Topic 6: Projected Dynamical Systems

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SCH-MGMT 825
Management Science Seminar
Variational Inequalities, Networks, and Game Theory
Spring 2018

Projected Dynamical Systems

A plethora of equilibrium problems, including network equilibrium problems, can be uniformly formulated and studied as finite-dimensional variational inequality problems, as we have seen in the preceding lectures.

Indeed, it was precisely the traffic network equilibrium problem, as stated by Smith (1979), and identified by Dafermos (1980) to be a variational inequality problem, that gave birth to the ensuing research activity in variational inequality theory and applications in transportation science, regional science, operations research/management science, and, more recently, in economics.

Projected Dynamical Systems

Usually, using this methodology, one first formulates the governing equilibrium conditions as a variational inequality problem. Qualitative properties of existence and uniqueness of solutions to a variational inequality problem can then be studied using the standard theory or by exploiting problem structure. Finally, a variety of algorithms for the computation of solutions to finite-dimensional variational inequality problems are now available (see, e.g., Nagurney (1999, 2006)).

Projected Dynamical Systems

Finite-dimensional variational inequality theory by itself, however, provides no framework for the study of the dynamics of competitive systems. Rather, it captures the system at its equilibrium state and, hence, the focus of this tool is static in nature.

Projected Dynamical Systems

Dupuis and Nagurney (1993) proved that, given a variational inequality problem, there is a naturally associated dynamical system, the stationary points of which correspond precisely to the solutions of the variational inequality problem. This association was first noted by Dupuis and Ishii (1991).

This dynamical system, first referred to as a *projected dynamical system* by Zhang and Nagurney (1995), is non-classical in that its right-hand side, which is a projection operator, is discontinuous. The discontinuities arise because of the constraints underlying the variational inequality problem modeling the application in question. Hence, classical dynamical systems theory is no longer applicable.

Projected Dynamical Systems

Nevertheless, as demonstrated rigorously in Dupuis and Nagurney (1993), a projected dynamical system may be studied through the use of the Skorokhod Problem (1961), a tool originally introduced for the study of stochastic differential equations with a reflecting boundary condition. Existence and uniqueness of a solution path, which is essential for the dynamical system to provide a reasonable model, were also established therein.

We present some recent results in the development of a new tool for the study of equilibrium problems in a dynamic setting, which has been termed *projected dynamical systems* theory (cf. Nagurney and Zhang (1996)).

Projected Dynamical Systems

One of the notable features of this tool, whose rigorous theoretical foundations were laid in Dupuis and Nagurney (1993), is its relationship to the variational inequality problem.

Projected Dynamical Systems

Projected dynamical systems theory, however, goes further than finite-dimensional variational inequality theory in that it extends the static study of equilibrium states by introducing an additional time dimension in order to allow for the analysis of disequilibrium behavior that precedes the equilibrium.

Projected Dynamical Systems

In particular, we associate with a given variational inequality problem, a nonclassical dynamical system, called a projected dynamical system.

The projected dynamical system is interesting both as a dynamical model for the system whose equilibrium behavior is described by the variational inequality, and, also, because its set of stationary points coincides with the set of solutions to a variational inequality problem.

In this framework, the feasibility constraints in the variational inequality problem correspond to discontinuities in the right-hand side of the differential equation, which is a projection operator.

Projected Dynamical Systems

Consequently, the projected dynamical system is not amenable to analysis via the classical theory of dynamical systems.

Projected Dynamical Systems

We first recall the variational inequality problem. We then present the definition of a projected dynamical system, which evolves within a constraint set K.

Its stationary points are identified with the solutions to the corresponding variational inequality problem with the same constraint set.

Projected Dynamical Systems

We then state in a theorem the fundamental properties of such a projected dynamical system in regards to the existence and uniqueness of solution paths to the governing ordinary differential equation.

We, subsequently, provide an interpretation of the ordinary differential equation that defines the projected dynamical system, along with a description of how the solutions may be expected to behave.

For additional qualitative results, in particular, stability analysis results, see Nagurney and Zhang (1996). For a discussion of the general iterative scheme and proof of convergence, see Dupuis and Nagurney (1993).

The Variational Inequality Problem and a Projected Dynamical System

Definition 1 (The Variational Inequality Problem)
For a closed convex set $K \subset R^n$ and vector function $F : K \mapsto R^n$, the variational inequality problem, $\mathrm{VI}(F, K)$, is to determine a vector $x^* \in K$, such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n.

As we have shown, the variational inequality has been used to formulate a plethora of equilibrium problems ranging from traffic network equilibrium problems to spatial oligopolistic market equilibrium problems.

The Variational Inequality Problem and a Projected Dynamical System

Finite-dimensional variational inequality theory, however, provides no framework for studying the underlying dynamics of systems, since it considers only equilibrium solutions in its formulation. Hence, in a sense, it provides a static representation of a system at its "steady state."

One would, therefore, like a theoretical framework that permits one to study a system not only at its equilibrium point, but also in a dynamical setting.

The Variational Inequality Problem and a Projected Dynamical System

The definition of a projected dynamical system (PDS) is given with respect to a closed convex set K, which is usually the constraint set underlying a particular application, such as, for example, network equilibrium problems, and a vector field F whose domain contains K.

As noted in Dupuis and Nagurney (1993), it is expected that such projected dynamical systems will provide mathematically convenient approximations to more "realistic" dynamical models that might be used to describe non-static behavior.

The Variational Inequality Problem and a Projected Dynamical System

The relationship between a projected dynamical system and its associated variational inequality problem with the same constraint set is then highlighted. For completeness, we also recall the fundamental properties of existence and uniqueness of the solution to the ordinary differential equation (ODE) that defines such a projected dynamical system.

Projected Dynamical Systems

Let $K \subset R^n$ be closed and convex. Denote the boundary and interior of K, respectively, by ∂K and K^0. Given $x \in \partial K$, define the set of inward normals to K at x by

$$N(x) = \{\gamma : \|\gamma\| = 1, \text{ and } \langle \gamma, x - y \rangle \leq 0, \forall y \in K\}. \quad (2)$$

We define $N(x)$ to be $\{\gamma : \|\gamma\| = 1\}$ for x in the interior of K.

Projected Dynamical Systems

When K is a convex polyhedron (for example, when K consists of linear constraints), K takes the form $\cap_{i=1}^{Z} K_i$, where each K_i is a closed half-space with inward normal N_i. Let P_K be the norm projection. Then P_K projects onto K "along N," in that if $y \in K$, then $P(y) = y$, and if $y \notin K$, then $P(y) \in \partial K$, and $P(y) - y = \alpha\gamma$ for some $\alpha > 0$ and $\gamma \in N(P(y))$.

Projected Dynamical Systems

Definition 2

Given $x \in K$ and $v \in R^n$, define the projection of the vector v at x (with respect to K) by

$$\Pi_K(x, v) = \lim_{\delta \to 0} \frac{(P_K(x + \delta v) - x)}{\delta}. \tag{3}$$

The class of ordinary differential equations that are of interest here take the following form:

$$\dot{x} = \Pi_K(x, -F(x)), \tag{4}$$

where K is a closed convex set, corresponding to the constraint set in a particular application, and $F(x)$ is a vector field defined on K.

Projected Dynamical Systems

Note that a classical dynamical system, in contrast, is of the form

$$\dot{x} = -F(x). \tag{5}$$

We have the following results (cf. [6]):
(i). If $x \in K^0$, then

$$\Pi_K(x, -F(x)) = -F(x). \tag{6}$$

(ii). If $x \in \partial K$, then

$$\Pi_K(x, -F(x)) = -F(x) + \beta(x)N^*(x), \tag{7}$$

where

$$N^*(x) = \arg \max_{N \in N(x)} \langle (-F(x)), -N \rangle, \tag{8}$$

and

$$\beta(x) = \max\{0, \ \ \langle (-F(x)), -N^*(x) \rangle \}. \tag{9}$$

Projected Dynamical Systems

Note that since the right-hand side of the ordinary differential equation is associated with a projection operator, it is discontinuous on the boundary of K. Therefore, one needs to explicitly state what one means by a solution to an ODE with a discontinuous right-hand side.

Projected Dynamical Systems

Definition 3

We say that the function $x : [0, \infty) \mapsto K$ *is a solution to the equation* $\dot{x} = \Pi_K(x, -F(x))$ *if* $x(\cdot)$ *is absolutely continuous and* $\dot{x}(t) = \Pi_K(x(t), -F(x(t)))$*, save on a set of Lebesgue measure zero.*

In order to distinguish between the pertinent ODEs from the classical ODEs with continuous right-hand sides, we refer to the above as ODE(F, K).

Definition 4 (An Initial Value Problem)
For any $x_0 \in K$ as an initial value, we associate with $\mathrm{ODE}(F, K)$ *an initial value problem,* $\mathrm{IVP}(F, K, x_0)$*, defined as:*

$$\dot{x} = \Pi_K(x, -F(x)), \quad x(0) = x_0. \tag{10}$$

Note that if there is a solution $\phi_{x_0}(t)$ to the initial value problem IVP(F, K, x_0), with $\phi_{x_0}(0) = x_0 \in K$, then $\phi_{x_0}(t)$ always stays in the constraint set K for $t \geq 0$.

Projected Dynamical Systems

We now present the definition of a projected dynamical system, governed by such an ODE(F, K), which, correspondingly, will be denoted by PDS(F, K).

Definition 5 (The Projected Dynamical System)
Define the projected dynamical system $\mathrm{PDS}\,(F, K)$ *as the map* $\Phi : K \times R \mapsto K$ *where*

$$\Phi(x, t) \;=\; \phi_x(t) \tag{11}$$

solves the $\mathrm{IVP}(F, K, x)$*, that is,*

$$\dot{\phi}_x(t) = \Pi_K(\phi_x(t), -F(\phi_x(t))), \quad \phi_x(0) = x. \tag{12}$$

Projected Dynamical Systems

The behavior of the dynamical system is now described. One may refer to Figure 1 for an illustration of this behavior. If $x(t) \in K^0$, then the evolution of the solution is directly given in terms of $F : \dot{x} = -F(x)$. However, if the vector field $-F$ drives x to ∂K (that is, for some t one has $x(t) \in \partial K$ and $-F(x(t))$ points "out" of K) the right-hand side of the ODE becomes the projection of $-F$ onto ∂K. The solution to the ODE then evolves along a "section" of ∂K, e. g., ∂K_i for some i. At a later time the solution may re-enter K^0, or it may enter a lower dimensional part of ∂K, e.g., $\partial K_i \cap \partial K_j$. Depending on the particular vector field F, it may then evolve within the set $\partial K_i \cap \partial K_j$, re-enter ∂K_i, enter ∂K_j, etc.

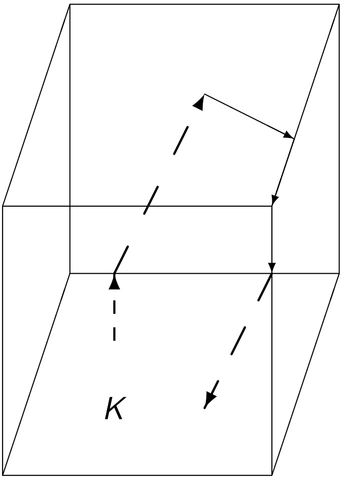

Figure: A trajectory of a projected dynamical system that evolves both on the interior and on the boundary of the constraint set K

Projected Dynamical Systems

We now define a stationary or an equilibrium point.

Definition 6 (A Stationary Point or an Equilibrium Point)
The vector $x^* \in K$ *is a stationary point or an equilibrium point of the projected dynamical system* $\mathrm{PDS}(F, K)$ *if*

$$0 = \Pi_K(x^*, -F(x^*)). \tag{13}$$

In other words, we say that x^* is a stationary point or an equilibrium point if, once the projected dynamical system is at x^*, it will remain at x^* for all future times.

Projected Dynamical Systems

From the definition it is apparent that x^* is an equilibrium point of the projected dynamical system PDS(F, K) if the vector field F vanishes at x^*. The contrary, however, is only true when x^* is an interior point of the constraint set K. Indeed, when x^* lies on the boundary of K, we may have $F(x^*) \neq 0$.

Projected Dynamical Systems

Note that for classical dynamical systems, the necessary and sufficient condition for an equilibrium point is that the vector field vanish at that point, that is, that $0 = -F(x^*)$.

Projected Dynamical Systems

The following theorem states a basic connection between the static world of finite-dimensional variational inequality problems and the dynamic world of projected dynamical systems.

Theorem 1 (Dupuis and Nagurney (1993))
Assume that K is a convex polyhedron. Then the equilibrium points of the $\mathrm{PDS}(F, K)$ *coincide with the solutions of* $\mathrm{VI}(F, K)$. *Hence, for* $x^* \in K$ *and satisfying*

$$0 = \Pi_K(x^*, -F(x^*)) \tag{14}$$

also satisfies

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \tag{15}$$

Projected Dynamical Systems

This Theorem establishes the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of a variational inequality problem.

Moreover, it provides a natural underlying dynamics (out of equilibrium) of such systems.

Before stating the fundamental theorem about projected dynamical systems, we introduce the following assumption needed for the theorem.

Projected Dynamical Systems

Assumption 1 (Linear Growth Condition) *There exists a* $B < \infty$ *such that the vector field* $-F : R^n \mapsto R^n$ *satisfies the linear growth condition:* $\|F(x)\| \leq B(1 + \|x\|)$ *for* $x \in K$*, and also*

$$\langle(-F(x) + F(y)), x - y\rangle \leq B\|x - y\|^2, \quad \forall x, y \in K. \quad (16)$$

Theorem 2 (Existence, Uniqueness, and Continuous Dependence) *Assume that the linear growth condition holds. Then*
(i). *For any* $x_0 \in K$*, there exists a unique solution* $x_0(t)$ *to the initial value problem;*
(ii). *If* $x_k \rightarrow x_0$ *as* $k \rightarrow \infty$*, then* $x_k(t)$ *converges to* $x_0(t)$ *uniformly on every compact set of* $[0, \infty)$*.*

Projected Dynamical Systems

The second statement of this Theorem 2 is sometimes called the *continuous dependence* of the solution path to ODE(F, K) on the initial value. By virtue of the Theorem, the PDS(F, K) is well-defined and inhabits K whenever the Assumption holds.

Projected Dynamical Systems

Lipschitz continuity is a condition that plays an important role in the study of variational inequality problems. It also is a critical concept in the classical study of dynamical systems.

Definition 7 (Lipschitz Continuity)

$F : K \mapsto R^n$ is locally Lipschitz continuous if for every $x \in K$ there is a neighborhood $\eta(x)$ and a positive number $L(x) > 0$ such that

$$\|F(x') - F(x'')\| \leq L(x)\|x' - x''\|, \quad \forall x', x'' \in \eta(x). \tag{17}$$

When this condition holds uniformly on K for some constant $L > 0$, that is,

$$\|F(x') - F(x'')\| \leq L\|x' - x''\|, \quad \forall x', x'' \in K, \tag{18}$$

then F is said to be Lipschitz continuous on K.

Lipschitz continuity implies the Assumption and is, therefore, a sufficient condition for the fundamental properties of projected dynamical systems stated in the Theorem.
We now present an example.

An Example (A Tatonnement or Adjustment Process)
Consider the market equilibrium model in which there are n commodities. We denote the price of commodity i by p_i, and group the prices into the n-dimensional column vector p. The supply of commodity i is denoted by $s_i(p)$, and the demand for commodity i is denoted by $d_i(p)$. We are interested in determining the equilibrium pattern that satisfies the following equilibrium conditions:

Market Equilibrium Conditions

For each commodity i; $i = 1, \ldots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} = 0, & \text{if } p_i^* > 0 \\ \geq 0, & \text{if } p_i^* = 0. \end{cases} \quad (19)$$

For this problem we propose the following adjustment or tatonnement process: For each commodity i; $i = 1, \ldots, n$:

$$\dot{p}_i = \begin{cases} d_i(p) - s_i(p), & \text{if } p_i > 0 \\ \max\{0, d_i(p) - s_i(p)\}, & \text{if } p_i = 0. \end{cases} \quad (20)$$

Projected Dynamical Systems

In other words, a price of an instrument will increase if the demand for that instrument exceeds the supply of that instrument; the price will decrease if the demand for that instrument is less than the supply for that instrument. However, if the price of an instrument is equal to zero, and the supply of that instrument exceeds the demand, then the price will not change since one cannot have negative prices according to equilibrium conditions.

Projected Dynamical Systems

In vector form, we may express the above as

$$\dot{p} = \Pi_K(p, d(p) - s(p)), \tag{21}$$

where $K = R^n_+$, $s(p)$ is the n-dimensional column vector of supply functions, and $d(p)$ is the n-dimensional column vector of demand functions. Note that this adjustment process can be put into the standard form of a PDS, if we define the column vectors: $x \equiv p$ and $F(x) \equiv s(p) - d(p)$.

Projected Dynamical Systems

On the other hand, if we do not constrain the instrument prices to be nonnegative, then $K = R^n$, and the above tatonnement process would take the form:

$$\dot{p} = d(p) - s(p). \tag{22}$$

This would then be an example of a classical dynamical system.

In the context of the Example, we have then that, according to the Theorem, the stationary point of prices, p^*, that is, those prices that satisfy

$$0 = \Pi_K(p^*, d(p^*) - s(p^*)) \tag{23}$$

also satisfy the variational inequality problem

$$\langle (s(p^*) - d(p^*)), p - p^* \rangle \geq 0, \quad \forall p \in K. \tag{24}$$

Projected Dynamical Systems

Hence, there is a natural underlying dynamics for the prices, and the equilibrium point satisfies the variational inequality problem; equivalently, is a stationary point of the projected dynamical system.

PDS General Iterative Scheme

We describe the general iterative scheme of Dupuis and Nagurney (1993) designed to estimate stationary points of the projected dynamical system; equivalently, to determine solutions to the variational inequality problem (1).

The algorithms induced by what we term the PDS general iterative scheme can be interpreted as discrete time approximations to the continuous time model given by.

PDS General Iterative Scheme

The PDS general iterative scheme for obtaining a solution to (2.1) takes the form:

PDS General Iterative Scheme

Step 0: Initialization:

Start with an $x^0 \in K$. Set $k := 0$.

Step 1: Computation:

Compute x^{k+1} by solving the variational inequality problem:

$$x^{k+1} = P_K(x^k - a_k F_k(x^k)), \tag{25}$$

where $\{a_k; k = 1, 2, \ldots\}$ is a sequence of positive scalars and the sequence of vector fields $\{F_k(\cdot); k = 1, 2, \ldots\}$ are "approximations" to $F(\cdot)$.

PDS General Iterative Scheme

Step 2: Convergence Verification

If $|x^{k+1} - x^k| \leq \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; otherwise, set $k := k + 1$, and go to Step 1.

PDS General Iterative Scheme

We first give the precise conditions for the convergence theorem and a general discussion of the conditions. Subsequently, several examples of the functions $\{F_k(\cdot); k = 1, 2, \ldots\}$ are given.

The following notation is needed for the statement of Assumption 2.

For each $x \in R^n$, let the set-valued function $\bar{F}(x)$ be defined as

$$\bar{F}(x) = \bigcap_{\epsilon > 0} \text{cov}\left(\overline{\{F(y) : \|x - y\| \leq \epsilon\}}\right)$$

where the overline indicates the closure and $\text{cov}(A)$ denotes the convex hull of the set A. Then $\bar{F}(x)$ is convex and upper semicontinuous, particularly, $\bar{F}(x) = F(x)$, when F is continuous at x.

PDS General Iterative Scheme

For any $z \in R^n, A \subset R^n$, let

$$d(z, A) := \inf_{y \in A} \|z - y\|$$

denote the distance between z and A. Then

$$d(z, A) = \|z - P_A(z)\|,$$

when A is closed and convex.

The conditions for the convergence theorem are now stated.

PDS General Iterative Scheme

Assumption 2

Suppose we fix an initial condition $x^0 \in K$ *and define the sequence* $\{x^k; k = 1, 2, \ldots\}$ *by (2.69). We assume the following conditions.*

(i). $\sum_{k=0}^{\infty} a_k = \infty, \quad a_k > 0, \quad a_k \rightarrow 0, \quad as \quad k \rightarrow \infty.$

(ii). $d(F_k(x), \bar{F}(x)) \rightarrow 0$ *uniformly on compact subsets of* K *as* $k \rightarrow \infty$*.*

PDS General Iterative Scheme

(iii). *Define* $y(\cdot)$ *to be the unique solution to* $\dot{x} = \Pi_K(x, -F(x))$ *that satisfies* $y(0) = y \in K$. *The* $w-$*limit set*

$$w(K) = \cup_{y \in K} \cap_{t \geq 0} \overline{\cup_{s \geq t} \{y(s)\}}$$

is contained in the set of stationary points of $\dot{x} = \Pi_K(x, -F(x))$.

(iv). *The sequence* $\{x^k; k = 1, 2, \ldots\}$ *is bounded.*

(v). *The solutions to* $\dot{x} = \Pi_K(x, -F(x))$ *are stable in the sense that given any compact set* K_1 *there exists a compact set* K_2 *such that* $\cup_{y \in K \cap K_1} \cup_{t \geq 0} \{y(t)\} \subset K_2$.

Examples

We now give examples for the vector field $F_k(x)$. The most obvious example is $F_k(x) = F(x)$ for $k = 1, 2, \ldots$ and $x \in K$.

This would correspond to the basic **Euler scheme** in the numerical approximation of classical ordinary differential equations. Another example is a Heun-type scheme given by

$$F_k(x) = \frac{1}{2}\left[F(x) + F(x + P_K(x - a_k F(x)))\right].$$

Convergence

Theorem 3 (Convergence of PDS General Iterative Scheme)
Let S denote the solutions to the variational inequality (1), and assume Assumption 1 and Assumption 2. Suppose $\{x^k; k = 1, 2, \ldots\}$ is the scheme generated by (25). Then $d(x^k, S) \to 0$ as $k \to \infty$.

Corollary 1
Assume the conditions of Theorem 3, and also that S consists of a finite set of points. Then $\lim_{k\to\infty} x^k$ exists and equals a solution to the variational inequality.

Summary

We have overviewed the fundamentals of projected dynamical systems theory with a focus on the relationships with variational inequality theory.

We also provided the general iterative scheme that induces algorithms such as the Euler method and the Heum method.

PDSs have been to-date use to solve problems from evolutionary games to neuroscience to dynamic predator-prey networks to supply chains. Moreover, all the models described in this seminar that were formulated as variational inequality problems have also been extended using projected dynamical systems theory.

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