OIM 413 Logistics and Transportation
Lecture 5: The System-Optimized (S-O) Problem

Professor Anna Nagurney

John F. Smith Memorial Professor
and
Director – Virtual Center for Supernetworks
Department of Operations & Information Management
User-optimization or decentralized decision-making behavior is relevant to congested urban transportation networks and to the Internet and electric power generation and distribution networks as well as certain financial networks.

System-optimization or centralized decision-making behavior is relevant to networks, including transportation and logistical ones, in which there is a central controller who seeks to route the traffic/flows so that the total cost in the network is minimized.

Examples of transportation networks in which system-optimization is the governing behavioral principle include freight networks – truck, rail and waterway.

Supply chains in the case of humanitarian operations and logistics would operate under the system-optimization principle.
Sea Freight

www.scanlogistics.com
Rail Freight

www.galleywinter.com
The system-optimized model is useful in many supply chain contexts including healthcare, blood supply chains, and humanitarian relief supply chains.
Healthcare Supply Chains
Blood Supply Chains
Humanitarian Relief
Amazon has revolutionized electronic commerce and is a key developer and user of models and algorithms based on operations research. At Amazon’s first Supply Chain Optimization Summit in October 2015 in Seattle with Dr. Greg Duncan, Chief Economist.

Below is a link to a video of an Amazon fulfillment center, complete with robots. https://www.youtube.com/watch?v=tMpsMt7ETi8
A central authority can route traffic according to his/her will (users can’t make their own choices).

Criterion or Objective: To minimize total cost in the network.

The total cost on a link \( a = \hat{c}_a(f_a) = c_a(f_a) \times f_a \),
where \( c_a(f_a) \) is the user link cost on \( a \) and \( f_a \) is the flow on link \( a \).

Recall that the total cost on a network can be expressed as:

\[
\sum_{a \in L} \hat{c}_a(f_a).
\]
The System-Optimization Problem

The system optimization (S-O) problem can, hence, be expressed as:

Minimize \( \sum_{a \in L} \hat{c}_a(f_a) \) [the total cost]

subject to the constraints:

\[ d_w = \sum_{p \in P_w} F_p, \text{ for all O/D pairs } w \in W, \]
\[ f_a = \sum_{p \in P} F_p \delta_{ap}, \text{ for all links } a \in L, \]
\[ F_p \geq 0, \text{ for all } p \in P. \]

Note: The constraints are identical to those in the U-O problem.
The System-Optimizing Conditions

The S-O Conditions

Theorem

A link flow pattern $f$, induced by a path flow pattern $F$, is system-optimizing, if and only if, there exists an ordering of paths: $p_1, \ldots, p_{s'}, p_{s'+1}, \ldots, p_{n_w}$ that connect each O/D pair $w \in W$, such that

\[
\hat{C}'_{p_1}(f) = \ldots = \hat{C}'_{p_{s'}}(f) = \mu_w \leq \hat{C}'_{p_{s'+1}}(f) \leq \ldots \leq \hat{C}'_{p_{n_w}}(f)
\]

\[
F_{pr} > 0; \ r = 1, \ldots, s';
\]

\[
F_{pr} = 0; \ r = s'+1, \ldots, n_w,
\]
where

\[ \hat{C}'_p(f) = \sum_{a \in L} \hat{c}'_a(f_a) \delta_{ap} = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}. \]

\[ \hat{C}'_p(f) \]: the marginal of the total cost on path \( p \),
\[ \hat{c}'_a(f_a) \]: the marginal of the total cost on link \( a \).

The above conditions must be satisfied for every O/D pair in the network.
The System-Optimizing Conditions

These conditions are equivalent to the Kuhn-Tucker conditions of the nonlinear optimization problem since the feasible set is convex and the user link cost functions are assumed to be continuous.

If the marginal cost functions are strictly increasing functions of the flows on the links, we are guaranteed a unique S-O link flow pattern since the objective function is strictly convex.
An Example

Determine the U-O and the S-O flow pattern for the following network:

\[ \begin{align*}
\text{The demand is } d_{xy} = 100 \text{ and the user link cost functions are:} \\
c_a(f_a) &= 3f_a + 1000, \quad c_b(f_b) = 2f_b + 1500.
\end{align*} \]

Recall that a flow pattern \( f^* \) would be U-O if: \( c_a = c_b \) and \( f_a^* > 0, f_b^* > 0 \), or \( c_a \geq c_b \) and \( f_a^* > 0, f_b^* = 0 \), or \( c_a \leq c_b \) and \( f_a^* > 0, f_b^* = 0 \).
Let’s apply the U-O EEA:

- Sort the $h_a$s: $1000 < 1500$.

- Compute

$$\lambda^1_w = \frac{d_w + \frac{h_{a_1}}{g_{a_1}}}{\frac{1}{g_{a_1}}} = \frac{100 + \frac{1000}{3}}{\frac{1}{3}} = 1300.$$  

- Check: Is: $1000 < 1300 \leq 1500$; Yes!

The critical $s = 1$, so

$$f^*_a = F^*_{p_1} = \frac{\lambda^1_w - h_{a_1}}{g_{a_1}} = \frac{1300 - 1000}{3} = 100, \quad f^*_b = F^*_{p_2} = 0.$$
A flow pattern would be S-O if:

\[ \hat{c}_a' = \hat{c}_b' \text{ and } f_a > 0, f_b > 0, \]

or \[ \hat{c}_a' \geq \hat{c}_b' \text{ and } f_b > 0, f_a = 0, \]

or \[ \hat{c}_a' \leq \hat{c}_b' \text{ and } f_a > 0, f_b = 0. \]

For this example, let’s see if both paths (which are links here) can be used.

Then we must have: \( \hat{c}_a' = \hat{c}_b' \).

**How do we construct these?**
The total costs on links $a$ and $b$ are:

$$\hat{c}_a(f_a) = (3f_a + 1000) \times f_a = 3f_a^2 + 1000f_a,$$

$$\hat{c}_b(f_b) = (2f_b + 1500) \times f_b = 2f_b^2 + 1500f_b.$$ 

Then the marginal total costs on these links are:

$$\hat{c}'_a = \frac{\partial \hat{c}_a}{\partial f_a} = 6f_a + 1000, \quad \hat{c}'_b = \frac{\partial \hat{c}_b}{\partial f_b} = 4f_b + 1500.$$ 

What else do we know?

$$f_a + f_b = d_w = 100 \Rightarrow f_b = 100 - f_a$$
Hence,

\[ \hat{c}'_a(f_a) = \hat{c}'_b(f_b) \] means that:

\[ 6f_a + 1000 = 4(100 - f_a) + 1500, \]
\[ 6f_a + 1000 = 1900 - 4f_a, \]
\[ 10f_a = 900, \ f_a = 90; \ f_b = 10. \]
The S-O pattern is distinct from the U-O pattern (except for certain very specific networks or a special cost structure).

The total cost under the **S-O pattern** is

\[ c_a = 1270; \quad c_b = 1520. \]

The total cost \( \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b = (1270)90 + (1520)10 = 129,500. \)

The total cost under the **U-O pattern** is:

\[ c_a = 1300; \quad c_b = 1500 \]

The total cost \( \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b = 1300(100) + 1500(0) = 130,000. \)

**Note:** The total cost under the S-O pattern is less than the total cost under the U-O pattern.

\[ 129,500 < 130,000! \]
The System-Optimizing (S-O) Exact Equilibration Algorithm

- **Step 1:** Sort the $h_{ai}$s in non-descending order and relabel accordingly. Set $r = 1$ and $h_{am+1} = \infty$.

- **Step 2:** Compute

$$\mu^r_w = \frac{d_w + \sum_{i=1}^{r} \frac{h_{ai}}{2g_{ai}}}{\sum_{i=1}^{r} \frac{1}{2g_{ai}}}.$$
**Step 3:** Check

If

\[ h_{ar} < \mu_w^r \leq h_{ar+1} \]

then STOP.

Set the critical \( s' = r \);

\[ F_{pr} = \frac{\mu_w^r - h_{ar}}{2g_{ar}} \quad r = 1, \ldots, s'; \]

\[ F_{pr} = 0 \quad r = s' + 1, \ldots, m. \]

Else, set \( r = r + 1 \) and goto Step 2.
The U-O Conditions

For each O/D pair \( w \), there exists an ordering of the paths:

\[
C_{p_1}(f^*) = \ldots = C_{p_s}(f^*) = \lambda_w \leq C_{p_{s+1}}(f^*) \leq \ldots \leq C_{p_{nw}}(f^*)
\]

\( F_{*p_r} > 0; \ r = 1, \ldots, s; \)

\( F_{*p_r} = 0; \ r = s + 1, \ldots, n_w. \)

Here user costs on used paths are “equilibrated or equal.”
The S-O Conditions

For each O/D pair \( w \), there exists an ordering of the paths:

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\hat{C}'_{p_1}(f) = \ldots = \hat{C}'_{p_{s'}}(f) = \mu_w \leq \hat{C}'_{p_{s'+1}}(f) \leq \ldots \leq \hat{C}'_{p_{m'}}(f)
\]

\( F_{p_r} > 0; \ r = 1, \ldots, s'; \)
\( F_{p_r} = 0; \ r = s' + 1, \ldots, m'. \)

Here the marginal total costs on used paths are “equilibrated” or equal.
Definition: U-O or Network Equilibrium – Fixed Demands
A path flow pattern \( F^* \), of nonnegative path flows, with O/D pair demand satisfaction, is said to be U-O or in equilibrium, if the following condition holds for each O/D pair \( w \in W \) and each path \( p \in P_w \):

\[
C_p(F^*) \begin{cases} 
= \lambda_w, & \text{if } F^*_p > 0, \\
\geq \lambda_w, & \text{if } F^*_p = 0.
\end{cases}
\]

Definition: S-O Conditions – Fixed Demands
A path flow pattern \( F \), of nonnegative path flows, with O/D pair demand satisfaction, is said to be S-O, if the following condition holds for each O/D pair \( w \in W \) and each path \( p \in P_w \):

\[
\hat{C}_p(F) \begin{cases} 
= \mu_w, & \text{if } F_p > 0, \\
\geq \mu_w, & \text{if } F_p = 0,
\end{cases}
\]

where \( \hat{C}_p(F) = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap} \).
**Question:** When does the U-O solution coincide with the S-O solution?

**Answer:** In a general network, when the user link cost functions are given by:

\[ c_a(f_a) = c_a^0 f_a^\beta, \]

for all links, with \( c_a^0 \geq 0 \), and \( \beta \geq 0 \).

In particular, if \( c_a(f_a) = c_a^0 \), that is, in the case of **uncongested networks**, this result always holds.
In a subsequent lecture we will show how policies, in the forms of tolls, that, once imposed, will guarantee that the S-O solution is, at the same time, U-O!

For more advanced formulations and associated theory, see Professor Nagurney’s Fulbright Network Economics lectures.

http://supernet.som.umass.edu/austria_lectures/fulmain.html