OIM 413 Logistics and Transportation Lecture 12: The Spatial Price Equilbrium Problem

Professor Anna Nagurney

John F. Smith Memorial Professor
and
Director – Virtual Center for Supernetworks
Department of Operations & Information Management

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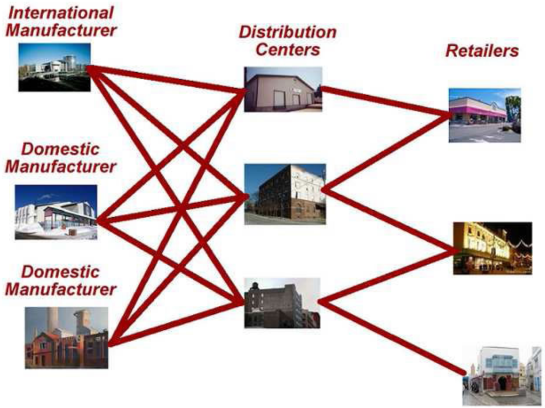
The Spatial Price Equilibrium Problem

Parallel to the study of the traffic network equilibrium problem (TNEP), researchers have studied the spatial price equilibrium problem (SPEP).

This problem dates to Samuelson (1952), and Takayama and Judge (1971) and is considered the **basic framework** for a variety of applications in energy, agriculture, and interregional / international trade.

New applications have also been proposed in **finance**.

Spatial price equilibrium problems are precursors to what we call today *supply chain network problems*, since they capture production, transportation, as well as consumption.



The Spatial Price Equilibrium Problem

The problem has a structure similar to the TNEP with elastic demands.

In fact, it has been shown that, given any SPE problem, one can construct a “TNEP,” which is “isomorphic” and, hence, any SPE problem can be solved as a TNEP.

Hence, what we have learned in the ocntext of transportation can be applied to spatial economic problems.

The Samuelson, Takayama and Judge Model

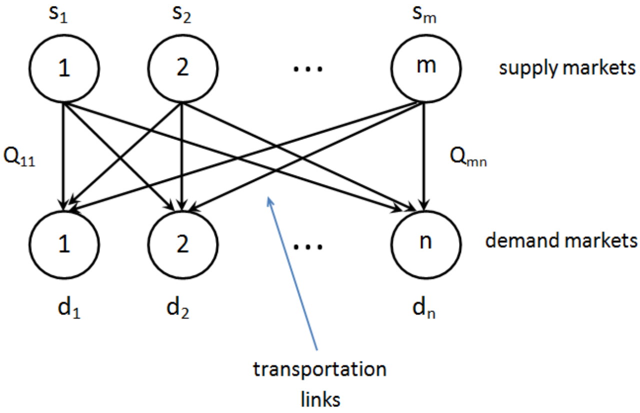
There are m supply markets and n demand markets involved in the production of a homogeneous commodity.

Let s_i denote the supply of the commodity at supply market i; $i = 1, \dots, m$.

Let d_j denote the demand for the commodity at demand market j; $j = 1, \dots, n$.

Let Q_{ij} denote the commodity shipment from i to j, for all supply and demand market pairs (i, j).

The basic network for the SPEP is the bipartite graph where the nodes represent the spatial locations of the supply and demand markets:



The Spatial Price Equilibrium Condition Statement

A vector of supplies, commodity shipments, and demands (s^*, Q^*, d^*) is said to constitute a spatial price equilibrium, if the demand price of the commodity at a demand market is equal to the supply price at the supply market plus the unit transportation cost **and** there is trade between this pair of supply and demand markets.

If there is no trade between a pair of markets, then the supply price plus transportation cost is greater than or equal to the demand price.

The Spatial Price Functions

Associated with each supply market i; $i = 1, \ldots, m$, is a supply price function:

$$\pi_i = \pi_i(s_i),$$

which is assumed to be increasing.

The Demand Price Functions

Associated with each demand market j; $j = 1, \ldots, n$, is a demand price function:

$$\rho_j = \rho_j(d_j),$$

which is assumed to be decreasing.

The Unit Transportation Cost Functions

Associated with each pair of supply and demand markets (i, j) there is a unit cost of shipping/transportation

$$c_{ij} = c_{ij}(Q_{ij}),$$

which is assumed to be increasing.

The Spatial Price Equilibrium Conditions

Mathematically, this equilibrium state is represented as follows:

For each pair (i, j):

$$\pi_i(s_i^*) + c_{ij}(Q_{ij}^*) \begin{cases} = \rho_j(d_j^*), & \text{if } \; Q_{ij}^* > 0 \\ \geq \rho_j(d_j), & \text{if } \; Q_{ij}^* = 0. \end{cases}$$

subject to the feasibility (conservation of flow) equations:

$s_i = \sum_{j=1}^{n} Q_{ij}$, for all supply markets i,

$d_j = \sum_{i=1}^{m} Q_{ij}$, for all demand markets j,

$Q_{ij} \geq 0$, for all pairs (i, j).

The Optimization Reformulation of the Classical Spatial Price Equilibrium Problem

These equilibrium conditions have an **equivalent optimization** formulation given by:

$$\text{Minimize} \sum_{i=1}^{m} \int_0^{s_i} \pi_i(x)dx + \sum_{ij} \int_0^{Q_{ij}} c_{ij}(y)dy - \sum_{j=1}^{n} \int_0^{d_j} \rho_j(z)dz$$

subject to the previous feasibility constraints.

The Spatial Price Equilibrium Problem

Indeed, the **Kuhn-Tucker** conditions for this optimization problem are equivalent to the spatial equilibrium conditions.

This objective function has been interpreted as a "Social Welfare Function" by many economists.

In the case where the Jacobians of the supply price, demand price, and transportation cost functions

$$[\frac{\partial \pi_i}{\partial s_j}], \quad [\frac{\partial \rho_j}{\partial d_i}], \quad [\frac{\partial c_{ij}}{\partial Q_{kl}}], \forall i, j,$$

are symmetric, one also has an optimization reformulation of the SPE conditions.

An SPEP Example: Agriculture

A Basic SPEP Numerical Example

Three agricultural firms transport their product to a single demand market. Their supply price functions are as follows:

$$\pi_1(s_1) = s_1 + 1, \quad \pi_2(s_2) = s_2 + 1, \quad \text{and} \quad \pi_3(s_3) = s_3 + 1.$$

The transportation cost functions are:

$c_{11}(Q_{11}) = 5Q_{11}+2$, $c_{21}(Q_{21}) = 5Q_{21}+2$, and $c_{31}(Q_{31}) = 5Q_{31}+2$.

Also, the demand price function is:

$$\rho_1(d_1) = -2d_1 + 195.$$

Solution

We have from the conservation of flow equations that:

$$d_1 = Q_{11} + Q_{21} + Q_{31}.$$

Due to the "symmetry" in the supply price functions and the transportation cost functions, we can conclude that:

$$d_1^* = 3Q_{11}^*.$$

The equilibrium condition for the first supply/demand market pair can be written as, since $Q_{11}^* > 0$:

$$\pi_1(s_1^*) + c_{11}(Q_{11}^*) = \rho_1(d_1^*),$$

or,

$$s_1^* + 1 + 5Q_{11}^* + 2 = -6Q_{11}^* + 195.$$

Solution

On the other hand, we also know from the conservation of flow equations that $s_1^* = Q_{11}^*$; thus, the equilibrium condition equation becomes:

$$12Q_{11}^* = 192.$$

So the solution to this SPE problem is:

$$s_1^* = s_2^* = s_3^* = 16,$$

$$Q_{11}^* = Q_{21}^* = Q_{31}^* = 16,$$

and

$$d_1^* = 3 \times 16 = 48.$$

Another Numerical Example

A fuel company is involved with the distribution of fuel to two demand markets. The supply price function corresponding to this company is:

$$\pi_1(s_1) = 4s_1 + 3.$$

The transportation cost functions are:

$$c_{11}(Q) = 3Q_{11} + Q_{12} + 5,$$

$$c_{12}(Q) = 4Q_{12} + 1.5Q_{11} + 7.$$

Also, the demand price functions at demand markets 1 and 2 are:

$$\rho_1(d) = -3d_1 - 2d_2 + 21,$$

$$\rho_2(d) = -5d_2 - 3d_1 + 30.$$

Note that, in this example, the price and cost functions are no longer separable, that is, the price at a supply market does not depend only on the quantity produced at the supply market; the same for the unit transportation cost functions, and the demand price functions.

Moreover, since the Jacobian matrices of the supply price, demand price, and transportation cost functions are no longer symmetric, there is not an optimization reformulation of the spatial price equilibrium conditions.

Hence, we must utilize more advanced theory, such as the theory of variational inequalities (cf. Nagurney (1999)).

However, this problem is small enough that we can explicitly determine the solution.

SPE Example: Fuel

Solution

Assuming that the spatial pricing equilibrium conditions hold, and that the equilibrium commodity shipments are both positive, we have the following equations corresponding to the two supply/demand market pairs:

$$\begin{cases} \pi_1(s^*) + c_{11}(Q^*) = \rho_1(d^*), \quad \text{and} \\ \pi_2(s^*) + c_{12}(Q^*) = \rho_2(d^*). \end{cases}$$

Or, equivalently:

$$\begin{cases} 4s_1^* + 3 + 3Q_{11}^* + Q_{12}^* + 5 & = -3d_1^* - 2d_2^* + 21, \quad \text{and} \\ 4s_1^* + 3 + 4Q_{12}^* + 1.5Q_{11}^* + 7 & = -5d_2^* - 3d_1^* + 30. \end{cases}$$

On the other hand, the conservation of flow equations yield:

$$\begin{cases} s_1 & = Q_{11} + Q_{12}, \\ d_1 & = Q_{11}, \quad \text{and} \\ d_2 & = Q_{12}. \end{cases}$$

Substituting the above conservation of flow equations into the equilibrium conditions and, then, solving the system of equations, we have:

$$Q_{11}^* = 0.41, \qquad Q_{12}^* = 1.27,$$

hence:

$$s_1^* = Q_{11}^* + Q_{12}^* = 1.68,$$

$$d_1^* = 0.41, \quad \text{and} \quad d_2^* = 1.27.$$

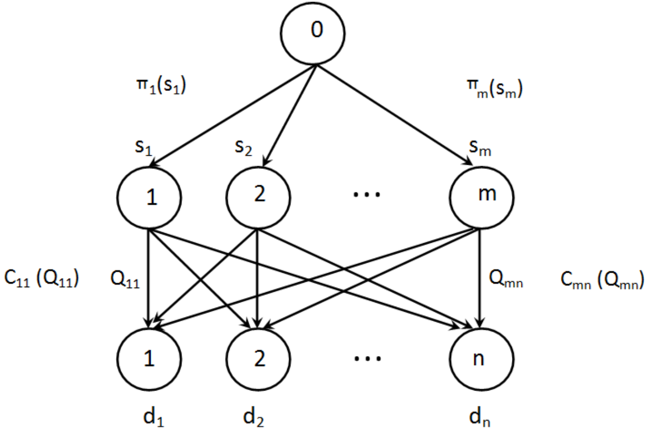
Relationship of the SPEP with the TNEP

The spatial price equilibrium problem can be reformulated as a transportation network equilibrium problem over an appropriately constructed abstract network or supernetwork.

By making such a transformation, we can then apply the methodological tools developed for transportation networks to the formulation, analysis, and computation of solutions to spatial price equilibrium problems.

We will now establish how this is done (due to Dafermos and Nagurney; see the book by Nagurney (1999) for more details).

Construction of the Isomorphic Traffic Network Equilibrium Problem



Construction of the Isomorphic Traffic Network Equilibrium Problem

The O/D pairs are:

$$w_1 = (0, 1), \ldots, w_n = (0, n).$$

The travel disutilities are:

$$\lambda_{w_1} = \rho_1(d_1), \ldots, \lambda_{w_n} = \rho_n(d_n).$$

The flow on a path $p_{ij} = Q_{ij}$.

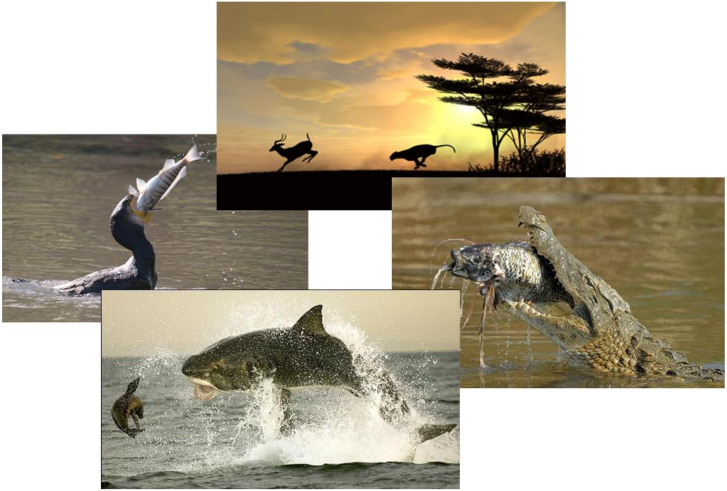
Extensions of the Classical Spatial Price Equilibrium Model

There have been numerous extensions developed, including multicommodity ones, as well as, spatial price equilibrium problems on general networks.

The SPEP has also had numerous applications, partially due to its flexibility and amenability to efficient solution.

One can include a variety of different policies into this framework, including tariffs and ad valorem tariffs.

Ecological Predator-Prey Networks Behave Like Spatial Price Networks



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Acknowledgments

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