\
\geq 0, & \text{if } x^k_r(t) = 0. 
\end{cases}$$
The standard form of the EVI that we work with is:

determine \( x^* \in \mathcal{K} \) such that \( \langle \langle F(x^*), x - x^* \rangle \rangle \geq 0, \ \forall x \in \mathcal{K} \).

**Theorem (Nagurney, Parkes, Daniele (2007))**

\( x^* \in \mathcal{K} \) is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:

\[
\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.
\]

The Time-Dependent (Demand-Varying) Braess Paradox and Evolutionary Variational Inequalities
Recall the Braess Network where we add the link e.
The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)

Equilibrium Path Flow

Braess Network with Time-Dependent Demands

Equilibrium Path Flow

Paths 1 and 2

Path 3

Demand(t) = t

I II III

0 10 20

0 5 10

0 0 10 10 20

Paths 1 and 2

Path 3
In Demand Regime I, only the new path is used. 
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off! 
In Demand Regime III, only the original paths are used.

Network 1 is the Original Braess Network - Network 2 has the added link.
The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!
Extension of the Network Efficiency Measure to Dynamic Networks

An Efficiency Measure for Dynamic Networks Modeled as Evolutionary Variational Inequalities with Applications to the Internet and Vulnerability Analysis, Nagurney and Qiang, Netnomics, in press.
The network efficiency for the network $G$ with time-varying demand $d$ for $t \in [0, T]$, denoted by $\mathcal{E}(G, d, T)$, is defined as follows:

$$\mathcal{E}(G, d, T) = \frac{\int_0^T \left[ \sum_{w \in W} \frac{d_w(t)}{\lambda_w(t)} \right] / n_W \, dt}{T}.$$ 

The above measure is the average network performance over time of the dynamic network.
Network Efficiency Measure for Dynamic Networks - Discrete Time

Let \( d_w^1, d_w^2, ..., d_w^H \) denote demands for O/D pair \( w \) in \( H \) discrete time intervals, given, respectively, by:
\([t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]\), where \( t_H = T \). We assume that the demand is constant in each such time interval for each O/D pair. Moreover, we denote the corresponding minimal costs for each O/D pair \( w \) at the \( H \) different time intervals by: \( \lambda_w^1, \lambda_w^2, ..., \lambda_w^H \). The demand vector \( d \), in this special discrete case, is a vector in \( R_{nw \times H} \). The dynamic network efficiency measure in this case is as follows:

**Dynamic Network Efficiency: Discrete Time Version**

The network efficiency for the network \( (G, d) \) over \( H \) discrete time intervals:
\([t_0, t_1], (t_1, t_2], ..., (t_{H-1}, t_H]\), where \( t_H = T \), and with the respective constant demands:
\( d_w^1, d_w^2, ..., d_w^H \) for all \( w \in W \) is defined as follows:

\[
\mathcal{E}(G, d, t_H = T) = \frac{\sum_{i=1}^{H} [(\sum_{w \in W} \frac{d_w^i}{\lambda_w^i})(t_i - t_{i-1})/n_W]}{t_H}.
\]
The importance of a network component $g$ of network $G$ with demand $d$ over time horizon $T$ is defined as follows:

$$I(g, d, T) = \frac{\mathcal{E}(G, d, T) - \mathcal{E}(G - g, d, T)}{\mathcal{E}(G, d, T)}$$

where $\mathcal{E}(G-g,d,T)$ is the dynamic network efficiency after component $g$ is removed.
The table shows the importance of nodes and links in the Dynamic Braess Network using the N-Q Measure when $T=10$.

<table>
<thead>
<tr>
<th>Link</th>
<th>Importance Value</th>
<th>Importance Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$c$</td>
<td>0.1784</td>
<td>2</td>
</tr>
<tr>
<td>$d$</td>
<td>0.2604</td>
<td>1</td>
</tr>
<tr>
<td>$e$</td>
<td>-0.1341</td>
<td>3</td>
</tr>
</tbody>
</table>

Link $e$ is never used after $t = 8.89$ and in the range $t \in [2.58, 8.89]$, it increases the cost, so the fact that link $e$ has a negative importance value makes sense; over time, its removal would, on the average, improve the network efficiency!
Where Are We Now?

Empirical Case Study

• New England electric power market and fuel markets
• 82 generators who own and operate 573 power plants
• 5 types of fuels: natural gas, residual fuel oil, distillate fuel oil, jet fuel, and coal
• Hourly demand/price data of July 2006 (24 × 31 = 744 scenarios)
• 6 blocks (L1 = 94 hours, and Lw = 130 hours; w = 2, ..., 6)
The New England Electric Power Supply Chain Network with Fuel Suppliers
Predicted Prices vs. Actual Prices ($/Mwh)
Summary and Conclusions

We have described a new network efficiency/performance measure that can be applied to fixed demand, elastic demand as well as dynamic network problems to identify the importance and rankings of network components.

We also demonstrated through a verity of complex network applications the suitability of the measure to investigate vulnerability as well as robustness of complex networks with a focus on transportation and related applications, including the Internet and electric power supply chains.

An analogue of the measure has been developed and applied to financial networks with intermediation and electronic commerce by Nagurney and Qiang -- to appear in Computational Methods in Financial Engineering (2008), Kontogiorghes, Rustem, and Winker, editors, Springer,
Ongoing Research and Questions

- How can time delays be incorporated into the measure?

- How do we capture multiclass user behavior; equivalently, behavior in multimodal networks?

- Can the framework be generalized to capture multicriteria decision-making?

- What happens if either system-optimizing (S-O) or user-optimizing (U-O) behavior needs to be assessed from a network system performance angle? We have some results in this dimension in terms of vulnerability and robustness analysis as well as from an environmental (emissions generated) perspective.

- Can we identify the most important nodes and links in large-scale electric power supply chains as in our empirical case study?
The Virtual Center for Supernetworks

Dr. Rob Engle 2004 Nobel Laureate in Economics Visits ISOM

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

http://supernet.som.umass.edu
Thank you!

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