Topic 2: Game Theory

Professor Anna Nagurney

John F. Smith Memorial Professor
Director – Virtual Center for Supernetworks
Isenberg School of Management
University of Massachusetts Amherst

SCH-MGMT 825 Management Science Seminar
Advances in Variational Inequalities, Game Theory, and Applications, Spring 2017

Game Theory

Game theory is a very relevant and powerful theory that enables one to capture competition or cooperation among decision-makers.

In this lecture, we will focus on noncooperative games but will turn to cooperation in the form of Nash Bargaining, when we address cybersecurity investment application later in this seminar.

Game theory has had wide application and impact in operations research and management science, in economics, as well as in engineering, and even in such fields as political science.

Nash Equilibrium

Nash (1950, 1951) conceptualized and defined an equilibrium for a behavioral model consisting of n agents or players, each acting in his/her own self-interest, which has come to be called a noncooperative game.

The Nobel Laureate John F. Nash

www.search.tvnz.co.nz

Nash Equilibrium

To construct a noncooperative game, one must identify **the players** in the game, **their strategies** (that is what they can control), **the constraints** their strategies must satisfy, as well as **the utility function** associated with each player (decision-maker) which he/she seeks to maximize.

Some examples can include firms seeking their optimal production quantities and competing for consumers at demand markets; disaster relief organizations competing for financial funds, and even blood service organizations competing for blood donations.

The noncooperative game theory application landscape is rich and new applications keep on being discovered.

Game Theory and Nash Equilibrium

Specifically, consider m players, each player i having at his/her disposal a strategy vector $X_i = \{X_{i1}, \ldots, X_{in}\}$ selected from a closed, convex set $K_i \subset R^n$, with a utility function $U_i : K \mapsto R^1$, where $K = K_1 \times K_2 \times \ldots \times K_m \subset R^{mn}$. K_i is the feasible set of constraints faced by player i.

The rationality postulate is that each player i selects a strategy vector $X_i \in K_i$ that maximizes his/her utility level $U_i(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_m)$ given the decisions $(X_j)_{j \neq i}$ of the other players.

Nash Equilibrium

In this framework one then has:

Definition (Nash Equilibrium)
A Nash equilibrium is a strategy vector

$$X^* = (X_1^*, \ldots, X_m^*) \in K,$$

such that

$$U_i(X_i^*, \hat{X}_i^*) \geq U_i(X_i, \hat{X}_i^*), \quad \forall X_i \in K_i, \forall i, \tag{1}$$

where $\hat{X}_i^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_m^*)$.

Game Theory and Nash Equilibrium

It is important to emphasize that, in the case of a Nash equilibrium, the utility function of each player depends not only on its own strategies but also on those of the other players.

However, the feasible set K_i of each player i; $i = 1, \ldots, m$, depends only on his/her own strategies, and not on those of the other players.

We will see later in this seminar that there are also applications in which the feasible sets of the players are interdependent. This will then yield a **Generalized Nash Equilibrium**.

Variational Inequality Formulation of Nash Equilibrium

It has been shown (cf. Hartman and Stampacchia (1966) and Gabay and Moulin (1980)) that Nash equilibria satisfy variational inequalities. In the present context, under the assumption that each U_i is continuously differentiable on K and concave with respect to X_i, one has

Theorem (Variational Inequality Formulation of Nash Equilibrium)

Under the previous assumptions, X^* is a Nash equilibrium if and only if $X^* \in K$ is a solution of the variational inequality

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K, \tag{2}$$

where $F(X) \equiv (-\nabla_{X_1} u_1(X), \ldots, -\nabla_{X_m} U_m(X))$ is a row vector and where $\nabla_{X_i} U_i(X) = (\frac{\partial U_i(X)}{\partial X_{i1}}, \ldots, \frac{\partial U_i(X)}{\partial X_{in}})$.

Variational Inequality Formulation of Nash Equilibrium

Proof: Since U_i is a continuously differentiable function and concave with respect to X_i, the equilibrium condition (1), for a fixed i, is equivalent to the variational inequality problem

$$-\langle \nabla_{X_i} U_i(X^*), X_i - X_i^* \rangle \geq 0, \quad \forall X_i \in K_i, \tag{3}$$

which, by summing over all players i, yields (2). □

Variational Inequality Formulation of Nash Equilibrium

Note that the variational inequality in (2), in expanded form is, simply, determine $X^* \in K$, such that

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial U_i(X^*)}{\partial X_{ij}} \times (X_{ij} - X_{ij}^*) \geq 0, \quad \forall X \in K.$$

Qualitative Properties

If the feasible set K is compact, then existence is guaranteed under the assumption that each U_i is continuously differentiable. Rosen (1965) proved existence under similar conditions. Karamardian (1969), on the other hand, relaxed the assumption of compactness of K and provided a proof of existence and uniqueness of Nash equilibria under the strong monotonicity condition.

As shown by Gabay and Moulin (1980), the imposition of a coercivity condition on $F(X)$ will guarantee existence of a Nash equilibrium X^* even if the feasible set is no longer compact. Moreover, if $F(X)$ satisfies the strict monotonicity condition, uniqueness of X^* is guaranteed, provided that the equilibrium exists.

Nash Equilibrium and Oligopolies

Classical examples of noncooperative games under the Nash equilibrium are oligopolies.

Oligopoly theory dates to Cournot (1838), who investigated competition between two producers, the so-called duopoly problem, and is credited with being the first to study noncooperative behavior.

In an oligopoly, it is assumed that there are several firms, which produce a product and the price of the product depends on the quantity produced.

Examples of oligopolies include large firms in computer, automobile, chemical or mineral extraction industries, airlines, as well as certain supernarkets, among others.

Classical Oligopoly Problems

We now consider the classical oligopoly problem in which there are m producers involved in the production of a homogeneous commodity. The quantity produced by firm i is denoted by q_i, with the production quantities grouped into a column vector $q \in R^m$. Let f_i denote the cost of producing the commodity by firm i, and let p denote the demand price associated with the good. Assume that

$$f_i = f_i(q_i), \tag{4}$$

$$p = p(\sum_{i=1}^{m} q_i). \tag{5}$$

The profit for firm i, U_i, can then be expressed as

$$U_i(q) = p(\sum_{i=1}^{m} q_i)q_i - f_i(q_i). \tag{6}$$

Variational Inequality Formulation of Nash Equilibrium

Theorem (Variational Inequality Formulation of Classical Cournot-Nash Oligopolistic Market Equilibrium)

Assume that the profit function $U_i(q)$ is concave with respect to q_i, and that $U_i(q)$ is continuously differentiable. Then $q^* \in R^m_+$ is a Nash equilibrium if and only if it satisfies the variational inequality

$$\sum_{i=1}^{m} \left[\frac{\partial f_i(q_i^*)}{\partial q_i} - \frac{\partial p(\sum_{i=1}^{m} q_i^*)}{\partial q_i} q_i^* - p(\sum_{i=1}^{m} q_i^*) \right] \times [q_i - q_i^*] \geq 0,$$

$$\forall q \in R^m_+. \quad (7)$$

A Famous Example

The oligopoly consists of five firms, each with a production cost function of the form

$$f_i(q_i) = c_i q_i + \frac{\beta_i}{(\beta_i + 1)} K_i^{-\frac{1}{\beta_i}} q_i^{\frac{(\beta_i+1)}{\beta_i}}, \tag{20}$$

with the parameters given in the Table. The demand price function is given by

$$p(\sum_{i=1}^{5} q_i) = 5000^{\frac{1}{1.1}} (\sum_{i=1}^{5} q_i)^{-\frac{1}{1.1}}.$$

A Famous Example

Table: Parameters for the five-firm oligopoly example

Firm i	c_i	K_i	β_i
1	10	5	1.2
2	8	5	1.1
3	6	5	1.0
4	4	5	.9
5	2	5	.8

Computation of the Nash Equilibrium

This problem has been solved by several different variational inequality algorithms, including the relaxation method and the projection method (cf. Nagurney (1999)), induced by the general iterative scheme of Dafermos (1983), for example.

Both these methods converged to the equilibrium production pattern q^*=(36.93,41.81,43.70,42.65,39.17).

Computation of Nash Equilibrium

There are many algorithms that have been applied to compute solutions to Nash equilibrium problems, from those based on the general iterative scheme of Dafermos (1983), to those induced by the general iterative scheme of Dupuis and Nagurney (1993), which exploit the connection between projected dynamical systems and variational inequalities.

In this seminar course, we will detail algorithms when we discuss specific applications more fully.

References Cited and Additional Ones

Cournot, A. A., **Researches into the Mathematical Principles of the Theory of Wealth**, 1838, English translation, MacMillan, London, England, 1897.

Dafermos, S., "An iterative scheme for variational inequalities," *Mathematical Programming* **26** (1983) 40-47.

Dafermos, S., and Nagurney, A., "Oligopolistic and competitive behavior of spatially separated markets," *Regional Science and Urban Economics* **17** (1987) 245-254.

Dupuis, P., and Nagurney, A., "Dynamical systems and variational inequalities," *Annals of Operations Research* **44(1)** (1993) 7-42.

Friedman, J., **Oligopoly and the Theory of Games**, North-Holland, Amsterdam, The Netherlands, 1977.

References Cited and Additional Ones

Gabay, D., and Moulin, H., "On the uniqueness and stability of Nash-equilibria in noncooperative games," in **Applied Stochastic Control in Econometrics and Management Science**, pp. 271-294, A. Bensoussan, P. Kleindorfer, and C. S. Tapiero, editors, North-Holland, Amsterdam, The Netherlands, 1980.

Gabzewicz, J., and Vial, J. P., "Oligopoly 'a la Cournot' in a general equilibrium analysis," *Journal of Economic Theory* **14** (1972) 381-400.

Harker, P. T., "A variational inequality approach fror the determination of oligopolistic market equilibrium," *Mathematical Programming* **30** (1984) 105-111.

Harker, P. T., "Alternative models of spatial competition," *Operations Research* **34** (1986) 410-425.

References Cited and Additional Ones

Hartman, P., and Stampacchia, G., “On some nonlinear elliptic differential functional equations,” *Acta Mathematica* **115** (1966) 271-310.

Haurie, A., and Marcotte, P., “On the relationship between Nash-Cournot and Wardrop equilibria,” *Networks* **15** (1985) 295-308.

Karamardian, S., “Nonlinear complementarity problem with applications, ” *Journal of Optimization Theory and Applications* **4** (1969), Part I, 87-98, Part II, 167-181.

Manas, M., “A linear oligopoly game,” *Econometrica* **40** (1972) 917-922.

Murphy, F.H., Sherali, H. D., and Soyster, A. L., “A mathematical programming approach for determining oligopolistic market equilibrium,” *Mathematical Programming* **24** (1982) 92-106.

References Cited and Additional Ones

Nagurney, A., “Algorithms for oligopolistic market equilibrium problems,” *Regional Science and Urban Economics* **18** (1988) 425-445.

Nagurney, A., **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

Nagurney, A., **Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits**, Edward Elgar Publishing, Cheltenham, England, 2006.

Nash, J. F., “Equilibrium points in n-person games,” *Proceedings of the National Academy of Sciences, USA* **36** (1950) 48-49.

Nash, J. F., “Noncooperative games,” *Annals of Mathematics* **54** (1951) 286-298.

References

Novshek, W., "Cournot equilibrium with free entry," *Review of Economic Studies* **47** (1980) 473-486.

Novshek, W., and Sonnenschein, H., "Walrasian equilibria as limits of noncooperative equilibria, Part II, Pure strategies," *Journal of Economic Theory* **30** (1983) 171-187.

Okuguchi, K., **Expectations and Stability in Oligopoly Models**, *Lecture Notes in Economics and Mathematical Systems* **138**, Springer-Verlag, Berlin, Germany, 1976.

Okuguchi, K., and Szidarovsky, F., **The Theory of Oligopoly with Multi - Product Firms**, *Lecture Notes in Economics and Mathematical Systems* **342**, Springer-Verlag, Berlin, Germany, 1990.

Rosen, J. B., "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica* **33** (1965) 520-533.

References Cited and Additional Ones

Spence, M., "The implicit maximization of a function in monopolistically competitive markets," Harvard Institute of Economic Research, Harvard University Discussion Paper 461 (1976).

Szidarovsky, F., and Yakowitz, S., "A new proof of the existence and uniqueness of the Cournot equilibrium, " *International Economic Review* **18** (1977) 787-789.

Tobin, R. L., "Sensitivity analysis for a Cournot equilibrium," *Operations Research Letters* **9** (1990) 345-351.