OIM 413 Logistics and Transportation Lecture 10: The Extended Model

Professor Anna Nagurney

John F. Smith Memorial Professor
and
Director – Virtual Center for Supernetworks
Department of Operations & Information Management

The Extended Model

Recall that the standard model assumes that the user cost on each link is a function solely of the flow on that link.

A more general model is the extended model in which the cost on a link can depend upon the entire vector of link flows.

This feature enables us to model more complex interactions on the transportation network.

Real-World Illustrations

Real-World Illustrations

The Extended Model

Hence, in the extended model, we have that the user link cost functions are of the form:

$$c_a = c_a(f), \qquad \forall a \in L,$$

where f denotes the vector of link flows such that $f = (f_a, f_b, ..., f_n)$.

The user travel cost on path p is, thus,

$$C_p = C_p(f) = \sum_{a \in L} c_a(f)\delta_{ap}.$$

The Extended Model

The Simplest Link Cost Structure (Congested)

The user link travel costs are linear:

$$c_a(f) = \sum_{b \in L} g_{ab} f_b + h_a, \qquad \forall a \in L.$$

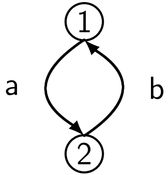
We expect that the main effect is from $g_{aa} f_a$.

The total link cost can then be expressed as:

$$\hat{c}_a(f) = \sum_{b \in L} g_{ab} f_b \times f_a + h_a f_a.$$

In particular, a cost on a link, c_a, may be independent of some of the link flows.

The Extended Model – An Example



Two way street: $a = (1, 2)$, $b = (2, 1)$, with user link costs and total link costs:

$$c_a(f) = g_{aa}f_a + g_{ab}f_b + h_a, \quad \hat{c}_a(f) = g_{aa}f_a^2 + g_{ab}f_bf_a + h_af_a,$$

$$c_b(f) = g_{bb}f_b + g_{ba}f_a + h_b, \quad \hat{c}_b(f) = g_{bb}f_b^2 + g_{ba}f_af_b + h_bf_b.$$

The Extended Model – The U-O / Equilibrium Conditions

Statement is the same as for the standard model, but user costs on links and paths require now more computation.

For every O/D pair w, there exists an ordering of the paths connecting w, such that:

$$C_{p_1}(f^*) = ... = C_{p_s}(f^*) = \lambda_w \leq C_{p_{s+1}}(f^*) \leq ... \leq C_{p_{n_w}}(f^*)$$

$F^*_{p_r} > 0; \; r = 1, ..., s;$
$F^*_{p_r} = 0; \; r = s+1, ..., n_w,$
where

$$C_p(f) = \sum_{a \in L} c_a(f)\delta_{ap}.$$

The Extended Model – The Associated Network

We pose the problem:

Given a transportation network $T = (G, d, c)$, where

G: graph of nodes and links,
d: demands,
c: cost structure,

is it possible to construct an associated network
$T^* = (G, d, c^*)$, with c^*: total link costs, so that

$C_p(f) = \hat{C}_p^{*'}(f)$ - marginal total costs, for all paths p?

The Extended Model

This is true if the Jacobian matrix of the user link cost functions, Λ, given by:

$$\Lambda = \begin{bmatrix} \frac{\partial c_a}{\partial f_a} & \frac{\partial c_a}{\partial f_b} & \cdots & \frac{\partial c_a}{\partial f_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial c_n}{\partial f_a} & \frac{\partial c_n}{\partial f_b} & \cdots & \frac{\partial c_n}{\partial f_n} \end{bmatrix}$$

is **symmetric.**

Hence, there exists an optimization reformulation of the user-optimized conditions if the Jacobian of the user link travel cost functions is symmetric.

Interpretation of the Symmetry Condition

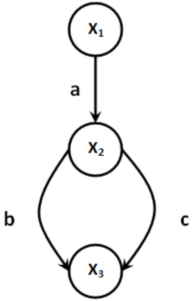
The **symmetry condition** means that

$$\frac{\partial c_a}{\partial f_b} = \frac{\partial c_b}{\partial f_a}, \quad \forall a, b.$$

In other words, the cost on a particular link is affected by the flow on another link in the same way that the cost on the other link is affected by the flow on the particular link.

A Numerical Example

Consider the following network:



The O/D pair is: $w_1 = (x_1, x_3)$ with $d_{w_1} = 100$. The paths are: $p_1 = (a, b)$ and $p_2 = (a, c)$. The user link cost functions are:

$$c_a(f) = 10f_a + 20,\ c_b(f) = 8f_b + 4f_c + 36,\ c_c(f) = 5f_c + 4f_b + 56.$$

Is there an optimization reformulation of the U-O conditions?

In other words, is there an optimization problem that, when solved, will give us the U-O flow pattern?

The answer is: **Yes.**

We construct the Jacobian of the user link cost functions, Λ, which is:

$$\Lambda = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

Clearly, Λ is symmetric.

Condition for Uniqueness of the U-O Link Flow Pattern

Moreover, if the Jacobian matrix of the link travel cost functions is positive definite, we are guaranteed a unique U-O link flow pattern.

How to verify if the Jacobian matrix is positive definite?

We know that **if the Jacobian matrix of the user link cost functions is symmetric and strictly diagonally dominant,** then it is positive definite, and the U-O (equilibrium) link flow pattern is unique.

Strict diagonal dominance of a square matrix with component $\{a_{ij}\}$ means that

$$a_{ii} > \sum_{j \neq i} |a_{ij}|$$

for each row of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

.

Recall that an $n \times n$ (square) matrix is positive definite if all of its eigenvalues are positive.

A matrix may still be positive definite even if it is not strictly diagonally dominant; however, that is an easy condition to check.

If a matrix is not symmetric, we can still use the strict diagonal dominance property to see whether it is positive definite.

For example, if the symmetric part of the Jacobian matrix Λ, given by $\frac{1}{2}(\Lambda + \Lambda^T)$, is strictly diagonally dominant then Λ is positive definite.

Is the U-O link flow pattern for the previous example unique?

Indeed, it is, since its Jacobian matrix:

$$\Lambda = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

is strictly diagonally dominant.

Note that each diagonal element in this matrix is strictly greater than the sum of the off-diagonal elements in its corresponding row.

The Extended Model – S-O Conditions

The S-O Conditions Correspond to the Kuhn-Tucker Conditions of Nonlinear Programming/Optimization

For every $w \in W$:

$$\hat{C}'_{p_1}(f) = \cdots = \hat{C}'_{p_{s'}}(f) = \lambda_w \leq \hat{C}'_{p_{s'+1}}(f) \leq \cdots \leq \hat{C}'_{p_{n_w}}(f)$$

$$F_{p_r} > 0; \;\; r = 1, \cdots, s';$$

$$F_{p_r} = 0; \;\; r = s' + 1, \cdots, n_w,$$

The Extended Model

where

by the Chain Rule

$$\hat{C}'_p(f) = \tfrac{\partial S}{\partial F_p} = \sum_{b \in L} \frac{\partial S}{\partial f_b} \times \frac{\partial f_b}{\partial F_p} =$$

$$\sum_{b \in L} \frac{\partial S}{\partial f_b} \times \delta_{bp} = \sum_{b \in L} \frac{\partial (\sum_{a \in L} \hat{c}_a(f))}{\partial f_b} \times \delta_{bp} = \sum_{a \in L} \sum_{b \in L} \frac{\partial \hat{c}_a}{\partial f_b} \delta_{bp}.$$

Equilibration Algorithms for the Extended Model

General equilibration algorithms have been developed to determine the S-O and the U-O (under the symmetry condition) flow patterns for the extended model.

In the case of asymmetric user link cost functions, in order to formulate and solve for the user-optimized flow patterns, one must apply the more advanced theory of variational inequalities (cf. Nagurney (1999) and the references therein).

Some Novel Applications of System-Optimization

In this course, we are providing the foundations for the formulation, analysis, and solution of complex network problems, operating under different behavioral principles.

Such principles can be applied to a spectrum of different applications.

Now, for illustration purposes, we highlight some recent applications of system-optimization.

Current Merger & Acquisition (M&A) Activity

Global merger and acquisition activity hit \$3.5 trillion in 2014, which is up 47% from the year before.

Some of the most visible recent mergers have occurred in the airline industry with Delta and Northwest completing their merger in October 2008 and United and Continental closing on the formation of United Continental Holdings Oct. 1, 2010.

American Airlines merged with US Airways in 2013.

Successful mergers can add tremendous value; **however, the failure rate is estimated to be between 74% and 83%.**

It is worthwhile to develop tools to better predict the associated strategic gains, which include, among others, cost savings.

Mergers and Acquisitions and Network Synergies

A successful merger depends on the ability to measure the anticipated synergy of the proposed merger.

◇ A. Nagurney (2009) "A System-Optimization Perspective for Supply Chain Network Integration: The Horizontal Merger Case," *Transportation Research E* **45**, 1-15.

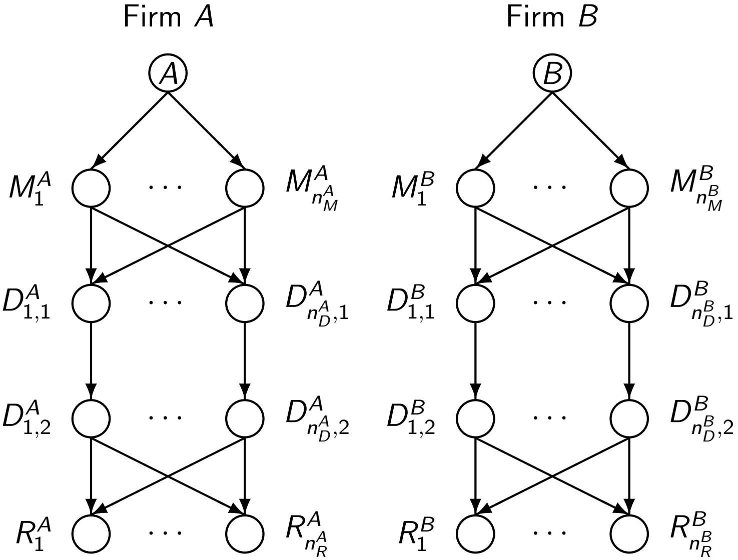


Figure: Case 0: Firms A and B Prior to Horizontal Merger

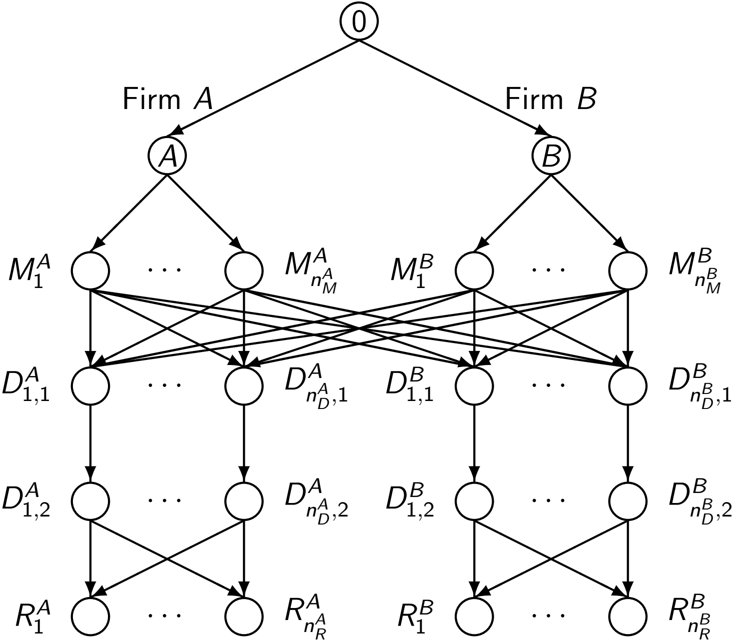


Figure: Case 1: Firms A and B Merge

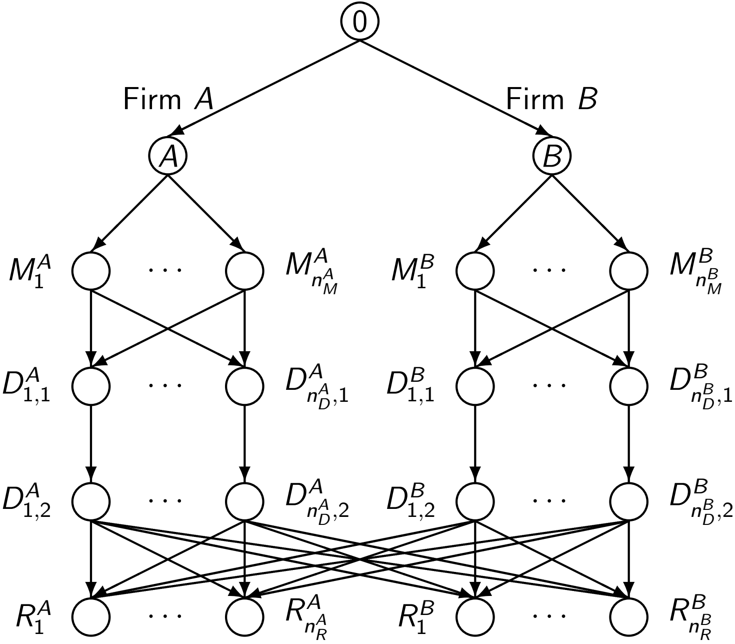


Figure: Case 2: Firms A and B Merge

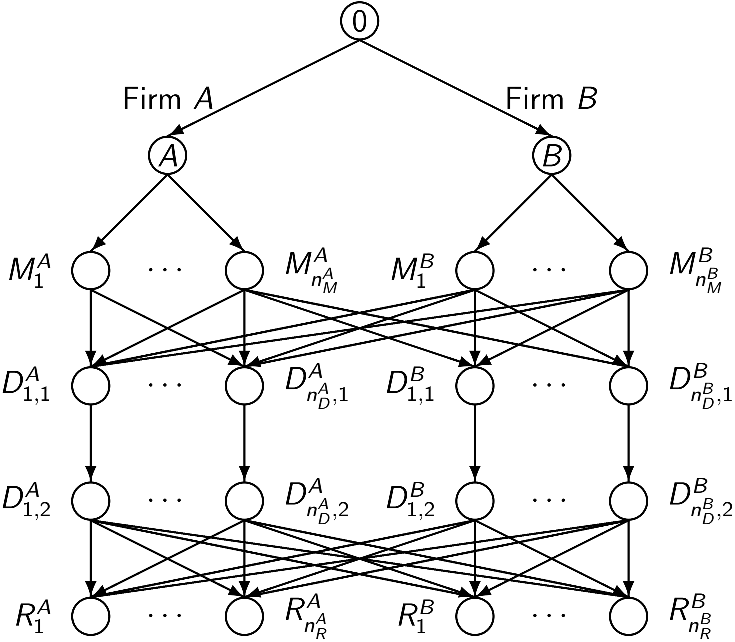


Figure: Case 3: Firms A and B Merge

Synergy Measure

The measure that we utilized in Nagurney (2009) to capture the gains, if any, associated with a horizontal merger Case i; $i = 1, 2, 3$ is as follows:

$$\mathcal{S}^i = \left[\frac{TC^0 - TC^i}{TC^0} \right] \times 100\%,$$

where TC^i is the total cost associated with the value of the objective function $\sum_{a \in L^i} \hat{c}_a(f_a)$ for $i = 0, 1, 2, 3$ evaluated at the optimal solution for Case i. Note that $\mathcal{S}^i$; $i = 1, 2, 3$ may also be interpreted as *synergy*.

This system-optimization model can also be applied to the teaming of organizations in the case of humanitarian operations.

Bellagio Conference on Humanitarian Logistics

See:
https://supernet.isenberg.umass.edu/hlogistics/hlogistics.html

References

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⇒ S. C. Dafermos and F. T. Sparrow (1969) The traffic assignment problem for a general network, *Journal of Research of the National Bureau of Standards*, 73B, pp. 91-118.

⇒ A. Nagurney (1999) *Network Economics: A Variational Inequality Approach*, second and revised edition, Kluwer Academic Publishers.

⇒ A. Nagurney and Q. Qiang (2009) *Fragile Networks: Identifying Vulnerabilities and Synergies in an Uncertain World*, John Wiley & Sons, Hoboken, New Jersey.