Topic 1: Variational Inequality Theory Fundamentals

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Background

In this seminar, we will be focusing on some recent advances in variational inequalities and game theory with a focus on novel applications ranging from disaster relief to blood supply chains to competition for shared resources in supply chains as well as perishable product supply chains and even cybersecurity.

The goal of this seminar is to illuminate some recent techniques that allow us to model, analyze, and solve a plethora of very timely problems today.

Students may have different background and interests but working together we will push the research forward.

Background

Of special relevance in this seminar this year will be a variety of game theory problems, beginning with noncooperative games under the Nash equilibrium to Generalized Nash equilibrium problems, in which not only does the utility of a player/decision-maker depend on the strategies of the other players, but their respective feasible sets do as well, and cooperative game theory in the form of Nash bargainaing games.

A big topic will be to construct general, relevant mathematical models that are grounded in reality and that also are computable.

Background

The results on variational inequality foundations are provided without proof. For proofs, please see the book by Nagurney (1999): *Network Economics: A Variational Inequality Approach*.

In addition, the underlying dynamics associated with such problems, additional qualitative analysis, as well as algorithms, can be found in the book by Nagurney and Zhang (1996): *Projected Dynamical Systems and Variational Inequalities with Applications*.

Background

When one considers decision-making, one, typically, has a decision-maker in mind, his/her objective function to be optimized, the decision variables, as well as the constraints that make up the feasible set. Such problems are cast as **optimization problems**.

On the other hand, if one has multiple decision-makers, who interact with one another, whether noncooperatively or cooperatively, one is then dealing with **game theory problems**.

In game theory problems, as well as numerous related problems, the generalization of an optimal solution is an **equilibrium solution**.

Background

Setting up the mathematical model is **both an art and a science** since one must decide on what essential information one wants to capture; the level of complexity and generality, and also make sure that data can be obtained, and that the model can actually be formulated appropriately and solved!

In order to formulate and solve numerous game theory and other problems, we use the powerful methodology of **variational inequality theory**.

Background

We emphasize that equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering.

Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included:

- systems of equations,
- optimization theory,
- complementarity theory, and
- fixed point theory.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems.

Background

Variational inequality theory was introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequalities were *infinite-dimensional* rather than *finite-dimensional* as we will be studying here.

The breakthrough in finite-dimensional theory occurred in 1980 when Dafermos recognized that the traffic network equilibrium conditions as stated by Smith (1979) had a structure of a variational inequality.

This unveiled this methodology for the study of problems in economics, management science/operations research, and also in engineering, with a focus on transportation.

Background

To-date problems which have been formulated and studied as variational inequality problems include:

- traffic network equilibrium problems
- spatial price equilibrium problems
- oligopolistic market equilibrium problems
- financial equilibrium problems
- migration equilibrium problems, as well as
- environmental network and ecology problems,
- knowledge network problems,
- electric power generation and distribution networks,
- supply chain network equilibrium problems, and even
- the Internet!

Observe that many of the applications explored to-date that have been formulated, studied, and solved as variational inequality problems are, in fact, network problems.

In addition, as we shall see, many of the advances in variational inequality theory have been spurred by needs in practice!

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• qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and

• providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

Variational Inequality Theory

Also, as shown by Dupuis and Nagurney (1993), there is associated with a variational inequality problem, a *projected dynamical system*, which provides a natural underlying dynamics until an equilibrium state is achieved, under appropriate conditions.

This result further enriches the scope and reach of variational inequalities in terms of theory and especially applications!

The Variational Inequality Problem

Definition 1 (Variational Inequality Problem)
The finite - dimensional variational inequality problem, $\mathrm{VI}(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K} \subset R^N$*, such that*

$$F(X^*)^T \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{K},$$

or, equivalently,

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K} \tag{1}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in N-dimensional Euclidean space, as does "*$\cdot$*".*

Here we assume that all vectors are column vectors, except where noted.

The Variational Inequality Problem

Another equivalent way of writing (1) is:

$$\sum_{i=1}^{N} F_i(X^*) \times (X_i - X_i^*) \geq 0, \quad \forall X \in \mathcal{K}. \tag{2}$$

$\mathcal{K}$ is the feasible set, X^* is the vector of solution values of the variables, and F is sometimes referred to as the function that enters the variational inequality.

The Variational Inequality Problem

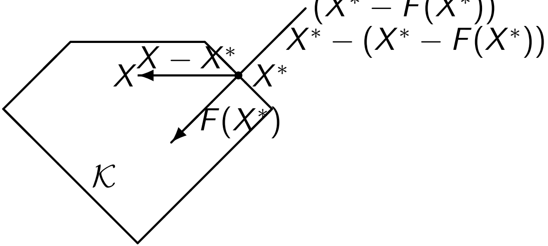

Figure: Geometric Depiction of the Variational Inequality Problem

In geometric terms, the variational inequality (1) states that $F(X^*)^T$ is "orthogonal" to the feasible set $\mathcal{K}$ at the point X^*.

Optimization Problems

An optimization problem is characterized by its specific objective function that is to be maximized or minimized, depending upon the problem and, in the case of a constrained problem, a given set of constraints.

Possible objective functions include expressions representing profits, costs, market share, portfolio risk, etc. Possible constraints include those that represent limited budgets or resources, nonnegativity constraints on the variables, conservation equations, etc. Typically, an optimization problem consists of a single objective function.

Optimization Problems

Both **unconstrained** and **constrained** optimization problems can be formulated as variational inequality problems. The subsequent two propositions and theorem identify the relationship between an optimization problem and a variational inequality problem.

Proposition

Let X^* be a solution to the optimization problem:

$$\text{Minimize} \quad f(X) \tag{3}$$

$$\text{subject to:} \quad X \in \mathcal{K},$$

where f is continuously differentiable and $\mathcal{K}$ is closed and convex. Then X^* is a solution of the variational inequality problem:

$$\nabla f(X^*)^T \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{K}. \tag{4}$$

Optimization Problems

Proposition

If $f(X)$ *is a convex function and* X^* *is a solution to* $\mathrm{VI}(\nabla f, \mathcal{K})$*, then* X^* *is a solution to the optimization problem (3).*

Optimization Problems

If the feasible set $\mathcal{K} = R^N$, then the unconstrained optimization problem is also a variational inequality problem.

Relationship Between Optimization Problems and Variational Inequalities

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem.

In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then fix this relationship in a theorem.

Relationship Between Optimization Problems and Variational Inequalities

Theorem

Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix} \quad (5)$$

is symmetric and positive-semidefinite.

Then there is a real-valued convex function $f : \mathcal{K} \mapsto R^1$ *satisfying*

$$\nabla f(X) = F(X)$$

with X^* *the solution of* $\mathrm{VI}(F, \mathcal{K})$ *also being the solution of the mathematical programming problem:*

$$\text{Minimize} \quad f(X) \tag{6}$$

$$\text{subject to:} \quad X \in \mathcal{K}.$$

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, only when the symmetry condition and the positive-semidefiniteness condition hold.

The variational inequality, therefore, is the more general problem in that it can also handle a function $F(X)$ with an asymmetric Jacobian.

Consequently, variational inequality theory allows for the modeling, analysis, and solution of multimodal traffic network equilibrium problems, multicommodity spatial price equilibrium problems, general economic equilibrium problems, and numerous competitive supply chain network equilibrium problems since one no longer has to make a "symmetry" assumption of $F(X)$.

Fixed Point Problems

Fixed point theory has been used to formulate, analyze, and compute solutions to economic equilibrium problems. The relationship between the variational inequality problem and a fixed point problem can be made through the use of a projection operator. First, the projection operator is defined.

Lemma

Let $\mathcal{K}$ be a closed convex set in R^n. Then for each $x \in R^N$, there is a unique point $y \in \mathcal{K}$, such that

$$\|x - y\| \leq \|x - z\|, \quad \forall z \in K,$$

and y is known as the orthogonal projection of X on the set $\mathcal{K}$ with respect to the Euclidean norm, that is,

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \|X - z\|.$$

Projection Operator

A property of the projection operator which is useful both in qualitative analysis of equilibria and their computation is now presented.

Corollary

Let $\mathcal{K}$ be a closed convex set. Then the projection operator $P_{\mathcal{K}}$ is nonexpansive, that is,

$$\|P_{\mathcal{K}}X - P_{K}X'\| \leq \|X - X'\|, \quad \forall X, X' \in R^N.$$

Geometric Interpretation of Projection

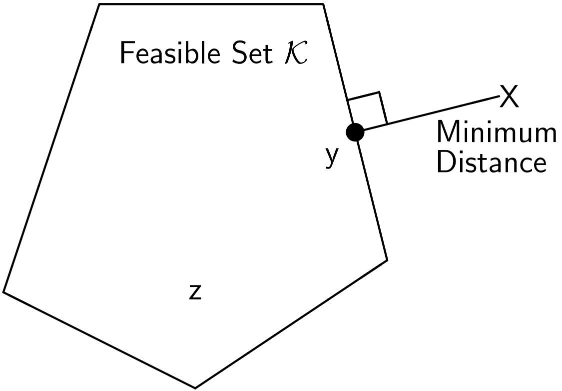

Figure: The projection y of X on the set $\mathcal{K}$

Additional Geometric Interpretation

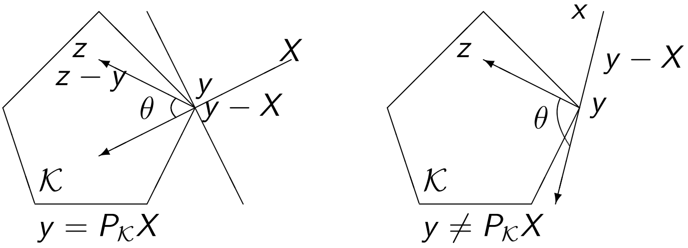

Figure: Geometric interpretation of $\langle (y - X), z - y \rangle \geq 0$, for $y = P_{\mathcal{K}}X$ and $y \neq P_{\mathcal{K}}X$

Relationship Between Fixed Point Problems and Variational Inequalities

The relationship between a variational inequality and a fixed point problem is as follows.

Theorem

Assume that $\mathcal{K}$ *is closed and convex. Then* $X^* \in \mathcal{K}$ *is a solution of the variational inequality problem* $\mathrm{VI}(F, \mathcal{K})$ *if and only if for any* $\gamma > 0$, X^* *is a fixed point of the map*

$$P_{\mathcal{K}}(I - \gamma F) : \mathcal{K} \mapsto \mathcal{K},$$

that is,

$$X^* = P_{\mathcal{K}}(X^* - \gamma F(X^*)).$$

Basic Existence and Uniqueness Results

Variational inequality theory is also a powerful tool in the qualitative analysis of equilibria. We now provide conditions for existence and uniqueness of solutions to $\mathrm{VI}(F, \mathcal{K})$ are provided.

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set $\mathcal{K}$ is compact. Indeed, we have the following:

Theorem (Existence Under Compactness and Continuity)
If $\mathcal{K}$ is a compact convex set and $F(X)$ is continuous on $\mathcal{K}$, then the variational inequality problem admits at least one solution X^*.

Basic Existence and Uniqueness Results

Let VI_R denote the variational inequality problem:
Determine $x_R^* \in K_R$, such that

$$F(X_R^*)^T \cdot (y - X_R^*) \geq 0, \quad \forall y \in K_R.$$

Theorem

$\mathrm{VI}(F, \mathcal{K})$ *admits a solution if and only if there exists an* $R > 0$ *and a solution of* VI_R, x_R^*, *such that* $\|x_R^*\| < R$.

Although $\|x_R^*\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.

Basic Existence and Uniqueness Results

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition (Monotonicity)
$F(X)$ is monotone on $\mathcal{K}$ if

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition (Strict Monotonicity)
$F(X)$ is strictly monotone on $\mathcal{K}$ if

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) > 0, \quad \forall X^1, X^2 \in \mathcal{K},\ X^1 \neq X^2.$$

Basic Existence and Uniqueness Results

Definition (Strong Monotonicity)
$F(X)$ is strongly monotone on $\mathcal{K}$ if for some $\alpha > 0$

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) \geq \alpha \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition (Lipschitz Continuity)
$F(X)$ is Lipschitz continuous on K if there exists an $L > 0$, such that

$$\|F(X^1) - F(X^2)\| \leq L\|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Basic Existence and Uniqueness Results

A uniqueness result is presented in the subsequent theorem.

Theorem (Uniqueness)

Suppose that $F(X)$ is strictly monotone on $\mathcal{K}$. Then the solution is unique, if one exists.

More on Monotonicity

Monotonicity is closely related to positive-definiteness.

Theorem
Suppose that $F(X)$ is continuously differentiable on $\mathcal{K}$ and the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix},$$

which need not be symmetric, is positive-semidefinite (positive-definite). Then $F(X)$ is monotone (strictly monotone).

More on Monotonicity

Proposition

Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that $\nabla F(X)$ is strongly positive-definite. Then $F(X)$ is strongly monotone.

One obtains a stronger result in the special case where $F(X)$ is linear.

Corollary

Suppose that $F(X) = MX + b$, where M is an $N \times N$ matrix and b is a constant vector in R^N. The function F is monotone if and only if M is positive-semidefinite. F is strongly monotone if and only if M is positive-definite.

Proposition

Assume that $F : \mathcal{K} \mapsto R^N$ is continuously differentiable at $\bar{X}$. Then $F(X)$ is locally strictly (strongly) monotone at $\bar{X}$ if $\nabla F(\bar{X})$ is positive-definite (strongly positive-definite), that is,

$$v^T F(\bar{X}) v > 0, \quad \forall v \in R^N, v \neq 0,$$

$$v^T \nabla F(\bar{X}) v \geq \alpha \|v\|^2, \quad \text{for some} \quad \alpha > 0, \quad \forall v \in R^N.$$

A Strong Result

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set $\mathcal{K}$ is made.

Theorem (Existence and Uniqueness)
Assume that $F(X)$ is strongly monotone. Then there exists precisely one solution X^* to $\mathrm{VI}(F, \mathcal{K})$.

Hence, in the case of an unbounded feasible set $\mathcal{K}$, strong monotonicity of the function F guarantees both existence and uniqueness. If $\mathcal{K}$ is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

A Contraction

Assume now that $F(X)$ is both strongly monotone and Lipschitz continuous. Then the projection $P_{\mathcal{K}}[X - \gamma F(X)]$ is a contraction with respect to X, that is, we have the following:

Theorem

Fix $0 < \gamma \leq \frac{\alpha}{L^2}$ where α and L are the constants appearing, respectively, in the strong monotonicity and the Lipschitz continuity condition definitions. Then

$$\|P_{\mathcal{K}}(X - \gamma F(X)) - P_{\mathcal{K}}(y - \gamma F(y))\| \leq \beta\|X - y\| \tag{31}$$

for all $X, y \in \mathcal{K}$, where

$$(1 - \gamma\alpha)^{\frac{1}{2}} \leq \beta < 1.$$

An immediate consequence of the Theorem and the Banach Fixed Point Theorem is:

Corollary

The operator $P_{\mathcal{K}}(X - \gamma F(X))$ *has a unique fixed point* X^*.

Summary

In this lecture, the fundamental qualitative tools for the formulation and analysis of finite-dimensional variational inequalities have been provided.

In subsequent lectures, we will describe algorithms and a plethora of applications of variational inequalities and game theory models.

Many of the applications will be network-based.

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Below are the citations referenced in the lecture as well as other relevant ones.

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