OIM 413 Logistics and Transportation
Lecture 4: The User-Optimized (U-O) Problem

Professor Anna Nagurney

John F. Smith Memorial Professor
and
Director – Virtual Center for Supernetworks
Department of Operations & Information Management

UMassAmherst | Isenberg School of Management

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There are two fundamental principles of flow (travel) behavior, due to Wardrop (1952), with terms coined by Dafermos and Sparrow (1969).

User-optimized (U-O) Problem – each user determines his/her cost minimizing route of travel between an origin/destination, until an equilibrium is reached, in which no user can decrease his/her cost of travel by unilateral action.

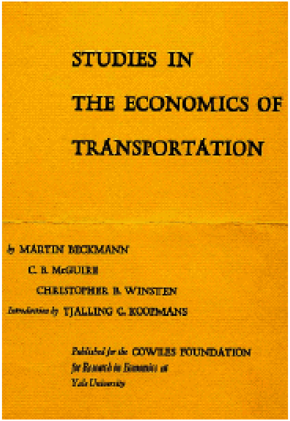
System-optimized (S-O) Problem – users are allocated among the routes so as to minimize the total cost in the system, where the total cost is equal to the sum over all the links of the link's user cost times its flow.

Decision-makers select their cost-minimizing routes.

		User-Optimized or Equilibrium
Decentralized	Selfish	U-O
vs. ⟺	vs. ⟺	vs.
Centralized	Unselfish	S-O
		System-Optimized

Flows are routed so as to minimize the total cost to society.

Decentralized (Selfish) versus Centralized (Unselfish) Behavior



In 1956, Yale University Press published *Studies in the Economics of Transportation* **by Beckmann, McGuire, and Winsten.**

We celebrated the 50th anniversary of its publication at the 2005 INFORMS Meeting, San Francisco. (Professor Nagurney with Professors Beckmann, McGuire and Boyce)

For more photos, see http://supernet.isenberg.umass.edu/bmw-informs-2005/bmw50.html

The User-Optimized (U-O) Problem

The U-O problem is also known as the Traffic Assignment or the Traffic Network Equilibrium Problem (TNEP).

We first consider the U-O Problem with Fixed Demands.

Problem Statement

Given the travel demands d_w associated with every O/D pair w and the user's travel cost $c_a(f_a)$ for every link a, determine the path flows F_p for all paths, and the link flows f_a for all links such that the network is U-O.

The Feasibility Conditions (Constraints)

We know that path and link flows must be feasible, i.e.

$$d_w = \sum_{p \in P_w} F_p, \quad \forall w \in W,$$

that is, the demands must be satisfied.

Also, $F_p \geq 0$, for all paths p.

A link flow pattern $f = f_a \; ; \; a \in L$, that is induced by a feasible flow pattern F via the equation:

$$f_a = \sum_{p \in P} F_p \delta_{ap},$$

for all $a \in L$ is called feasible.

A Non-Cooperative Game (Nash Equilibrium)

John F. Nash

www.search.tvnz.co.nz

A rational user will make a unilateral decision to change his path, if, assuming that all other users are fixed, he can decrease his travel cost by changing his path. If no user has any incentive to change paths, the traffic assignment (flow pattern) is said to be:

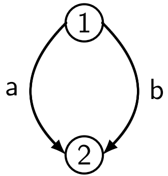
USER-OPTIMIZED

or in

EQUILIBRIUM

Before giving a mathematical definition of an equilibrium state, we will look at some examples.

A Network Example



$w_1 = (1, 2)$, $d_{w_1} = 30$, $c_a(f_a) = 3f_a + 30$, and $c_b(f_b) = 2f_b + 20$.

The paths are: $p_1 = a$ and $p_2 = b$.

Claim: The user-optimized traffic pattern is $f_a^* = F_{p_1}^* = 10$, $f_b^* = F_{p_2}^* = 20$. Note that here links and paths, and, hence, link flows and path flows coincide.

With the computed travel costs along paths p_1, p_2:

$$c_a = C_{p_1} = 60$$
$$c_b = C_{p_2} = 60.$$

$*$ **Observe that the costs on the paths are equal**.

Assume that the flow $\delta > 0$ switches from p_1 to p_2. Then, the new user cost along path p_2 (or link b) would be:

$$c_b(f_b') = 2\ (20 + \delta) + 20$$
$$= 60 + 2\delta > c_b(f_b) = 60.$$

So not cost-wise to have a switch from path p_2 to p_1.

Suppose we also want to evaluate a switch from p_2 to p_1. Then, the new user cost along path p_1 (or link a) would be:

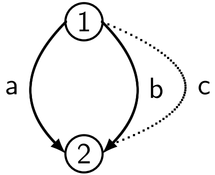
$$c_a(f_a') = 3\ (10 + \delta) + 30$$

$$= 60 + 3\delta > c_b(f_b) = 60.$$

Also, not cost-wise!

$*$ **Hence, the pattern (10,20) satisfies the definition and it is user-optimized.**

Suppose that we add another link, c, to the network:



with user link travel cost: $c_c(f_c) = f_c + 80$.

Is the original flow pattern still U-O? Why or why not?

Definition of U-O (Equilibrium)

We will now define when f, the vector of link flows, is a U-O pattern.

To do this, we must compare f with other link flow patterns f' which we will construct as follows:

Consider any path r such that $F_r > 0$ (a used path) and choose a number δ, such that $0 \leq \delta \leq F_r$.

Also, consider a path q that connects the same O/D pair of nodes as r and construct a new path flow pattern F' from F as follows:

The new flow pattern F':

$$F'_r = F_r - \delta$$

$$F'_q = F_q + \delta$$

$F'_p = F_p$, for all other paths $p \neq q \cup r$.

Observe that F' is feasible, that is, it satisfies the constraints.

Definition

We say that a U-O path flow pattern F has been reached if, for every pair of paths q, r with $F_r > 0$, (and w fixed), and for every possible $\delta > 0$, the following inequalities hold:

$C_r(f) \leq C_q(f')$ true for $q, r \in P_w$ and for all w.

COMMENT

This is a difficult test. Moreover, we **can't compute** the equilibrium via this method.

Necessary and sufficient conditions for a User-Optimized (U-O) flow pattern

A feasible path flow pattern F^* and its induced link flow pattern f^* is U-O, if and only if, for every O/D pair w there exists a numbering of the paths $p_1, ..., p_s, p_{s+1}, ..., p_{n_w}$ that connect w such that:

$$\underline{C_{p_1}(f^*) = C_{p_2}(f^*) = ... = C_{p_s}(f^*) = \lambda_w \leq C_{p_s+1}(f^*) \leq ... C_{p_{n_w}}(f^*)}$$

and

$$F^*_{p_r} > 0, \text{ for } p_r;\ r = 1, ..., s.$$

$$\underline{F^*_{p_r} = 0, \text{ for } p_r;\ r = s+1, ..., n_w.}$$

Hence, under User-Optimization (U-O), all used paths, that is those with positive flow, for a given O/D pair, have equal and minimal travel costs!

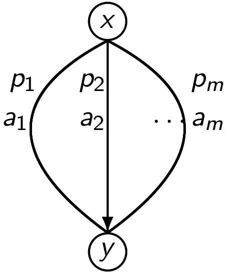
This is precisely Wardrop's first principle of travel behavior.

We can utilize the U-O conditions to develop algorithms to determine the path (and link) flows.

Efficient algorithms are very important for solving large-scale transportation networks.

Derivation of the U-O Exact Equilibration Algorithm (U-O EEA)

Computation of the U-O flow pattern for the following **special** network.



- O/D pair $w = (x, y)$
- m-disjoint paths
- fixed travel demand d_w
- travel cost on each link/path is linear and a function of its own flow: $c_{a_i}(f_{a_i}) = g_{a_i} f_{a_i} + h_{a_i}$, $C_{p_i}(F_{p_i}) = c_{a_i}(f_{a_i})$; $i = 1, \ldots, m$.

Suppose that we know the critical s. Then the U-O conditions take the form:

$$g_{a_1} f^*_{a_1} + h_{a_1} = g_{a_2} f^*_{a_2} + h_{a_2} = \ldots = g_{a_s} f^*_{a_s} + h_{a_s} = \lambda_w$$
$$\leq g_{a_{s+1}} f^*_{a_{s+1}} + h_{a_{s+1}} \leq \ldots \leq g_{a_m} f^*_{a_m} + h_{a_m} \qquad \textbf{(I)}$$

$$f^*_{a_r} = F^*_{p_r} > 0; \;\; r = 1, \ldots, s,$$
$$f^*_{a_r} = F^*_{p_r} = 0; \;\; r = s+1, \ldots, m,$$

Observe that (I) can be rewritten as:

$$\lambda_w = \frac{f^*_{a_1} + \frac{h_{a_1}}{g_{a_1}}}{\frac{1}{g_{a_1}}} = \ldots = \frac{f^*_{a_s} + \frac{h_{a_s}}{g_{a_s}}}{\frac{1}{g_{a_s}}} \leq \frac{f^*_{a_{s+1}} + \frac{h_{a_{s+1}}}{g_{a_{s+1}}}}{\frac{1}{g_{a_{s+1}}}} \leq \ldots \leq \frac{f^*_{a_m} + \frac{h_{a_m}}{g_{a_m}}}{\frac{1}{g_{a_m}}}.$$

A fact from linear algebra:

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{\sum_i x_i}{\sum_i y_i}.$$

By using this fact from linear algebra, we obtain:

$$\lambda_w = \frac{\sum_{i=1}^{s} f^*_{a_i} + \sum_{i=1}^{s} \frac{h_{a_i}}{g_{a_i}}}{\sum_{i=1}^{s} \frac{1}{g_{a_i}}} = \frac{d_w + \sum_{i=1}^{s} \frac{h_{a_i}}{g_{a_i}}}{\sum_{i=1}^{s} \frac{1}{g_{a_i}}}.$$

How do we compute the critical s?

Suppose that we have ordered/sorted the h_{a_i}s from smallest to largest and we knew the critical s, then:

$$f^*_{a_r} = \frac{\lambda_w - h_{a_r}}{g_{a_r}}, \;\; r = 1, ..., s;$$

$$f^*_{a_r} = 0, \;\; r = s+1, ..., m.$$

We know that

$$\frac{\lambda_w - h_{a_r}}{g_{a_r}} > 0; \;\; r = 1, ..., s,$$

$\lambda_w > h_{a_r} \Rightarrow \lambda_w > h_{a_r} > h_{a_s}$

$\lambda_w - h_{a_r} \leq 0$, for $r = s+1, ..., m \Rightarrow \lambda_w \leq h_{a_{s+1}}$.

Hence,

$$(*)\ h_{a_1} \leq h_{a_2} \leq \ldots \leq h_{a_{s-1}} \leq h_{a_s} \leq \lambda_w \leq h_{a_{s+1}} \leq \ldots \leq h_{a_m}.$$

Since

$$\lambda_w = \frac{d_w + \displaystyle\sum_{i=1}^{s} \frac{h_{a_i}}{g_{a_i}}}{\displaystyle\sum_{i=1}^{s} \frac{1}{g_{a_i}}},$$

we can define

$$\lambda_w^r = \frac{d_w + \displaystyle\sum_{i=1}^{r} \frac{h_{a_i}}{g_{a_i}}}{\displaystyle\sum_{i=1}^{r} \frac{1}{g_{a_i}}}.$$

The Exact Equilibration Algorithm (U-O)

Step 1: Sort the h_{a_i}s in non-decreasing order and relabel accordingly.

Set iteration count $r = 1$ and $h_{a_{m+i}} = \infty$.

Step 2: Compute

$$\lambda_w^r = \frac{d_w + \sum_{i=1}^{r} \frac{h_{a_i}}{g_{a_i}}}{\sum_{i=1}^{r} \frac{1}{g_{a_i}}}.$$

The Exact Equilibration Algorithm (U-O) (continued)

Step 3: Check

If

$$h_{a_r} < \lambda_w^r \leq h_{a_{r+1}},$$

then STOP.

Set the critical $s = r$;

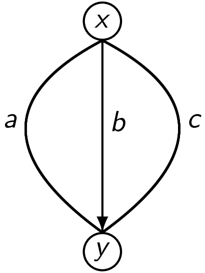
$$F_{p_r}^* = f_{a_r}^* = \frac{\lambda_w^r - h_{a_r}}{g_{a_r}}; \quad r = 1, \ldots, s;$$
$$F_{p_r}^* = f_{a_r}^* = 0; \qquad r = s+1, \ldots, m.$$

Else set $r = r + 1$ and goto Step 2.

The U-O EEA is guaranteed to converge to the exact U-O solution in a finite number of steps.

A Numerical Example Solved Using U-O EEA

Consider the following transportation network:



with a single O/D pair (x, y) and with travel demand $d_{xy} = 15$.
The user link cost functions are given by:

$$c_a(f_a) = f_a + 30, \qquad c_b(f_b) = f_b + 15, \qquad c_c(f_c) = f_c + 20.$$

What is the U-O link flow pattern?

The Solution Using the U-O EEA

Sorting the h_a terms, we have that
$h_{a_1} = h_b = 15 \leq h_{a_2} = h_c = 20 \leq h_{a_3} = h_a = 30.$

Iteration 1: (r=1) $\lambda_w^1 = \frac{d_w + \frac{h_{a_1}}{g_{a_1}}}{\frac{1}{g_{a_1}}} = \frac{15+\frac{15}{1}}{\frac{1}{1}} = 30.$

We need to check if $h_{a_1} < \lambda_w^1 \leq h_{a_2}$, or if $15 < 30 \leq 20$? The answer is NO. Set $r = r + 1 = 2$, and go to the second iteration:

Iteration 2: (r=2) $\lambda_w^2 = \frac{d_w + \frac{h_{a_1}}{g_{a_1}} + \frac{h_{a_2}}{g_{a_2}}}{\frac{1}{g_{a_1}} + \frac{1}{g_{a_2}}} = \frac{15+\frac{15}{1}+\frac{20}{1}}{\frac{1}{1}+\frac{1}{1}} = 25.$

Now, we need to check if $h_{a_2} < \lambda_w^2 \leq h_{a_3}$, or if $20 < 25 \leq 30$?

The answer is YES. Stop, and set the critical s=2. This means that only two of the three links/paths - the two with lowest h_as - will be used; therefore, $f_a^* = 0$.

For the two used links, that is, those with positive flow, we can set the user cost on paths equal to $\lambda_w^2 = 25$, and find the U-O link flows:

$$c_b = f_b^* + 15 = 25 \Rightarrow f_b^* = 25 - 15 = 10,$$

and

$$c_c = f_c^* + 20 = 25 \Rightarrow f_c^* = 25 - 20 = 5.$$

.

As a result, the U-O link flow pattern for this problem is:

$$f_a^* = 0, \quad f_b^* = 10, \quad f_c^* = 5.$$

Let's verify whether this flow pattern satisfies the U-O conditions.

Note that $C_{p_2} = c_b = 25$ and $C_{p_3} = c_c = 25$; also, $C_{p_1} = c_a = 30$, and the U-O conditions are satisfied, since all used paths, that is paths p_2 and p_3, have equal and minimal travel costs.

References

⇒ S. C. Dafermos and F. T. Sparrow (1969) The Traffic Assignment Problem for a General Network. *Journal of Research of the National Bureau of Standards*, 73B, pp. 91-118

⇒ D. E. Boyce, H. S. Mahmassani, and A. Nagurney (2005) A Retrospective on Beckmann, McGuire and Winsten's Studies in the Economics of Transportation. *Papers in Regional Science*, 84, pp. 85-103

http://supernet.isenberg.umass.edu/articles/jrs.pdf

For podcasts and videos on transportation topics and supply chains, including interviews with Professor Nagurney, and videos of Professor Martin Beckmann and David E. Boyce, see: http://supernet.isenberg.umass.edu/audiovideo.htm