OIM 413 Logistics and Transportation Lecture 6: Equilibration Algorithms for a General Network

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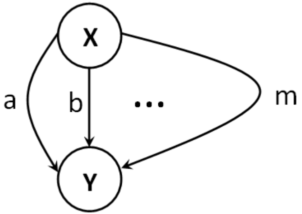
Equilibration Algorithms for a General Transportation Network

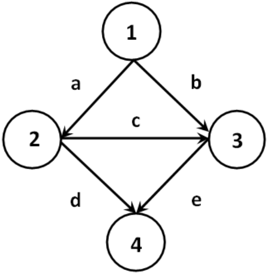
Assume that the user cost functions on the links are still linear and increasing, that is,

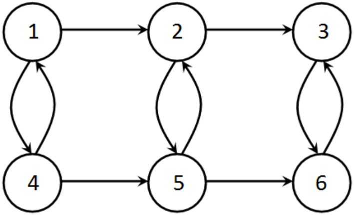
$$c_a(f_a) = g_a f_a + h_a, \qquad g_a, h_a > 0$$

but now the transportation network need no longer look like this:



The network can have any topology:





Some Real-World Transportation Network Examples

The Massachusetts Turnpike

Photo by: Central Artery / Tunnel Project [Massachusetts Turnpike Authority]

Some Real-World Transportation Network Examples

The Los Angeles I110-I105 Interchange

Some Real-World Transportation Network Examples

The Atlanta I85-I285 Highway

We will now present **General Equilibration Algorithms** (GEAs) for the determination of either the U-O or the S-O flow pattern on a transportation network. These algorithms can be applied to compute the flows for networks with *any* topology.

We assume that the user link travel costs are given by:

$$c_a(f_a) = g_a f_a + h_a, \quad \forall a \in L.$$

The total link travel costs are then:

$$\hat{c}_a(f_a) = g_a f_a^2 + h_a, \quad \forall a \in L.$$

These algorithms are iterative algorithms and are guaranteed to converge to the unique link flow pattern. They were developed by Dafermos and Sparrow (1969), where convergence was also established, and have an appealing intuitive interpretation as dynamic adjustment processes.

A General Equilibration Algorithm (U-O)

1. Initialization

Initialize with a feasible path flow and link flow pattern. Set $i = 1$.

2. Equilibration of O/D pair w_i:

Set
r = path p with $F_p > 0$ and maximum C_p in w_i,
q = path p with minimum C_p in w_i.

Compute the reallocation of path flow

$$\Delta' = \frac{C_r(f) - C_q(f)}{\sum_{a \in L} g_a(\delta_{aq} - \delta_{ar})^2}.$$

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The reallocation: $\Delta = \min\{\Delta', F_r\}$.

Update: new $F_q = F_q + \Delta$,

new $F_r = F_r - \Delta$,

all other F_ps stay the same.

Continue Step 2 until O/D pair w_i satisfies the U-O conditions within a prespecified tolerance.

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3. Go to next O/D pair:

Set $i = i + 1$; If $i \leq J$, where $J = \#$ of O/D pairs, goto Step 2; else goto Step 4.

4. Overall Convergence Verification

If all O/D pairs are equilibrated (within the prespecified convergence tolerance) then stop; the U-O pattern has been computed; else, set $i = 1$, and goto Step 2.

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A General Equilibration Algorithm (S-O)

1. Initialization

Initialize with a feasible path flow and link flow pattern. Set $i = 1$.

2. Equilibration of O/D pair w_i:

Set

$r =$ path p with $F_p > 0$ and maximum $\hat{C}'_p$ in w_i,

$q =$ path p with minimum $\hat{C}'_p$ in w_i.

Compute the reallocation of path flow

$$\Delta' = \frac{\hat{C}'_r(f) - \hat{C}'_q(f)}{2 \sum_{a \in L} g_a (\delta_{aq} - \delta_{ar})^2}.$$

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The reallocation: $\Delta = \min\{\Delta', F_r\}$.

Update: new $F_q = F_q + \Delta$,

new $F_r = F_r - \Delta$,

all other F_p's stay the same.

Continue Step 2 until O/D pair w_i satisfies the S-O conditions within a prespecified tolerance.

Equilibration Algorithms for a General Transportation Network

3. Go to next O/D pair:

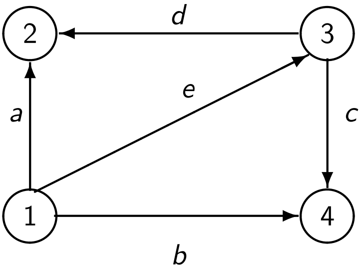
Set $i = i + 1$; If $i \leq J$, where $J = \#$ of O/D pairs, goto Step 2; else goto Step 4.

4. Overall Convergence Verification

If all O/D pairs are equilibrated (within the prespecified tolerance) then stop; the S-O pattern has been computed; else, set $i = 1$, and goto Step 2.

A Numerical Example

We now present an example for which we computed both the U-O and the S-O flow patterns using the respective general equilibration algorithm just given. The network topology is given below.



The user link cost functions are:

$$c_a(f_a) = 6f_a + 1,\ c_b(f_b) = f_b + 4,\ c_c(f_c) = 2f_c + 3,\ c_d(f_d) = 3f_d + 1,$$

$$c_e(f_e) = 2f_e + 1.$$

The U-O Solution

The O/D pairs are: $w_1 = (1,2)$ and $w_2 = (1,4)$, with demands $d_{w_1} = 40$ and $d_{w_2} = 80$.
The paths are: $p_1 = a$, $p_2 = (e,d)$, $p_3 = b$, and $p_4 = (e,c)$.
Both the U-O and the S-O general equilibration algorithms (GEAs) were implemented in software.
The computed U-O solution obtained via the U-O GEA: the equilibrium path flows are:

$$F^*_{p_1} = 19.7059, \; F^*_{p_2} = 20.2941, \; F^*_{p_3} = 72.1176, \; F^*_{p_4} = 7.8824,$$

and the equilibrium link flows:

$$f^*_a = 19.7059, \; f^*_b = 72.1176, \; f^*_c = 7.8824, \; f^*_d = 20.2941, \; f^*_e = 28.1765.$$

The user path costs for paths connecting the first O/D pair are: $C_{p_1} = C_{p_2} = 119.2353$. The user path costs for paths connecting the second O/D pair are: $C_{p_3} = C_{p_4} = 76.1176$.

The S-O Solution

The S-O solution, in turn, computed via the S-O GEA is: the optimal path flows are:

$$F_{p_1} = 19.6569, \; F_{p_2} = 20.3431, \; F_{p_3} = 72.1373, \; F_{p_4} = 7.8627,$$

and the optimal link flows are:

$$f_a = 19.6569, \; f_b = 72.1373, \; f_c = 7.8627, \; f_d = 20.3431, \; f_e = 28.2059.$$

The marginal total path costs evaluated at the S-O solution are as follows: For O/D pair w_1:

$$\hat{C}'_{p_1} = \hat{C}'_{p_2} = 236.8824;$$

for O/D pair w_2:

$$\hat{C}'_{p_3} = \hat{C}'_{p_4} = 148.2745.$$

Additional Reading

- S. C. Dafermos and F. T. Sparrow (1969) The Traffic Assignment Problem for a General Network. *Journal of Research of the National Bureau of Standards*, 73B, pp. 91-118.

- A. Nagurney (1990) Equilibrium Modeling, Analysis and Computation: the Contributions of Stella Dafermos. *Operations Research*, 39(1), pp. 9-12.

- A. Nagurney (2005) Supernetworks; Invited Chapter for the *Handbook of Optimization in Telecommunications*, P. M. Pardalos and M. G. C. Resende, Editors, Springer-Verlag.

- A. Nagurney (2010) Optimal Supply Chain Network Design and Redesign at Minimal Total Cost and with Demand Satisfaction. *International Journal of Production Economics*, 128, pp. 200-208.