Operations Research and the Captivating Study of Networks and Complex Decision-Making

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Outline of Presentation:

- Background - Operations Research
- Brief History of the Science of Networks
- Interdisciplinary Impact of Networks
- The Transportation Network Equilibriumn Problem and Methodological Tools
- The Braess Paradox
- Some Interesting Applications of Variational Inequalities
- The Time-Dependent (Demand-Varying) Braess Paradox and Evolutionary Variational Inequalities
- A New Network Performance/Efficiency Measure with Applications to Critical Infrastructure Networks

What is Operations Research (OR)?

Operations research is a scientific approach of using mathematical models and algorithms to aid in decision-making.

It is usually used to *analyze* complex real-world systems and to improve or *optimize performance*.

It is, by its nature, *interdisciplinary*.

We are in a New Era of Decision-Making Characterized by:

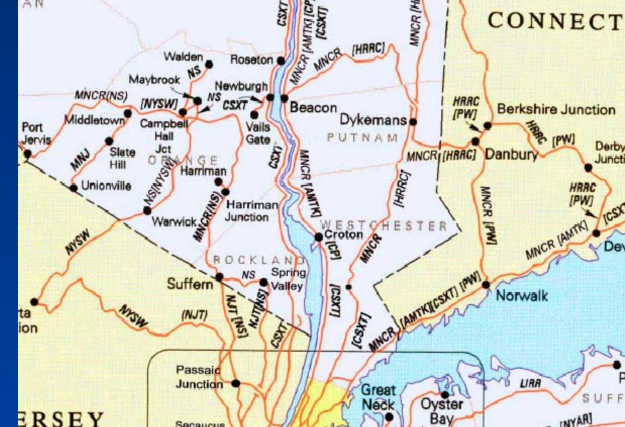
- complex interactions among decision-makers in organizations;
- alternative and at times conflicting criteria used in decision-making;
- constraints on resources: natural, human, financial, time, etc.;
- global reach of many decisions;
- high impact of many decisions;
- increasing risk and uncertainty, and
- the *importance of dynamics* and realizing a fast and sound response to evolving events.

Network problems are their own class of problems and they come in various forms and formulations, i.e., as optimization (linear or nonlinear) problems or as equilibrium problems and even dynamic network problems.

Network problems will be the focus of this talk.

Transportation, Communication, and Energy Networks

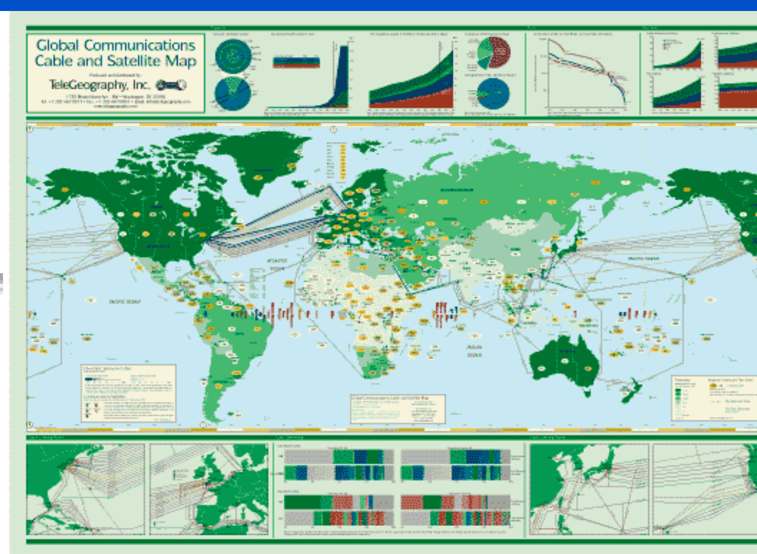
Subway Network



Railroad Network

Iridium Satellite Constellation Network

Satellite and Undersea Cable Networks

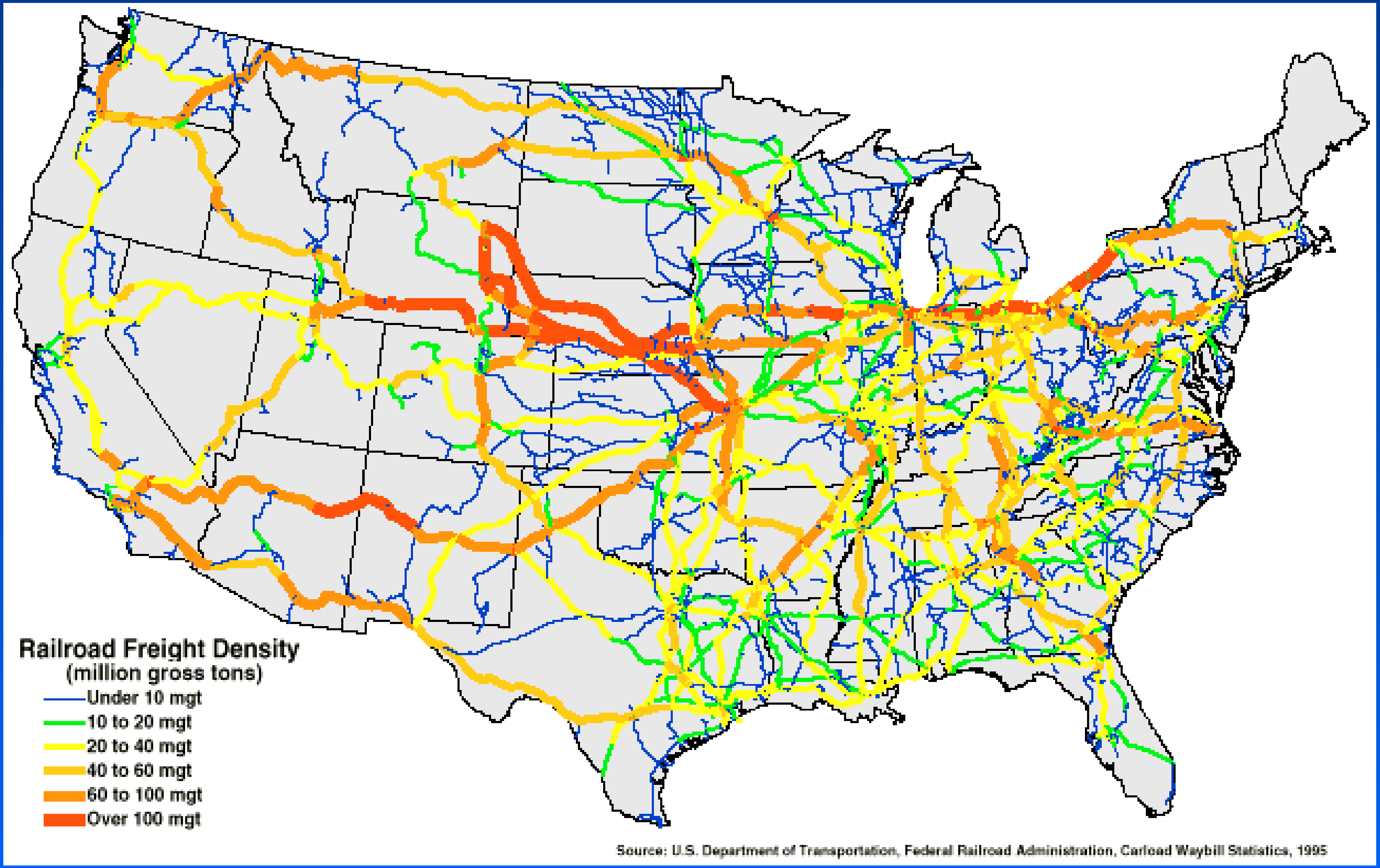


Duke Energy Gas Pipeline Network

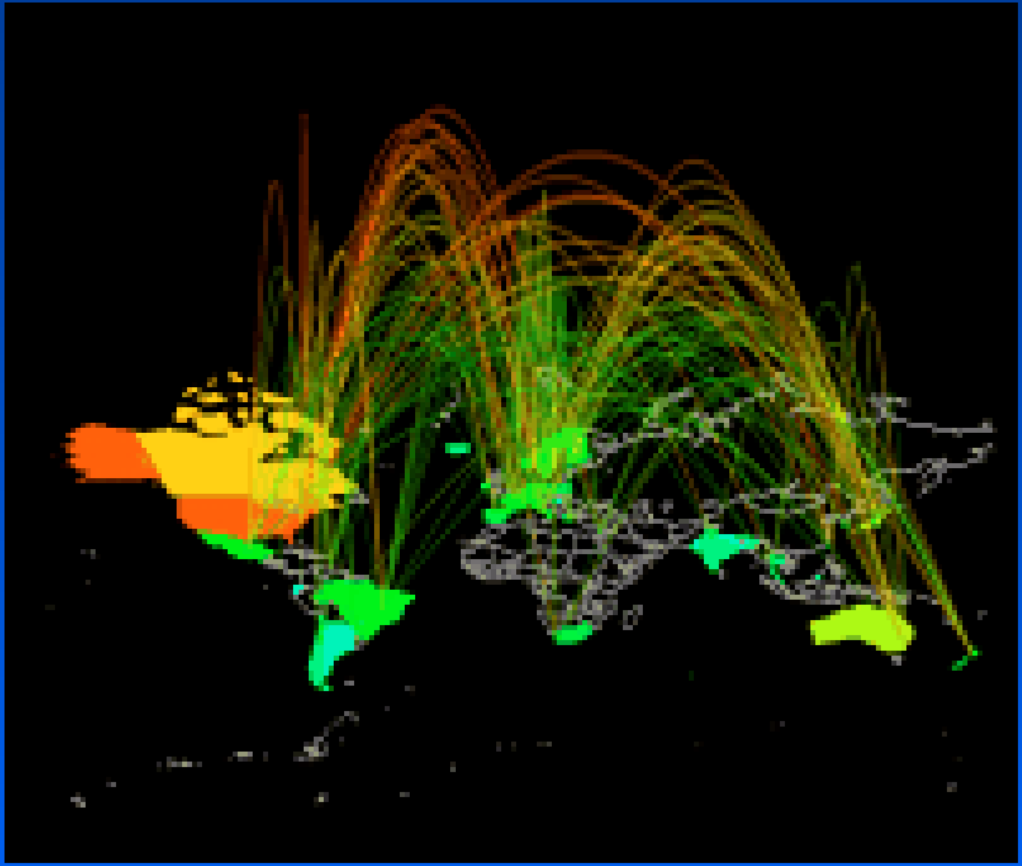
Components of Common Physical Networks

Network System	Nodes	Links	Flows
Transportation	Intersections, Homes, Workplaces, Airports, Railyards	Roads, Airline Routes, Railroad Track	Automobiles, Trains, and Planes,
Manufacturing and logistics	Workstations, Distribution Points	Processing, Shipment	Components, Finished Goods
Communication	Computers, Satellites, Telephone Exchanges	Fiber Optic Cables Radio Links	Voice, Data, Video
Energy	Pumping Stations, Plants	Pipelines, Transmission Lines	Water, Gas, Oil, Electricity

US Railroad Freight Flows

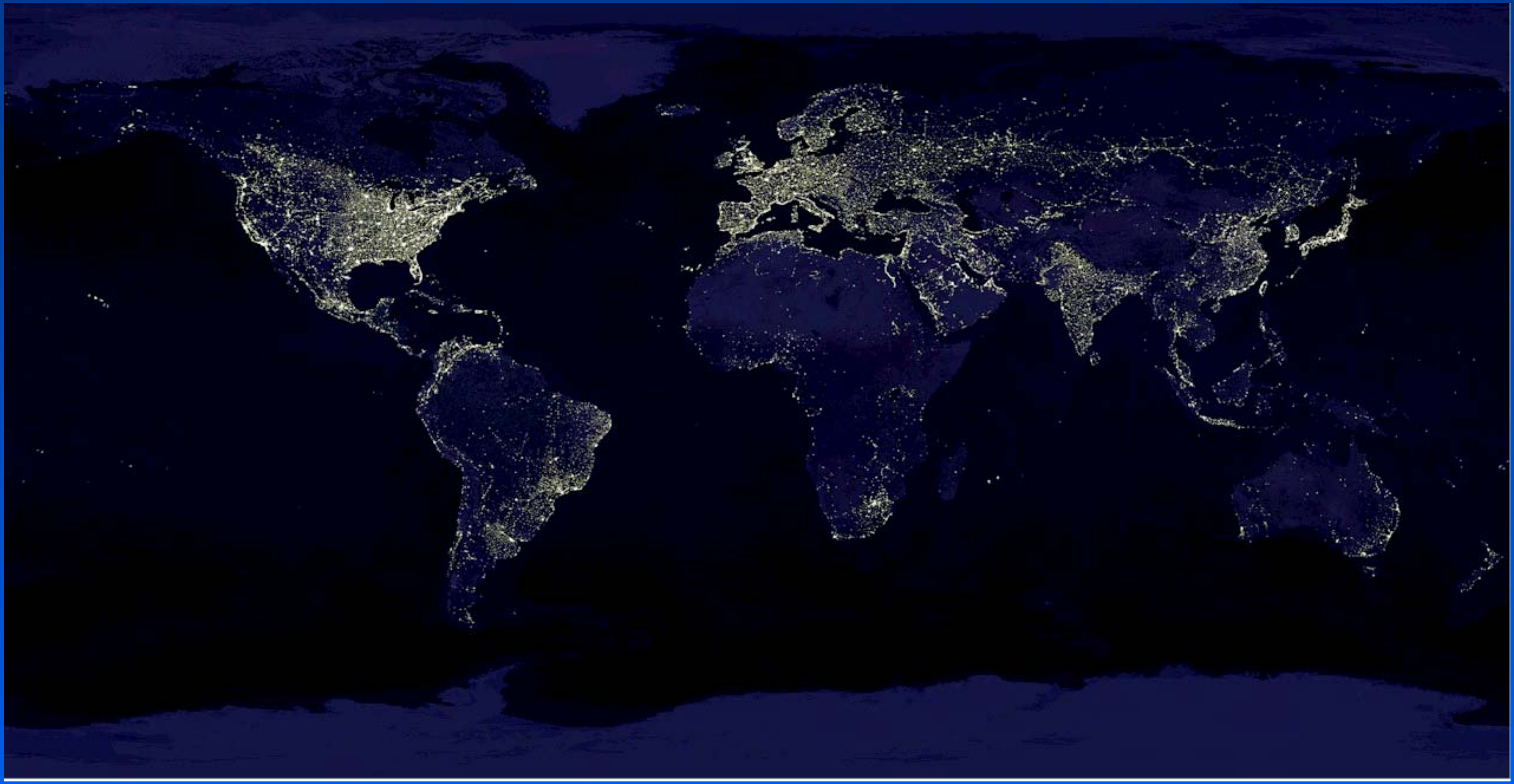


Internet Traffic Flows Over One 2 Hour Period



from Stephen Eick, Visual Insights

Electricity is Modernity

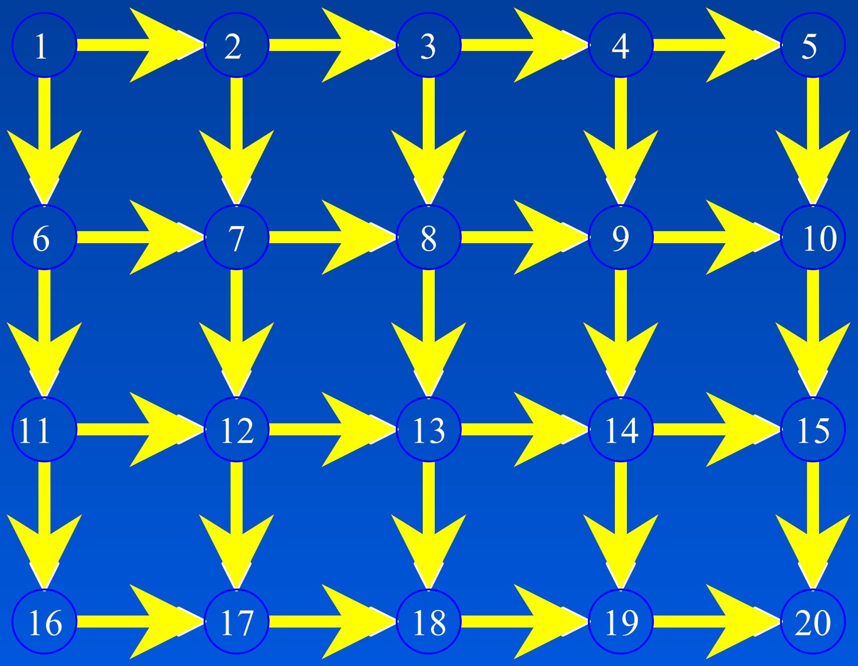


The scientific study of networks involves:

- how to **model** such applications as **mathematical entities**,
- how to **study the models** qualitatively,
- how to design **algorithms** to solve the resulting models.

The Basic Components of Networks

Nodes Links Flows



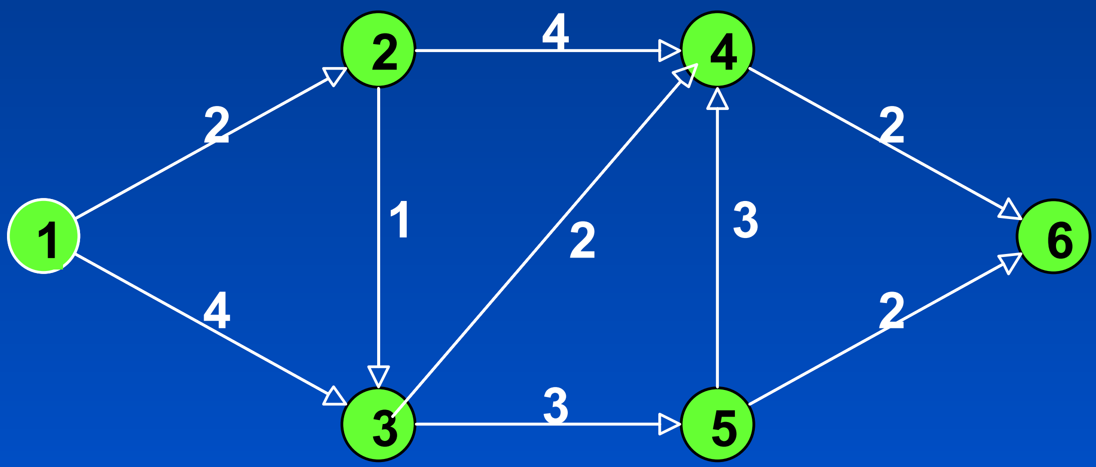
Classic Examples of Network Problems

The Shortest Path Problem

The Maximum Flow Problem

The Minimum Cost Flow Problem.

The Shortest Path Problem



What is the shortest path from 1 to 6?

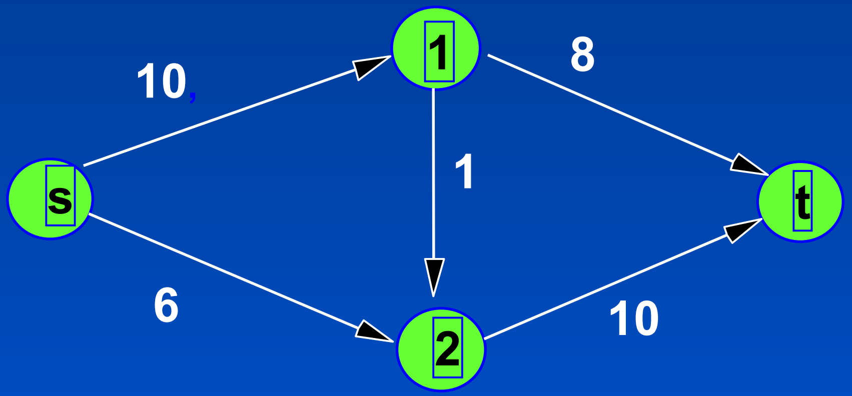
Applications of the Shortest Path Problem

Arise in transportation and telecommunications.

Other applications include:

- simple building evacuation models
- DNA sequence alignment
- assembly line balancing
- compact book storage in libraries.

The Maximum Flow Problem



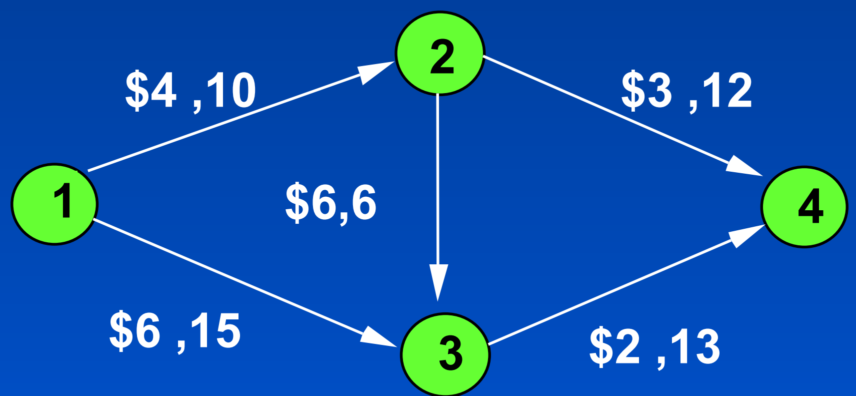
Each link has a maximum capacity.

How does one Maximize the flow from s to t, subject to the link capacities?

Applications of the Maximum Flow Problem

- machine scheduling
- network reliability testing
- building evacuation

The Minimum Cost Flow Problem



Each link has a linear cost and a maximum capacity.

How does one Minimize Cost for a given flow from 1 to 4?

The Optimization Formulation

Flow out of node i - Flow into node i = b(i)

Minimize $\Sigma_{i,j}\, c_{ij}\, x_{ij}$

s.t. $\Sigma_j\, x_{ij} - \Sigma_j\, x_{ji} = b(i)$ for each node i

$0 \preceq x_{ij} \preceq u_{ij}$ for all i,j

$\Sigma_i\, b(i) = 0$

Applications of the Minimum Cost Flow Problem

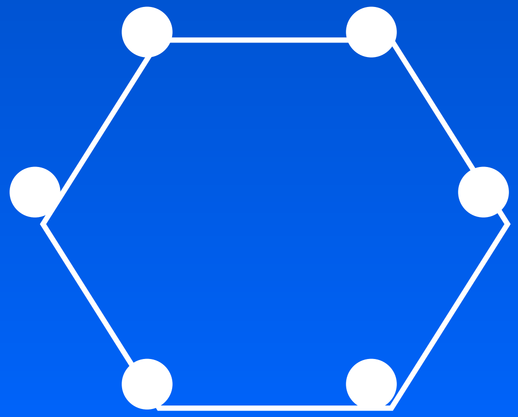
- warehousing and distribution
- vehicle fleet planning
- cash management
- automatic chromosome classification
- satellite scheduling

The study of the efficient operation on transportation networks dates to *ancient Rome* with a classical example being the publicly provided Roman road network and the *time of day chariot policy,* whereby chariots were banned from the ancient city of Rome at particular times of day.

Brief History of the Science of Networks

1736 - Euler - the earliest paper on graph theory - Konigsberg bridges problem.

1758 - Quesnay in his *Tableau Economique* introduced a graph to depict the circular flow of financial funds in an economy.



1781 - Monge, who had worked under Napoleon Bonaparte, publishes what is probably the first paper on transportation in minimizing cost.

1838 - Cournot states that a competitive price is determined by the intersection of supply and demand curves in the context of spatially separate markets in which transportation costs are included.

1841 - Kohl considered a two node, two route transportation network problem.

1845 - Kirchhoff wrote Laws of Closed Electric Circuits.

1920 - Pigou studied a transportation network system of two routes and noted that the decision-making behavior of the users on the network would result in different flow patterns.

1936 - Konig published the first book on graph theory.

1939, 1941, 1947 - Kantorovich, Hitchcock, and Koopmans considered the network flow problem associated with the classical minimum cost transportation problem and provided insights into the special network structure of these problems, which yielded special-purpose algorithms.

1948, 1951 - Dantzig published the simplex method for linear programming and adapted it for the classical transportation problem.

1951 - Enke showed that spatial price equilibrium problems can be solved using electronic circuits

1952 - Copeland in his book asked, *Does money flow like water or electricity?*

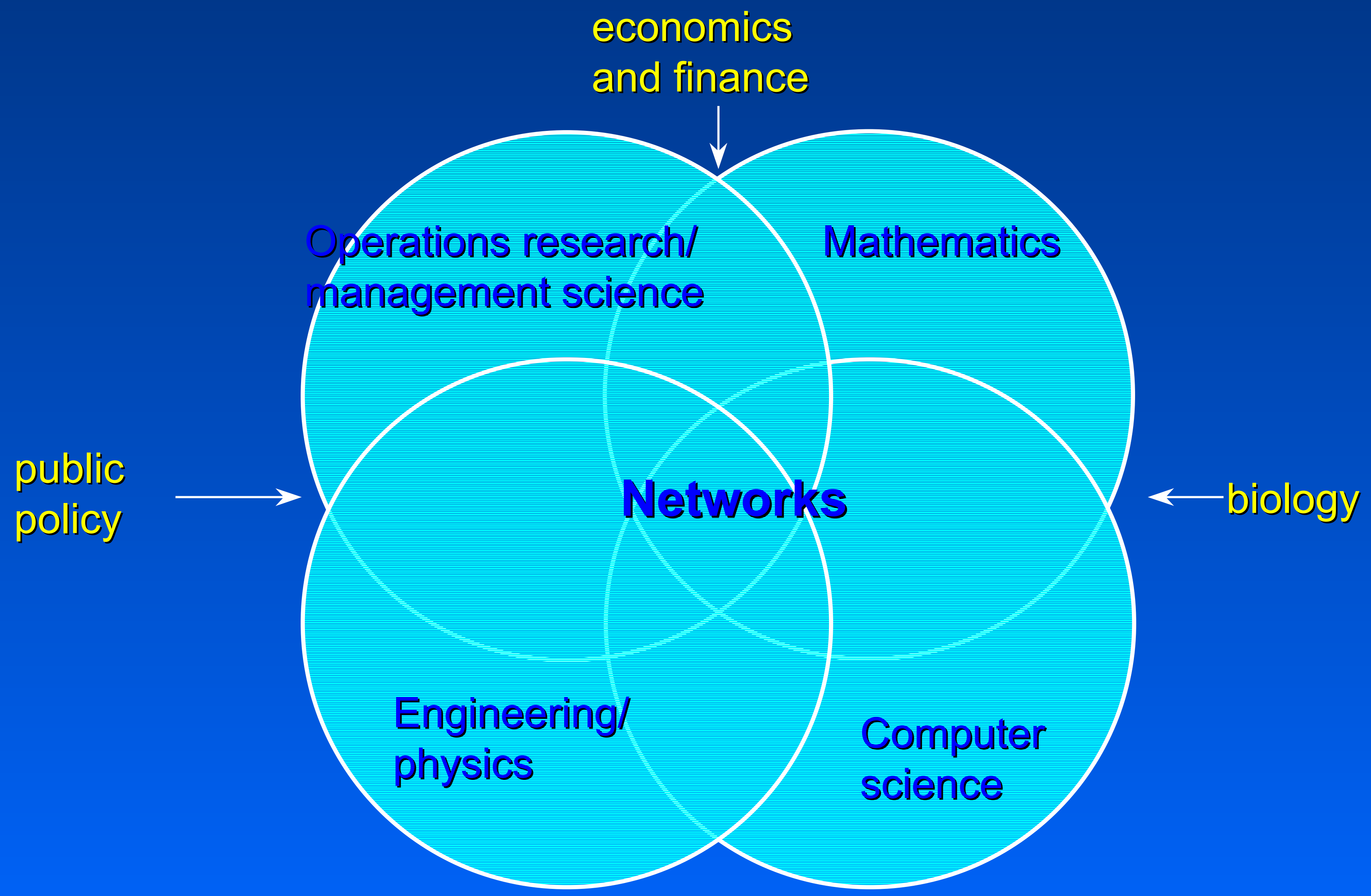
1952 - Samuelson gave a rigorous mathematical formulation of spatial price equilibrium and emphasized the network structure.

1956 - Beckmann, McGuire, and Winsten in their book, *Studies in the Economics of Transportation*, provided a rigorous treatment of congested urban transportation systems under different behavioral mechanisms due to Wardrop (1952).

1962 - Ford and Fulkerson publish *Flows in Networks*.

1969 - Dafermos and Sparrow coined the terms *user-optimization* and system-optimization and develop algorithms for the computation of solutions that exploit the network structure of transportation problems.

Networks in Different Disciplines



Interdisciplinary Impact of Networks

Mathematics

Networks

Economics

Interregional Trade

General Equilibrium

Industrial Organization

Portfolio Optimization

Flow of Funds Accounting

Engineering

Energy

Manufacturing

Telecommunications

Transportation

Sociology

Social Networks

Organizational Theory

Computer Science

Routing Algorithms

Biology

DNA Sequencing

Targeted Cancer Therapy

Characteristics of Networks Today

- *large-scale nature* and complexity of network topology;
- *congestion*;
- alternative behavior of users of the network, which may lead to *paradoxical phenomena*;
- the *interactions among networks* themselves such as in transportation versus telecommunications;
- *policies* surrounding networks today may have a *major impact* not only economically but also *socially, politically, and security-wise*.

There are *two fundamental principles of travel behavior*, due to Wardrop (1952), which we refer to as user-optimization (or network equilibrium) or system-optimization. These terms were coined by Dafermos and Sparrow (1969); see also Beckmann, McGuire, and Winsten (1956).

In a *user-optimized (network equilibrium) problem*, each user of a network system seeks to determine his/her cost-minimizing route of travel between an origin/destination pair, until an equilibrium is reached, in which no user can decrease his/her cost of travel by unilateral action.

In a *system-optimized network problem*, users are allocated among the routes so as to minimize the total cost in the system. Both classes of problems, under certain imposed assumptions, possess optimization formulations.

The Transportation Social - Knowledge Network

Professors Beckmann and Dafermos at Anna Nagurney's Post-Ph.D. Defense Party in Barus Holley

Professor Beckmann with Professor Michael Florian of Montreal

On the Beach in Mallacoota, Austrailia

INFORMS Honoring the 50th Anniversary of the Publication of ***Studies in the Economics of Transportation***

Professors Beckman and McGuire

The (U-O) Transportation Network Equilibrium Problem

Consider a general network $G = [N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote an acyclic path consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. P_w denotes the set of paths connecting the O/D pair of nodes w and P the set of all paths.

Let x_p represent the flow on path p and let f_a denote the flow on link a. The following conservation of flow equations must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. This expression states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if we let d_w denote the demand associated with O/D pair w, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair w must be equal to the given demand d_w.

Let c_a denote the user cost associated with traversing link a, and C_p the user cost associated with traversing the path p. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising the path. In the classical model, $c_a = c_a(f_a)$, $\forall a \in L$. In the most general case, $c_a = c_a(f), \forall a \in L$, where f is the vector of link flows.

The (U-O) Transportation Network Equilibrium Conditions

The network equilibrium conditions are then given by: For each path $p \in P_w$ and every O/D pair w:

$$C_p \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0 \end{cases}$$

where λ_w is an indicator, whose value is not known a priori. The equilibrium conditions state that the user costs on all used paths connecting a given O/D pair will be minimal and equalized. This is Wardrop's first principle of travel behavior.

As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that $\frac{\partial c_a}{\partial c_b} = \frac{\partial c_b}{\partial c_a}$, for all links a, b in the network then the solution to the above U-O problem can be reformulated as the solution to an associated optimization problem. For example, if we have that $c_a = c_a(f_a)$, for all links $a \in L$, then the solution to the U-O problem can be obtained by solving:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y) dy$$

subject to:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in L,$$

$$x_p \geq 0, \quad \forall p \in P.$$

The S-O Problem

The above discussion focused on the user-optimized (U-O) problem. We now turn to the system-optimized (S-O) problem in which a central controller, say, seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in L} \hat{c}_a(f_a)$$

where it is assumed that the total cost function on a link a is defined as:

$$\hat{c}_a(f_a) \equiv c_a(f_a) \times f_a,$$

subject to the conservation of flow constraints, and the nonnegativity assumption on the path flows. Here separable link costs have been assumed, for simplicity, and other total cost expressions may be used, as mandated by the particular application.

The S-O Optimality Conditions

Under the assumption of strictly increasing user link cost functions, the optimality conditions are: For each path $p \in P_w$, and every O/D pair w:

$$\hat{C}'_p \begin{cases} = \mu_w, & \text{if} \quad x_p > 0 \\ \geq \mu_w, & \text{if} \quad x_p = 0, \end{cases}$$

where $\hat{C}'_p$ denotes the marginal total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}.$$

The above conditions correspond to Wardrop's second principle of travel behavior.

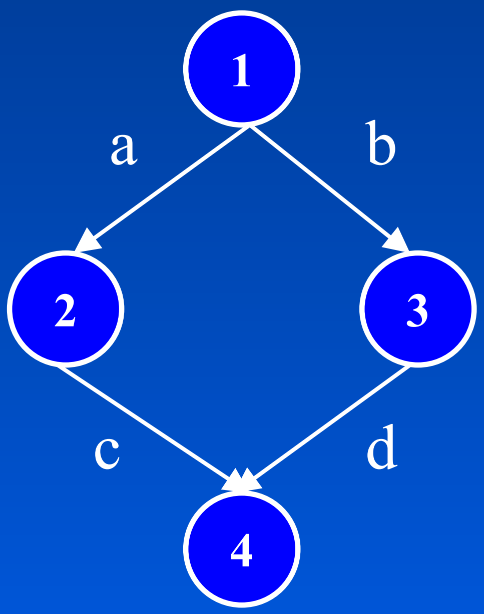
The Braess (1968) Paradox

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $\mathbf{p_1=(a,c)}$ and $\mathbf{p_2=(b,d)}$.

For a travel demand of **6,** the equilibrium path flows are $\mathbf{x_{p_1}^* = x_{p_2}^* = 3}$ and

The equilibrium path travel cost is

$\mathbf{C_{p_1} = C_{p_2} = 83}$.



$$c_a(f_a)=10\, f_a \quad c_b(f_b) = f_b+50$$

$$c_c(f_c) = f_c+50 \quad c_d(f_d) = 10\, f_d$$

Adding a Link Increases Travel Cost for All!

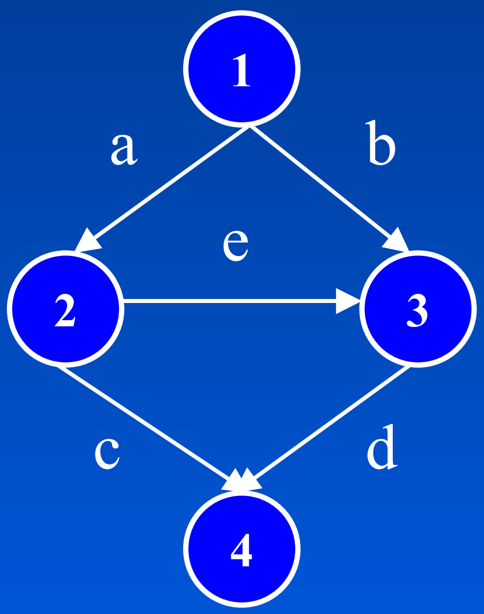
Adding a new link creates a new path **p_3=(a,e,d).**

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path **p_3, C_{p_3}=70.**

The new equilibrium flow pattern network is

$x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2$.

The equilibrium path travel costs: **$C_{p_1} = C_{p_2} = C_{p_3} = 92$.**



$c_e(f_e) = f_e + 10$

What is the S-O solution for the two Braess networks (before and after the addition of the new link e)?

Before the addition of the link e, we may write:

$$\tilde{c}'_a = 20 f_a, \quad \tilde{c}'_b = 2 f_b + 50,$$

$$\tilde{c}'_c = 2 f_c + 50, \quad \tilde{c}'_d = 20 f_d.$$

It is easy to see that, in this case, the S-O solution is identical to the U-O solution with $x_{p_1} = x_{p_2} = 3$ and $\widehat{C}'_{p_1} = \widehat{C}'_{p_2} = 116$.

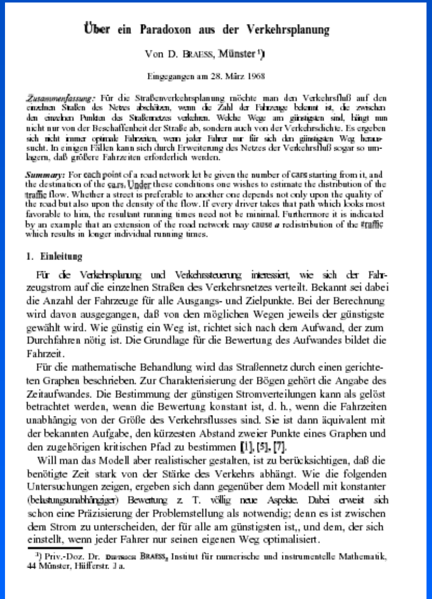
Furthermore, after the addition of link e, we have that $\tilde{c}'_e = 2 f_e + 10$. The new path p_3 is not used in the S-O solution, since with zero flow on path p_3, we have that $\widehat{C}'_{p_3} = 170$ and $\widehat{C}'_{p_1} = \widehat{C}'_{p_2}$ remains at 116.

The 1968 Braess article has been translated from German to English and appears as

On a Paradox of Traffic Planning

by Braess, Nagurney, Wakolbinger

in the November 2005 issue of *Transportation Science.*



Über ein Paradoxon aus der Verkehrsplanung

Von D. BRAESS, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

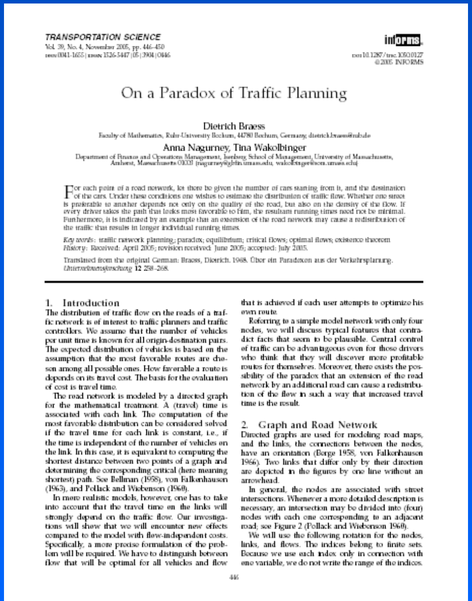
1. Einleitung

Für die Verkehrsplanung und Verkehrssteuerung interessiert, wie sich der Fahrzeugstrom auf die einzelnen Straßen des Verkehrsnetzes verteilt. Bekannt sei dabei die Anzahl der Fahrzeuge für alle Ausgangs- und Zielpunkte. Bei der Berechnung wird davon ausgegangen, daß von den möglichen Wegen jeweils der günstigste gewählt wird. Wie günstig ein Weg ist, richtet sich nach dem Aufwand, der zum Durchfahren nötig ist. Die Grundlage für die Bewertung des Aufwandes bildet die Fahrzeit.

Für die mathematische Behandlung wird das Straßennetz durch einen gerichteten Graphen beschrieben. Zur Charakterisierung der Bögen gehört die Angabe des Zeitaufwandes. Die Bestimmung der günstigen Stromverteilungen kann als gelöst betrachtet werden, wenn die Bewertung konstant ist, d. h., wenn die Fahrzeiten unabhängig von der Größe des Verkehrsflusses sind. Sie ist dann äquivalent mit der bekannten Aufgabe, den kürzesten Abstand zweier Punkte eines Graphen und den zugehörigen kritischen Pfad zu bestimmen [1], [5], [7].

Will man das Modell aber realistischer gestalten, ist zu berücksichtigen, daß die benötigte Zeit stark von der Stärke des Verkehrs abhängt. Wie die folgenden Untersuchungen zeigen, ergeben sich dann gegenüber dem Modell mit konstanter (belastungsunabhängiger) Bewertung z. T. völlig neue Aspekte. Dabei erweist sich schon eine Präzisierung der Problemstellung als notwendig; denn es ist zwischen dem Strom zu unterscheiden, der für alle am günstigsten ist,, und dem, der sich einstellt, wenn jeder Fahrer nur seinen eigenen Weg optimalisiert.

[1]) Priv.-Doz. Dr. DIETRICH BRAESS, Institut für numerische und instrumentelle Mathematik, 44 Münster, Hüfferstr. 3 a.



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On a Paradox of Traffic Planning

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For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Key words: traffic network planning; paradox; equilibrium; critical flows; optimal flows; existence theorem
History: Received: April 2005; revision received: June 2005; accepted: July 2005.

Translated from the original German: Braess, Dietrich. 1968. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* **12** 258–268.

1. Introduction

The distribution of traffic flow on the roads of a traffic network is of interest to traffic planners and traffic controllers. We assume that the number of vehicles per unit time is known for all origin-destination pairs. The expected distribution of vehicles is based on the assumption that the most favorable routes are chosen among all possible ones. How favorable a route is depends on its travel cost. The basis for the evaluation of cost is travel time.

The road network is modeled by a directed graph for the mathematical treatment. A (travel) time is associated with each link. The computation of the most favorable distribution can be considered solved if the travel time for each link is constant, i.e., if the time is independent of the number of vehicles on the link. In this case, it is equivalent to computing the shortest distance between two points of a graph and determining the corresponding critical (here meaning shortest) path. See Bellman (1958), von Falkenhausen (1963), and Pollack and Wiebenson (1960).

In more realistic models, however, one has to take into account that the travel time on the links will strongly depend on the traffic flow. Our investigations will show that we will encounter new effects compared to the model with flow-independent costs. Specifically, a more precise formulation of the problem will be required. We have to distinguish between flow that will be optimal for all vehicles and flow that is achieved if each user attempts to optimize his own route.

Referring to a simple model network with only four nodes, we will discuss typical features that contradict facts that seem to be plausible. Central control of traffic can be advantageous even for those drivers who think that they will discover more profitable routes for themselves. Moreover, there exists the possibility of the paradox that an extension of the road network by an additional road can cause a redistribution of the flow in such a way that increased travel time is the result.

2. Graph and Road Network

Directed graphs are used for modeling road maps, and the links, the connections between the nodes, have an orientation (Berge 1958, von Falkenhausen 1966). Two links that differ only by their direction are depicted in the figures by one line without an arrowhead.

In general, the nodes are associated with street intersections. Whenever a more detailed description is necessary, an intersection may be divided into (four) nodes with each one corresponding to an adjacent road; see Figure 2 (Pollack and Wiebenson 1960).

We will use the following notation for the nodes, links, and flows. The indices belong to finite sets. Because we use each index only in connection with one variable, we do not write the range of the indices.

If no such symmetry assumption holds for the user link costs functions, then the equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a *variational inequality problem!*

Smith (1979), Dafermos (1980)

VI Formulation of Transportation Network Equilibrium (Dafermos (1980), Smith (1979))

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

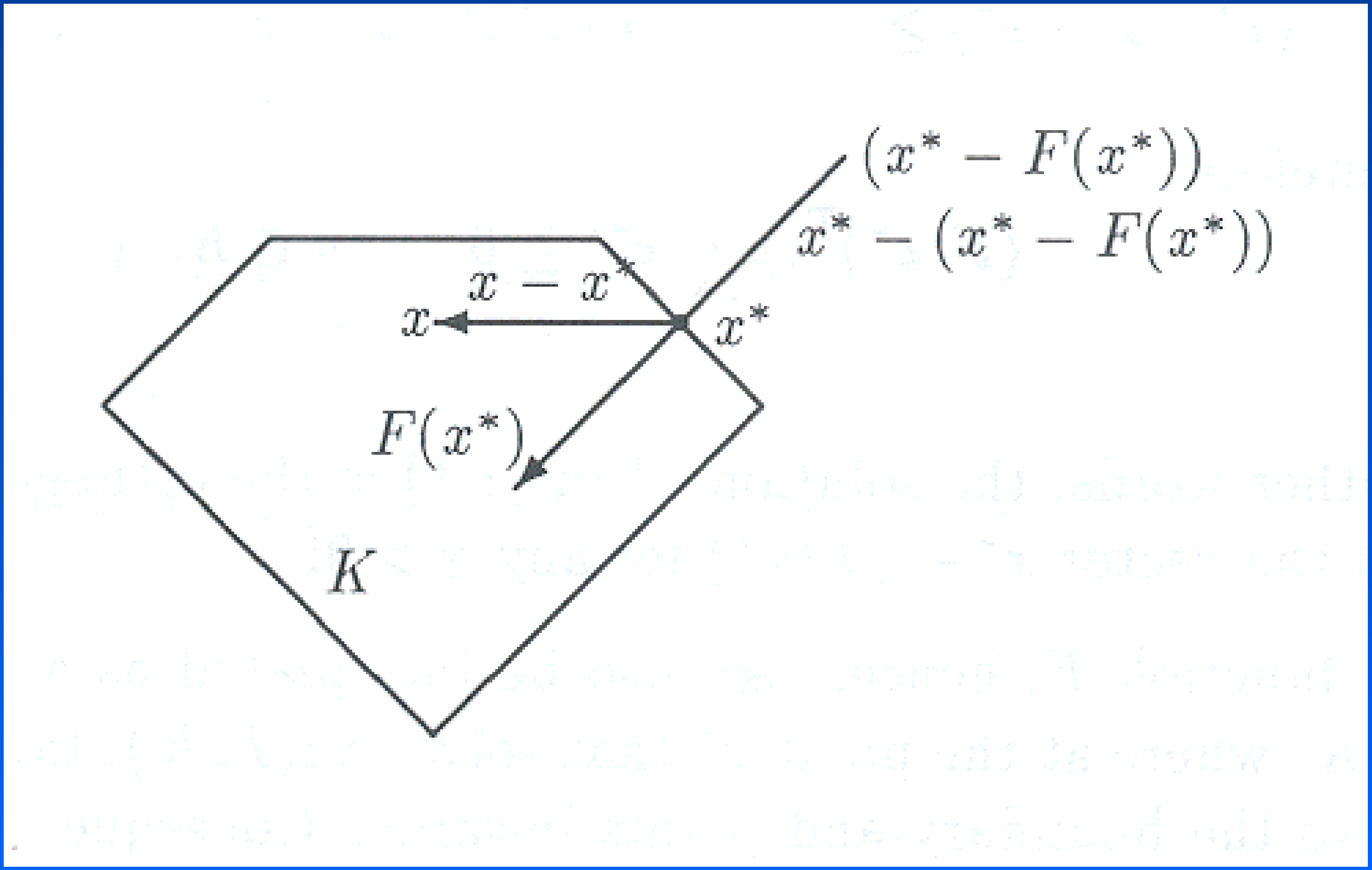
Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

A Geometric Interpretation of a Variational Inequality



The variational inequality problem, contains, as special cases, such classical problems as:

- systems of equations
- optimization problems
- complementarity problems

and is also closely related to fixed point problems.

Hence, it is a unifying mathematical formulation for a variety of mathematical programming problems.

In particular, variational inequalities have been used to formulate such equilibrium problems as:

- transportation network equilibrium problems
- spatial price equilibrium problems
- oligopolistic market equilibrium problems operating under Nash equilibrium
- migration equilibrium problems
- a variety of financial equilibrium problems.

Moreover, all such problems have network structure, which can be further exploited for computational purposes.

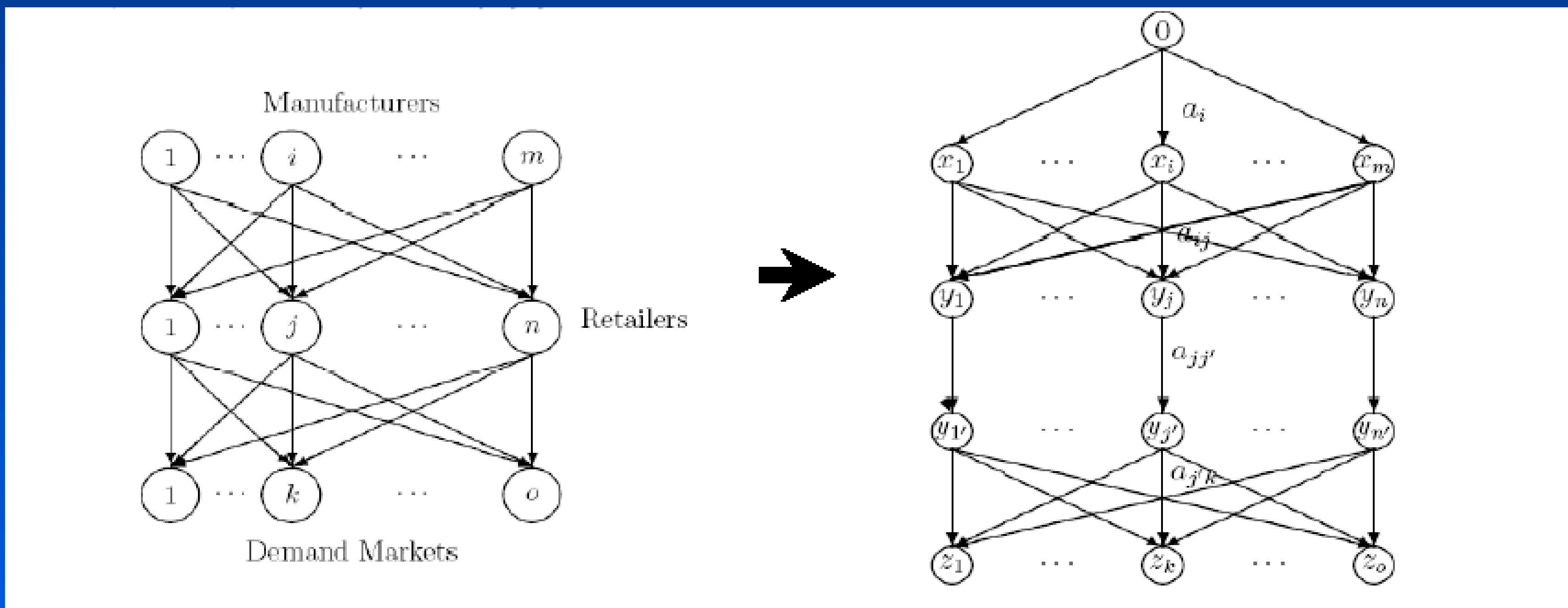
In addition, with the advent of the Internet, there are numerous new models and applications, in which variational inequalities have become a very powerful tool for formulation, qualitative analysis, and computations. Some of these application, we will be discussing in this presentation.

Indeed, the concept of network equilibrium is as relevant to the Internet as it is to transportation!

Some Interesting Applications

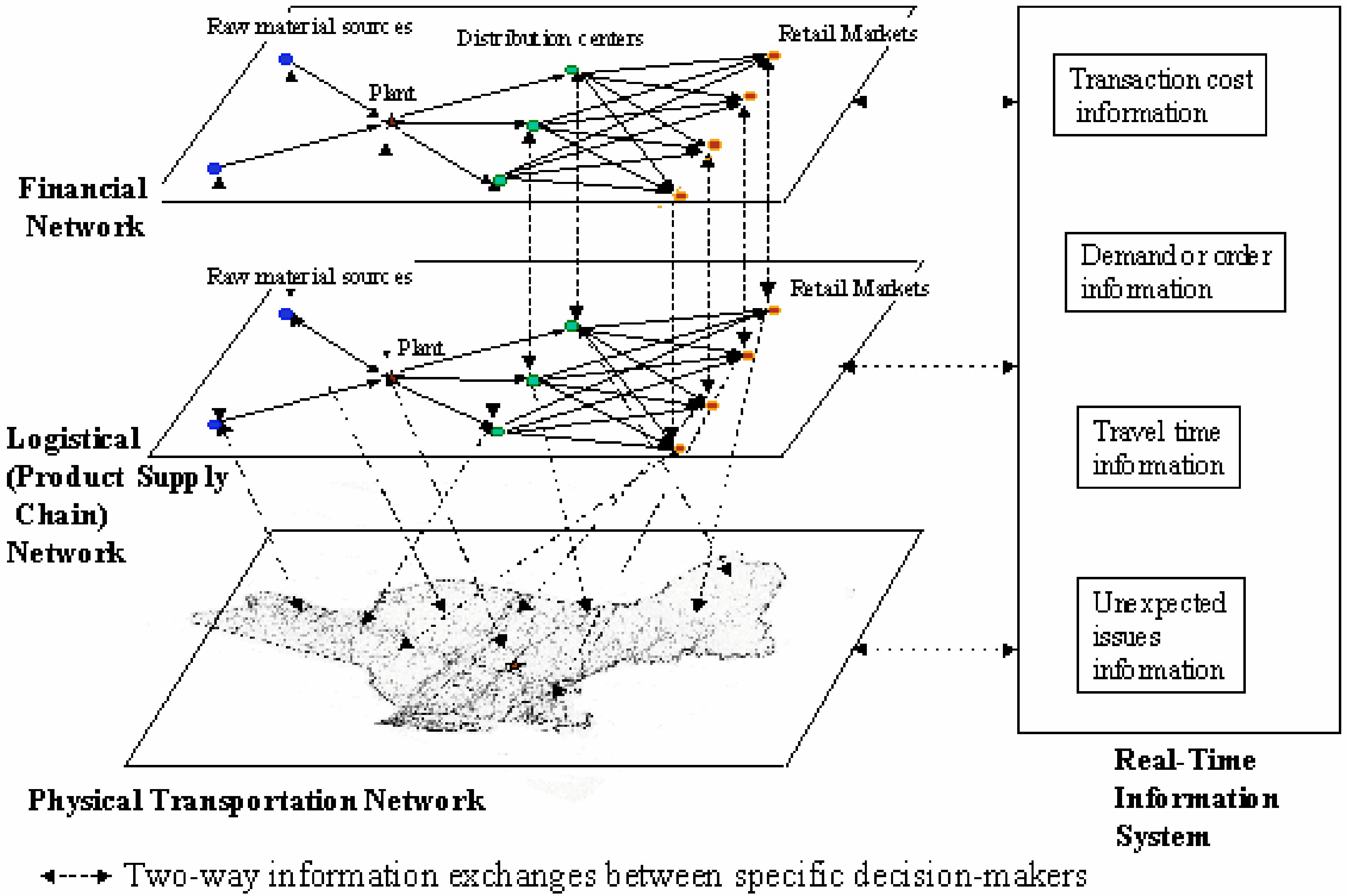
- Telecommuting/Commuting Decision-Making
- Teleshopping/Shopping Decision-Making
- Supply Chain Networks with Electronic Commerce
- Financial Networks with Electronic Transactions
- Reverse Supply Chains with E-Cycling
- Knowledge Networks
- Energy Networks/Power Grids
- Social Networks integrated with Economic Networks

The Equivalence of Supply Chain Networks and Transportation Networks



Nagurney, *Transportation Research E* (2006)

Supply Chain -Transportation Supernetwork Representation

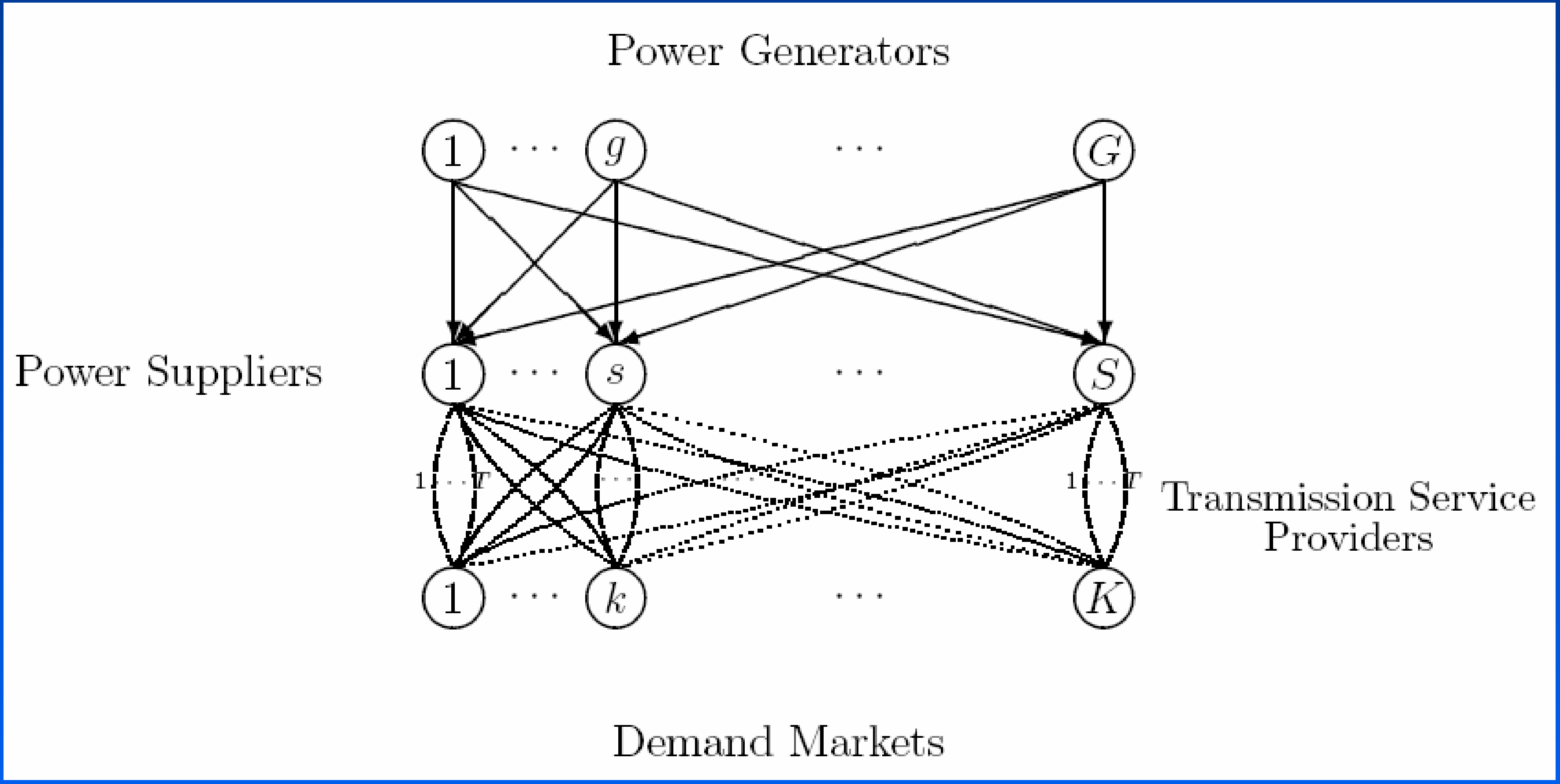


Nagurney, Ke, Cruz, Hancock, Southworth, *Environment and Planning B* (2002)

The fifth chapter of Beckmann, McGuire, and Winsten’s book, **Studies in the Economics of Transportation** (1956) describes some *unsolved problems* including a single commodity network equilibrium problem that the authors imply could be generalized to capture electric power networks.

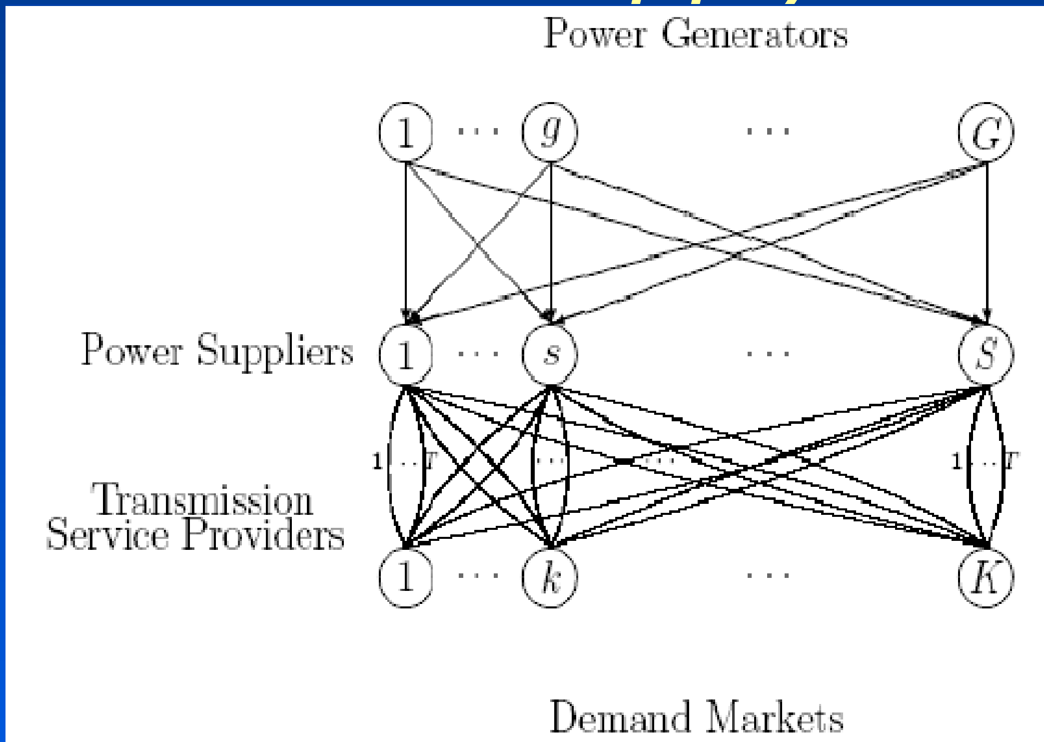
Specifically, they ask whether electric power generation and distribution networks can be reformulated as transportation network equilibrium problems.

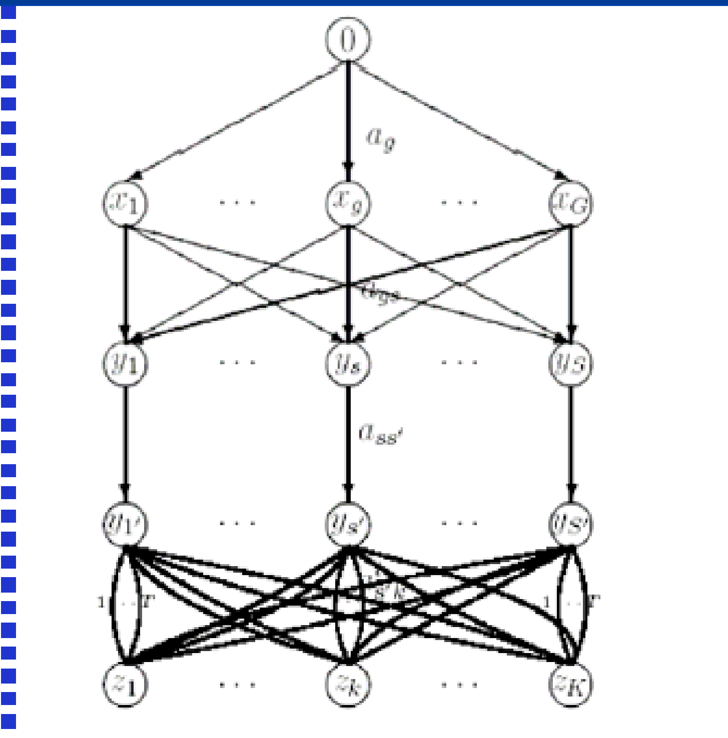
The Electric Power Supply Chain Network



Nagurney and Matsypura, *Proceedings of the CCCT* (2004)

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks





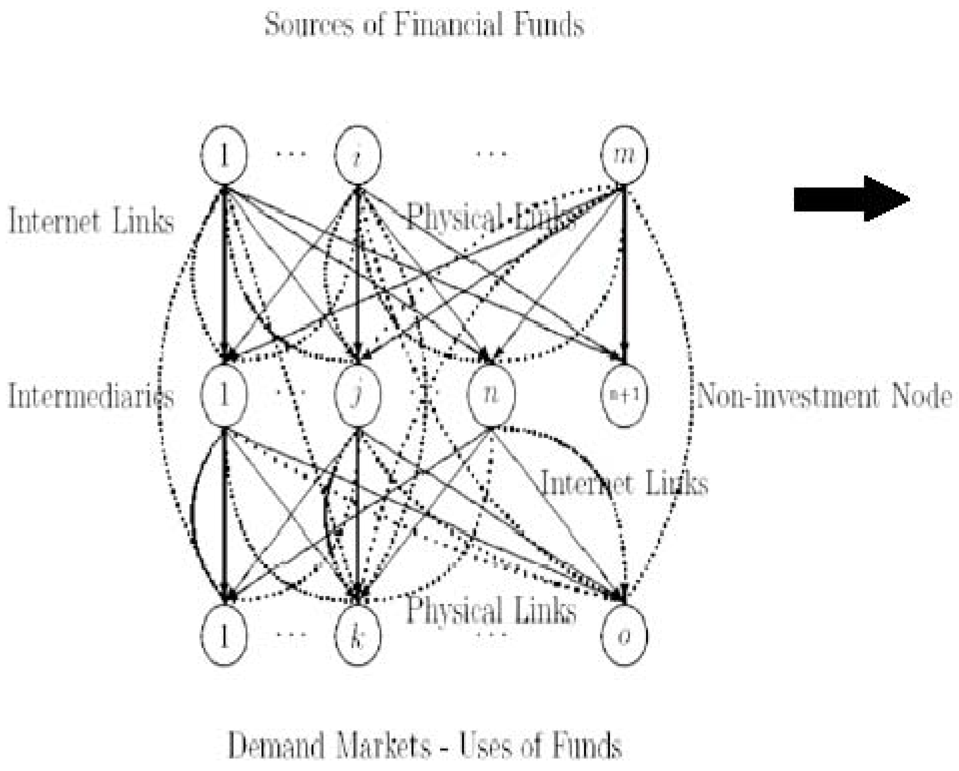
Electric Power Supply Chain Network

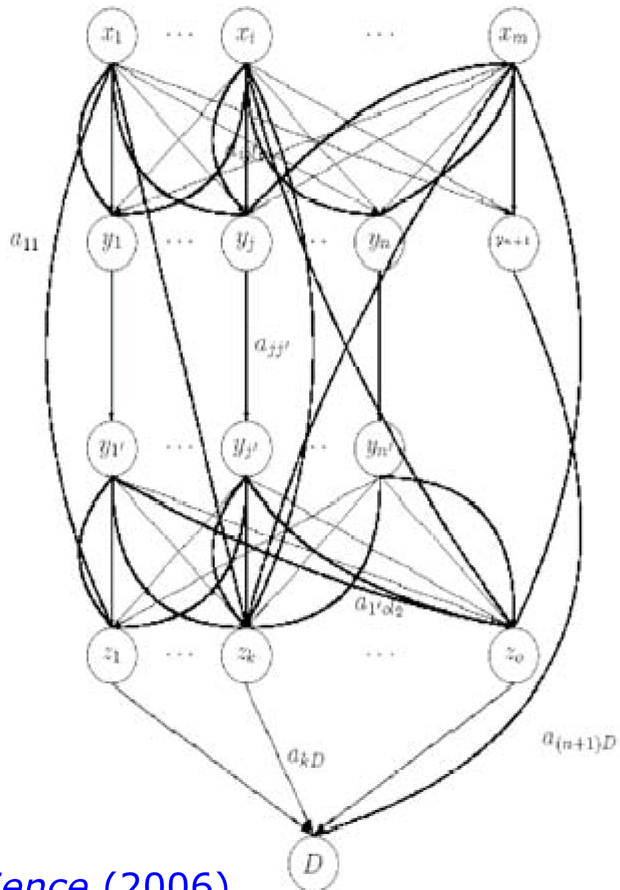
Transportation Network

Nagurney et al, *Transportation Research E (2007)*

In 1952, Copeland wondered whether money flows like water or electricity.

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation





Liu and Nagurney, *Computational Management Science* (2006)

We have shown that *money* as well as *electricity* flow like *transportation* and have answered questions posed fifty years ago by Copeland and Beckmann, McGuire, and Winsten!

We are using evolutionary variational inequalities to model dynamic networks with:

- *dynamic (time-dependent)* supplies and demands
- *dynamic (time-dependent)* capacities
- *structural changes* in the networks themselves.

Such issues are important for robustness, resiliency, and reliability of networks (including supply chains and the Internet).

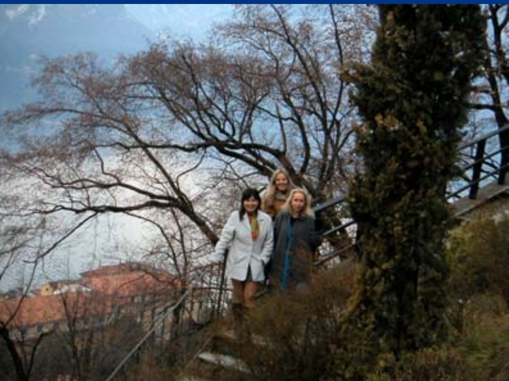
Evolutionary Variational Inequalities

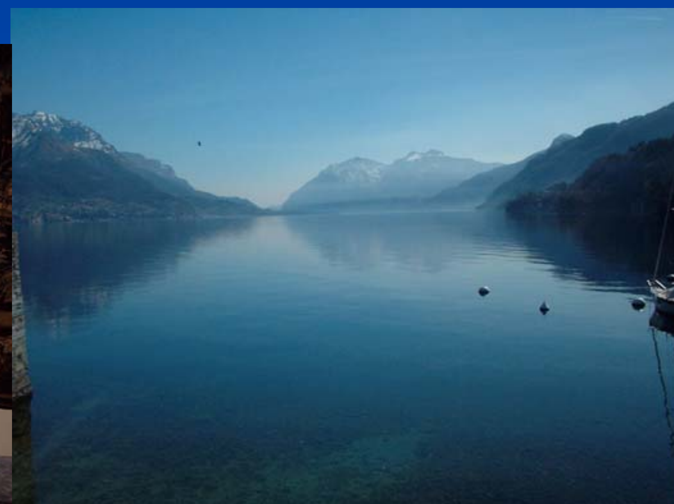
Evolutionary variational inequalities, which are infinite dimensional, were originally introduced by Lions and Stampacchia (1967) and by Brezis (1967) in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems.

Steinbach (1998) studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity.

Daniele, Maugeri, and Oettli (1998, 1999), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others. See also Ran and Boyce(1996).

Bellagio Research Team Residency March 2004





Evolutionary Variational Inequalities Transportation and the Internet

We model the Internet as a network $\mathcal{G} = [N, L]$, consisting of the set of nodes N and the set of directed links L. The set of origin/destination (O/D) pairs of nodes is denoted by W and consists of n_W elements. We denote the set of routes (with a route consisting of links) joining the origin/destination (O/D) pair w by P_w. We assume that the routes are acyclic. We let P with n_P elements denote the set of all routes connecting all the O/D pairs in the Internet. Links are denoted by a, b, etc; routes by r, q, etc., and O/D pairs by w_1, w_2, etc. We assume that the Internet is traversed by "jobs" or "classes" of traffic and that there are K "jobs" with a typical job denoted by k.

Let $d_w^k(t)$ denote the demand, that is, the traffic generated, between O/D pair w at time t by job class k. The flow on route r at time t of class k, which is assumed to be nonnegative, is denoted by $x_r^k(t)$ and the flow on link a of class k at time t by $f_a^k(t)$.

Since the demands over time are assumed known, the following conservation of flow equations must be satisfied at each t:

$$d_w^k(t) = \sum_{r \in P_w} x_r^k(t), \quad \forall w \in W, \forall k,$$

that is, the demand associated with an O/D pair and class must be equal to the sum of the flows of that class on the routes that connect that O/D pair. We assume that the traffic associated with each O/D pair is divisible and can be routed among multiple routes/paths. Also, we must have that

$$0 \le x_r^k(t) \le \mu_r^k(t), \quad \forall r \in P, \forall k,$$

where $\mu_r^k(t)$ denotes the capacity on route r of class k at time t.

We group the demands at time t of classes for all the O/D pairs into the Kn_W-dimensional vector $d(t)$. Similarly, we group all the class route flows at time t into the Kn_P-dimensional vector $x(t)$. The capacities on the routes at time t are grouped into the Kn_P-dimensional vector $\mu(t)$.

The link flows are related to the route flows, in turn, through the following conservation of flow equations:

$$f_a^k(t) = \sum_{r \in P} x_r^k(t)\delta_{ar}, \quad \forall a \in L, \forall k,$$

where $\delta_{ar} = 1$ if link a is contained in route r, and $\delta_{ar} = 0$, otherwise. Hence, the flow of a class on a link is equal to the sum of the flows of the class on routes that contain that link. All the link flows at time t are grouped into the vector $f(t)$, which is of dimension Kn_L.

The cost on route r at time t of class k is denoted by $C_r^k(t)$ and the cost on a link a of class k at time t by $c_a^k(t)$.

We allow the cost on a link to depend upon the entire vector of link flows at time t, so that

$$c_a^k(t) = c_a^k(f(t)), \quad \forall a \in L, \forall k.$$

We may write the link costs as a function of route flows, that is,

$$c_a^k(x(t)) \equiv c_a^k(f(t)), \quad \forall a \in L, \forall k.$$

The costs on routes are related to costs on links through the following equations:

$$C_r^k(x(t)) = \sum_{a \in L} c_a^k(x(t))\delta_{ar}, \quad \forall r \in P, \forall k.$$

We group the route costs at time t into the vector $C(t)$, which is of dimension Kn_P.

We now define the feasible set $\mathcal{K}$. We consider the Hilbert space $\mathcal{L} = L^2([0,T], R^{Kn_P})$ (where $[0,T]$ denotes the time interval under consideration) given by

$$\mathcal{K} = \Big\{ x \in L^2([0,T], R^{Kn_P}) : 0 \le x(t) \le \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{p \in P_w} x_p^k(t) = d_w^k(t), \forall w, \forall k \text{ a.e. in } [0,T] \Big\}.$$

We assume that the capacities $\mu_r^k(t)$, for all r and k, are in $\mathcal{L}$, and that the demands, $d_w^k \ge 0$, for all w and k, are also in $\mathcal{L}$. Further, we assume that

$$0 \le d(t) \le \Phi\mu(t), \text{ a.e. on } [0,T],$$

where Φ is the $Kn_W \times Kn_P$-dimensional O/D pair-route incidence matrix, with element (kw, kr) equal to 1 if route r is contained in P_w, and 0, otherwise. The feasible set $\mathcal{K}$ is nonempty. It is easily seen that $\mathcal{K}$ is also convex, closed, and bounded.

The dual space of $\mathcal{L}$ will be denoted by $\mathcal{L}^*$. On $\mathcal{L} \times \mathcal{L}^*$ we define the canonical bilinear form by

$$\langle\langle G, x \rangle\rangle := \int_0^T \langle G(t), x(t) \rangle dt, \quad G \in \mathcal{L}^*, \quad x \in \mathcal{L}.$$

Furthermore, the cost mapping $C : \mathcal{K} \rightarrow \mathcal{L}^*$, assigns to each flow trajectory $x(\cdot) \in \mathcal{K}$ the cost trajectory $C(x(\cdot)) \in \mathcal{L}^*$.

The conditions below are a generalization of the Wardrop's (1952) first principle of traffic behavior.

Definition: Dynamic Multiclass Network Equilibrium

A multiclass route flow pattern $x^* \in \mathcal{K}$ is said to be a dynamic network equilibrium (according to the generalization of Wardrop's first principle) if, for every O/D pair $w \in W$, every route $r \in P_w$, every class k; $k = 1, \ldots, K$, and a.e. on $[0, T]$:

$$C_r^k(x^*(t)) - \lambda_w^{k*}(t) \begin{cases} \leq 0, & \text{if} \quad x_r^{k*}(t) = \mu_r^k(t), \\ = 0, & \text{if} \quad 0 < x_r^{k*}(t) < \mu_r^k(t), \\ \geq 0, & \text{if} \quad x_r^{k*}(t) = 0. \end{cases}$$

The standard form of the EVI that we work with is:

determine $x^* \in \mathcal{K}$ such that $\langle\langle F(x^*), x - x^* \rangle\rangle \geq 0, \ \forall x \in \mathcal{K}$.

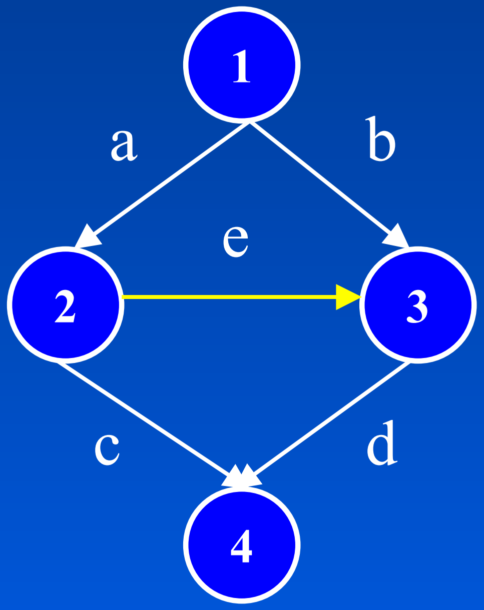
Theorem (Nagurney, Parkes, and Daniele (2007))

$x^* \in \mathcal{K}$ is an equilibrium flow according to the Definition if and only if it satisfies the evolutionary variational inequality:

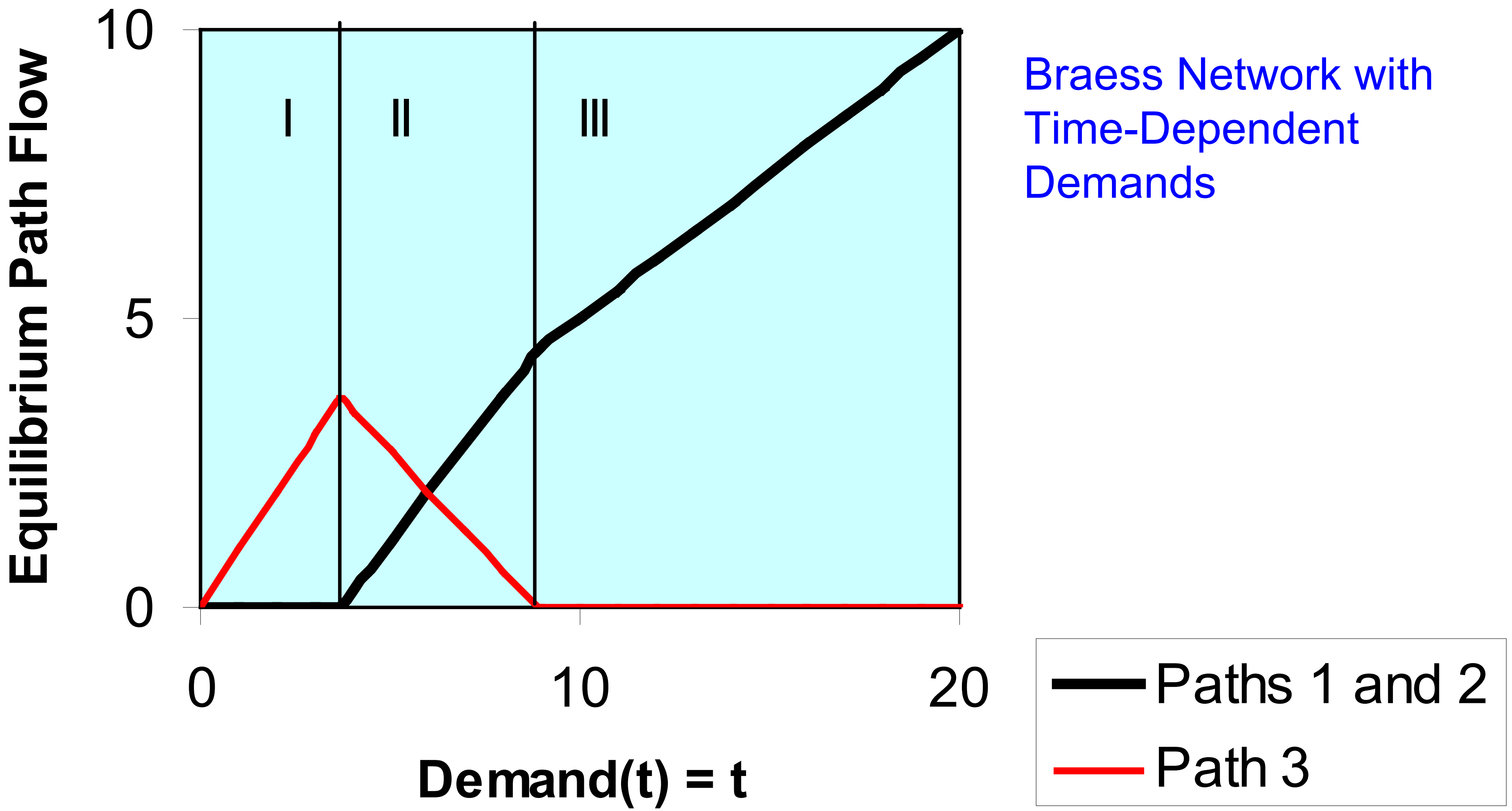
$$\int_0^T \langle C(x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in \mathcal{K}.$$

2005-2006 Radcliffe Institute for Advanced Study Fellowship Year at Harvard Collaboration Professors David Parkes, Patrizia Daniele, and Anna Nagurney

Recall the Braess Network
where we add the link e.

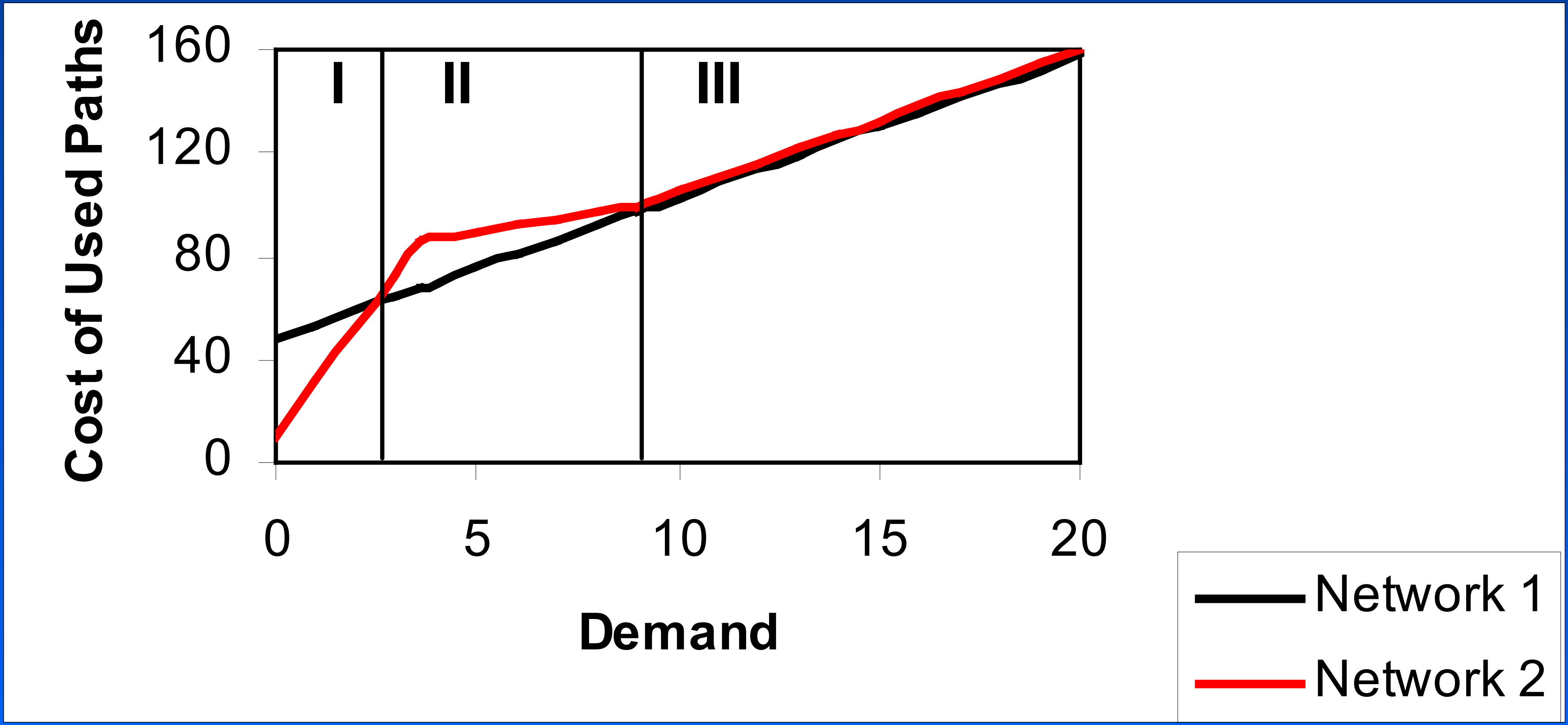


The Solution of an Evolutionary (Time-Dependent) Variational Inequality for the Braess Network with Added Link (Path)



Braess Network with Time-Dependent Demands

In Demand Regime I, only the new path is used.
In Demand Regime II, the Addition of a New Link (Path) Makes Everyone Worse Off!
In Demand Regime III, only the original paths are used.



Network 1 is the Original Braess Network - Network 2 has the added link.

The new link is NEVER used after a certain demand is reached even if the demand approaches infinity.

Hence, in general, except for a limited range of demand, building the new link is a complete waste!

Recent disasters have demonstrated the importance as well as the vulnerability of network systems.

For example:

_ The Minneapolis bridge collapse, August 1, 2007

- Hurricane Katrina, August 23, 2005
- The biggest blackout in North America, August 14, 2003
- 9/11 Terrorist Attacks, September 11, 2001

Disasters in Transportation Networks

www.salem-news.com

www.boston.com

The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007) proposed a network efficiency measure which captures demand, flow, and cost information under network equilibrium. It is defined as follows:

Definition

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2006), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w.

Importance of a Network Component

Definition **Importance of a Network Component**

The importance, $I(g)$ of a network component $g \in G$, is measured by the relative network efficiency drop after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where $G - g$ is the resulting network after component g is removed from network G.

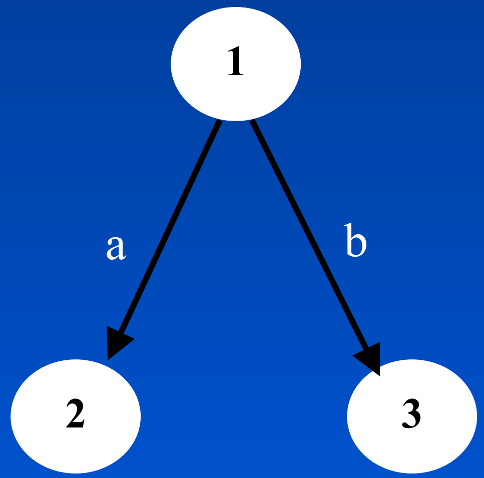
The Approach to Study the Importance of Network Components

The elimination of a link is treated in the Nagurney and Qiang network efficiency measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node.

In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks.

Example 1

Assume a network with two O/D pairs:
w_1=(1,2) and w_2=(1,3) with demands:
d_{w_1}=100 and d_{w_2}=20.



$c_a(f_a)=0.01f_a+19$
$c_b(f_b)=0.05f_b+19$

The paths are:

for w_1, $p_1=a$; for w_2, $p_2=b$.

The equilibrium path flows are:

$x_{p_1}^*$= 100, $x_{p_2}^*$=20.

The equilibrium path travel costs are:

$C_{p_1}=C_{p_2}$=20.

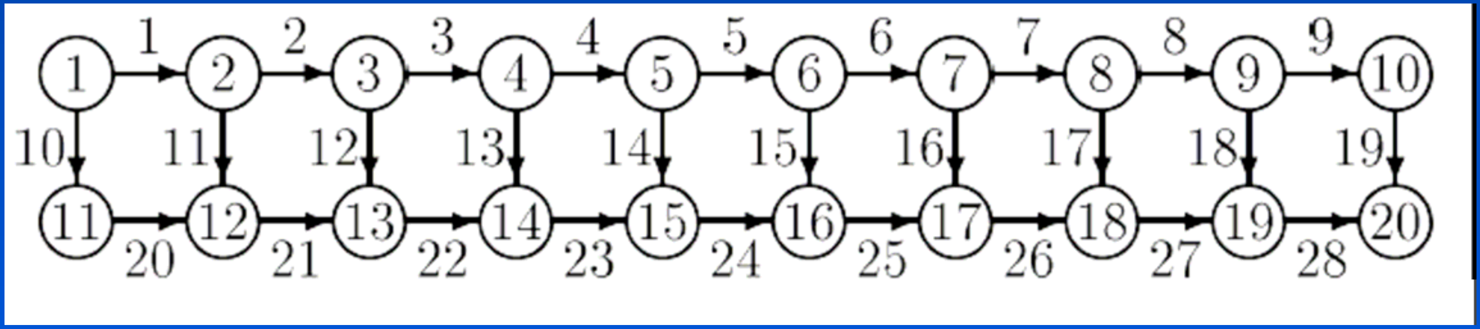
Importance and Ranking of Links and Nodes

Link	Importance Value from Our Measure	Importance Ranking from Our Measure
a	0.8333	1
b	0.1667	2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure
1	1	1
2	0.8333	2
3	0.1667	3

Example 2

The network is given by:



Link Cost Functions

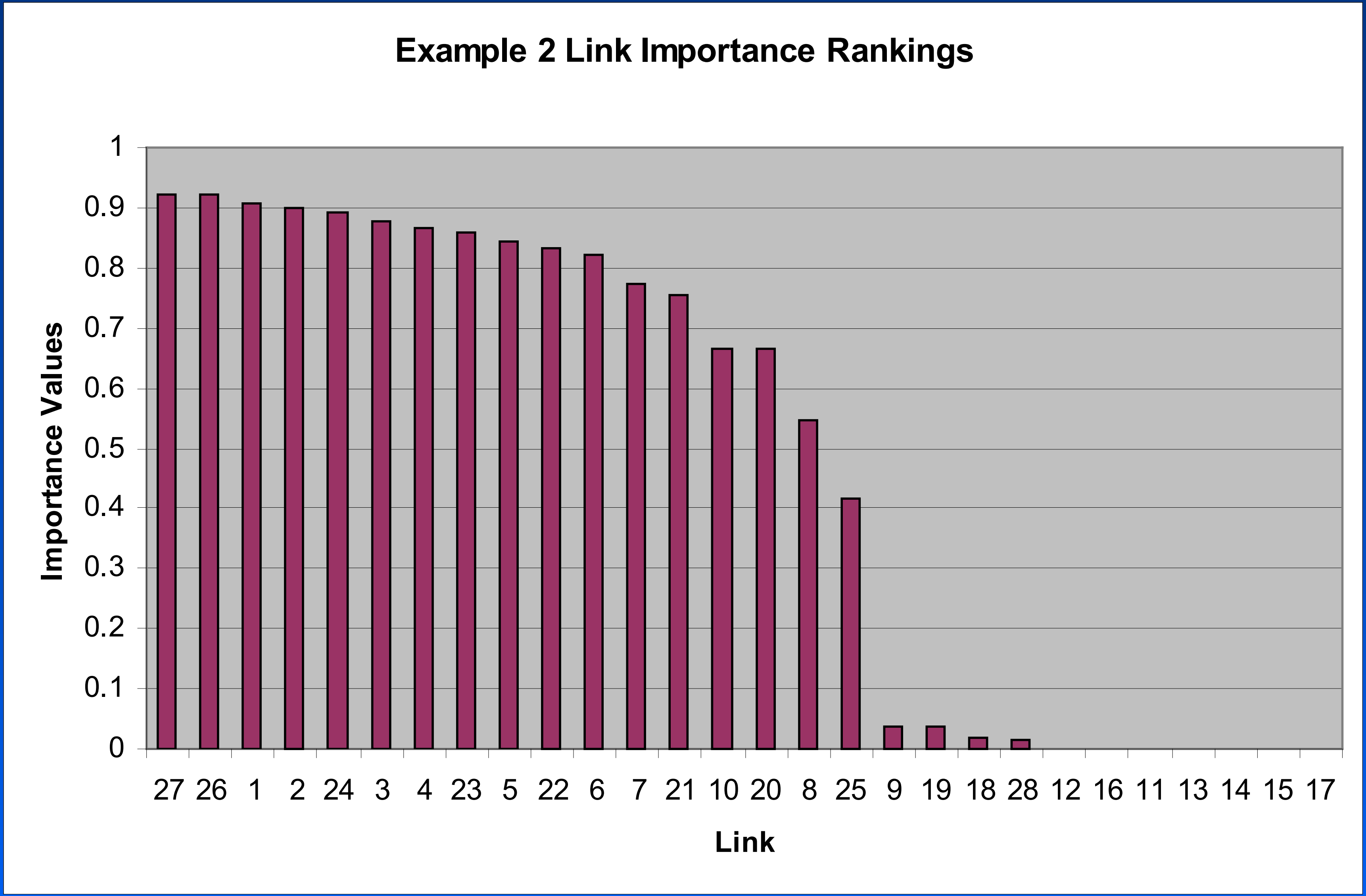
Link a	Link Cost Function $c_a(f_a)$
1	$.00005f_1^4 + 5f_1 + 500$
2	$.00003f_2^4 + 4f_2 + 200$
3	$.00005f_3^4 + 3f_3 + 350$
4	$.00003f_4^4 + 6f_4 + 400$
5	$.00006f_5^4 + 6f_5 + 600$
6	$7f_6 + 500$
7	$.00008f_7^4 + 8f_7 + 400$
8	$.00004f_8^4 + 5f_8 + 650$
9	$.00001f_9^4 + 6f_9 + 700$
10	$4f_{10} + 800$
11	$.00007f_{11}^4 + 7f_{11} + 650$
12	$8f_{12} + 700$
13	$.00001f_{13}^4 + 7f_{13} + 600$
14	$8f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003f_{15}^4 + 9f_{15} + 200$
16	$8f_{16} + 300$
17	$.00003f_{17}^4 + 7f_{17} + 450$
18	$5f_{18} + 300$
19	$8f_{19} + 600$
20	$.00003f_{20}^4 + 6f_{20} + 300$
21	$.00004f_{21}^4 + 4f_{21} + 400$
22	$.00002f_{22}^4 + 6f_{22} + 500$
23	$.00003f_{23}^4 + 9f_{23} + 350$
24	$.00002f_{24}^4 + 8f_{24} + 400$
25	$.00003f_{25}^4 + 9f_{25} + 450$
26	$.00006f_{26}^4 + 7f_{26} + 300$
27	$.00003f_{27}^4 + 8f_{27} + 500$
28	$.00003f_{28}^4 + 7f_{28} + 650$

Importance and Ranking of Links

Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19



Example 2 Link Importance Rankings
Importance Values
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
27 26 1 2 24 3 4 23 5 22 6 7 21 10 20 8 25 9 19 18 28 12 16 11 13 14 15 17
Link

The Advantages of the Nagurney and Qiang Network Efficiency Measure

- The measure captures demands, flows, costs, and behavior of users, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both; and
- It can be applied to assess the efficiency/performance of a wide range of network systems.

The Virtual Center for Supernetworks

Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

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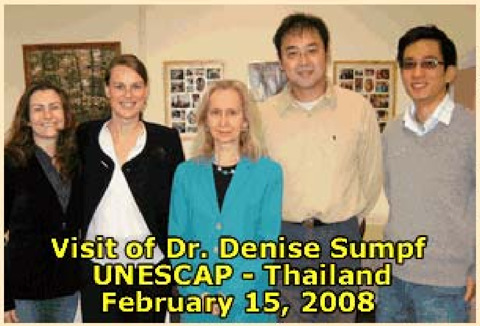

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

Updated: February 23, 2008

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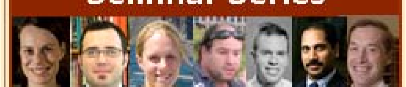
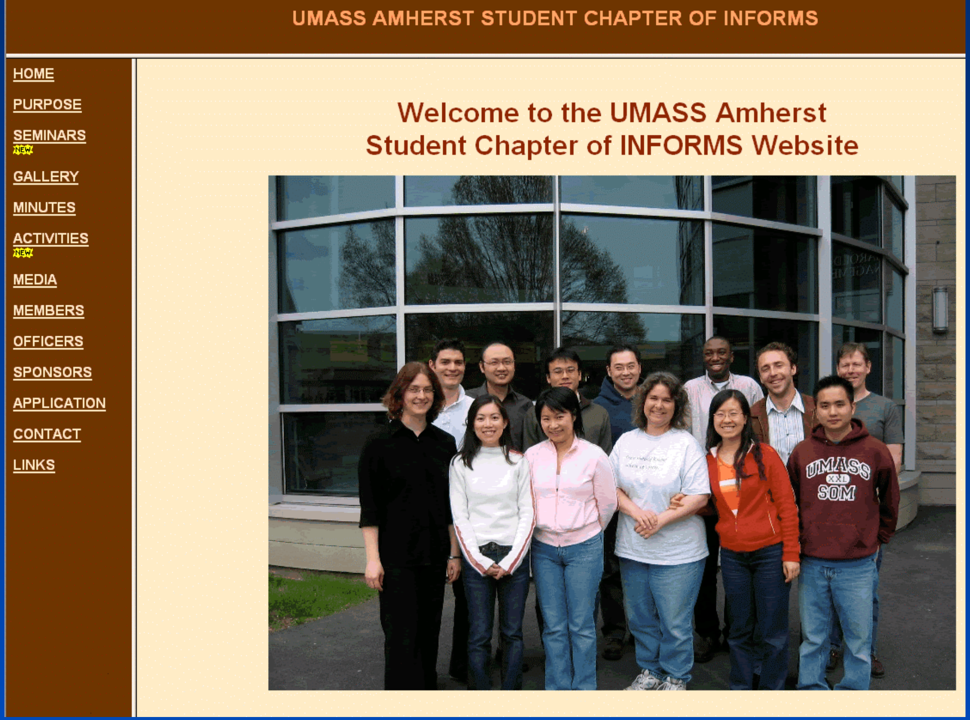


FELLOWS AWARD
Anna Nagurney
For outstanding contributions and dedicated leadership to the Field of Regional Science
Roger R. Stough, President
Savannah, November 10, 2007
Regional Science Association International



Thank you!

For more information, see

http://supernet.som.umass.edu

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