Network Equilibrium

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Outline of Masterclass on Network Equilibrium

Lecture 1: Overview of Network Equilibrium and Variational Inequalities

• Fundamental Theory, Traffic Network Equilibrium, and Nash Equilibrium

Lecture 2: Game Theory Network Models for Disaster Relief

• Generalized Nash Equilibrium

• Case Studies on Real-Life Disasters

Lecture 3: Perishable Product Supply Chains

• Capturing Perishability via Generalized Networks

• Applications to Food Supply Chains and More

Lecture 4: Cybercrime and Cybersecurity

• Nash Equilibrium, Nash Bargaining, System-Optimization

• Network Vulnerability

• Case Studies on Retail and Energy Sectors.

Overview of Network Equilibrium and Variational Inequalities

Outline

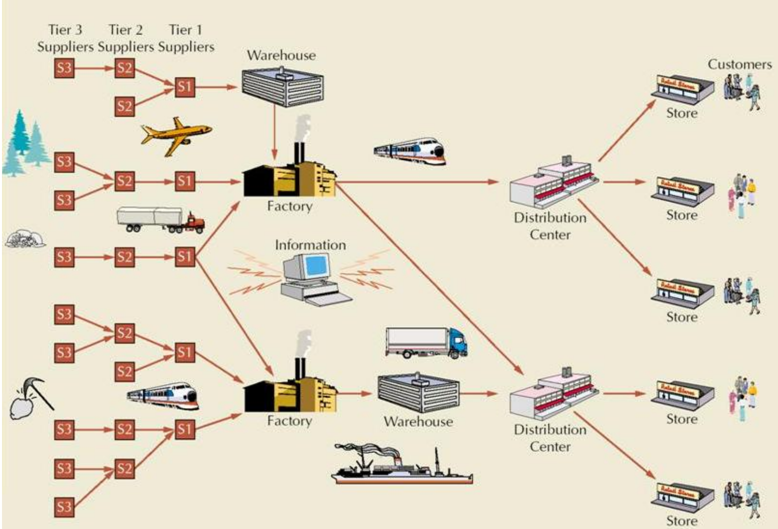
- Background and Motivation
- User-Optimization versus System-Optimization
- The Braess Paradox
- Variational Inequalities
- Variational Inequality Formulations of Traffic Network Equilibrium
- Nash Equilibrium and Oligopolies
- Summary

Background and Motivation

Characteristics of Networks Today

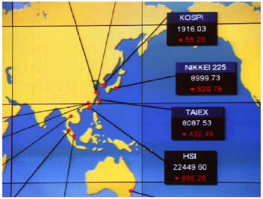
- **large-scale nature** and complexity of network topology;
- **congestion**, which leads to nonlinearities;
- **alternative behavior of users of the networks**, which may lead to paradoxical phenomena;
- **interactions among networks themselves**, such as the Internet with electric power networks, financial networks, and transportation and logistical networks;
- recognition of **the fragility and vulnerability of network systems**;
- policies surrounding networks today may have major impacts not only economically, but also **socially, politically, and security-wise**.

A General Supply Chain Network



Electric Power Generation and Distribution Networks

Financial Networks



Social Networks

In this Master Class we will be covering a variety of **nonlinear network flow problems**. The concept of *network equilibrium* owes much to **the study of congested transportation networks**, so we will begin with this topic, since this area of application has also driven many methodological advances.

Interestingly, the topic of congestion and its management was even a major issue in Roman times.

The Most Congested Cities in the World

Inrix recently released their comprehensive list of the most congested cities in the world, based on 2017 statistics.

Around the world, traffic congestion is increasingly problematic, with drivers in some cities spending upwards of 100 hours per year sitting in peak time traffic. Not only is this costing valuable time, but it is costing billions of dollars as well.

The Most Congested Cities in the World

- Los Angeles, California holds onto the #1 spot for most congested city in the world, with drivers averaging 102 hours spent in congestion during peak hours. NYC is in 3rd place, with NY drivers spending 91 hours sitting in traffic.

- London, England is the 7th most congested city in the world (and the second-most in Europe, after Moscow) with drivers spending about 74 additional hours per year driving in congestion.

The Study of Congested Transportation Networks Must Capture the Behavior of Users

Decentralized (Selfish) versus Centralized (Unselfish) Behavior

User-Optimization versus System-Optimization

Decision-makers select their cost-minimizing routes.

		User-Optimized
Decentralized	Selfish	U–O
vs. ⟺	vs. ⟺	vs.
Centralized	Unselfish	S–O
		System-Optimized

Flows are routed so as to minimize the total cost to society.

Two fundamental principles of travel behavior, due to Wardrop (1952), with terms coined by Dafermos and Sparrow (1969).

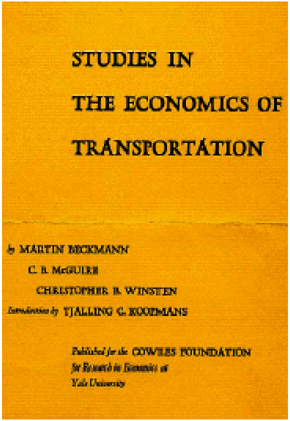
User-optimized (U-O) (network equilibrium) Problem – each user determines his/her cost minimizing route of travel between an origin/destination, until an equilibrium is reached, in which no user can decrease his/her cost of travel by unilateral action (in the sense of Nash).

System-optimized (S-O) Problem – users are allocated among the routes so as to minimize the total cost in the system, where the total cost is equal to the sum over all the links of the link's user cost times its flow.

The U-O problems, under certain simplifying assumptions, possess optimization reformulations. **But now we can handle cost asymmetries, multiple modes of transport, and different classes of travelers, without such assumptions.**

We Can State These Conditions Mathematically!

First Rigorous Formulation of U-O (Decentralized) and S-O (Centralized) Behavior



In 1956, Yale University Press published *Studies in the Economics of Transportation* **by Beckmann, McGuire, and Winsten.**

We celebrated the 50th anniversary of its publication at the 2005 INFORMS Meeting, San Francisco. (Professor Nagurney with Professors Beckmann, McGuire and Boyce)

The topology is represented by a mathematical graph consisting of nodes and links:

1. finite set of nodes N,

2. set of directed links (arcs, branches, edges) L represented by arrows.

Examples

In a **road network,** nodes are where traffic is generated or attracted to, or intermediate points. Links are the roads.

In an **airline network,** nodes are the airports, links are the air routes.

In a **railroad freight network,** nodes are loading/unloading points and switching points. Links are made up of tracks.

An Airline Example

Freight Trains

Origin/Destination Pairs of Nodes and Travel Demands

In addition to the set of nodes N and the set of links L used to represent the topology of a transportation network, we also denote the set of origin/destination (O/D) pairs by W and the set of all paths connecting all O/D pairs by P. An origin/destination pair of nodes represents where traffic originates and is destined to.

We denote the individual O/D pairs by w_1, w_2, etc., for a particular network, and we associate a travel demand d_w with each O/D pair w. We let P_w denote the set of all paths connecting O/D pair w.

In the first type of transportation network models that we will be studying we assume that the travel demands are known and fixed over the time horizon of interest, such as the morning or evening commuting period.

Path: A path (or route) is a sequence of links connecting an O/D pair $w = (x, y)$ of nodes. It can be represented by linking all the distinct links from the origin to the destination.

We exclude all cycles or loops.

We assume that the travel demand (rate) is constant in time over the time horizon under analysis (such as the commuting period).

Hence, the flows are constant in time. We are focusing on steady-state phenomena.

Some Real-World Network Sizes

In Chicago's Regional Transportation Network, there are **12,982 nodes, 39,018 links, and 2,297,945 origin/destination (O/D) pairs**.

In the Southern California Association of Governments model there are **3,217 origins and/or destinations, 25,428 nodes, and 99,240 links, plus 6 distinct classes of users**.

Notation:

x_p: flow on path p (measured in units/hrs., users/unit time).

$\Rightarrow$ *Path flows are always assumed to be nonnegative.*

f_a: flow on link a (measured in users/unit time).

Since the path flows are nonnegative, the link flows will be as well and this makes sense since we are dealing with traffic flows (vehicles, freight, messages, energy, etc.).

The general expression relating the travel demands and path flows:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,$$

that is, the travel demand for each O/D pair must be equal to the sum of the flows on paths that connect that O/D pair.

The general expression relating link flows and path flows:

$$f_a = \sum_p x_p \delta_{ap}, \quad \forall a \in L,$$

where

$$\delta_{ap} = \begin{cases} 1, & \text{if link } a \text{ is contained in path } p; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the flow on a link is equal to the sum on flows on paths that use / share that link.

These are the conservation of flow equations, which guarantee that every traveler arrives at his/her destination.

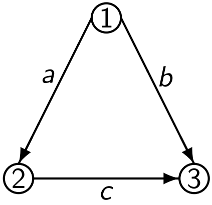


Figure: Network Example

Nodes: $1, 2, 3$; Links: a, b, c; O/D pair: $w_1 = (1, 3)$ with $d_{w_1} = 300$ vehicles/hr.

P_{w_1} denotes the set of paths connecting O/D pair w_1, where: $P_{w_1} = \{p_1, p_2\}$, with $p_1 = (a, c)$ and $p_2 = (b)$. Hence, $f_a = x_{p_1}$, $f_b = x_{p_2}$, and $f_c = x_{p_1}$.

If there are 100 cars per hour using path p_1 the volume of traffic on link a and link c is also 100. If there are 200 cars per hour using path p_2 then the traffic on link b is 200.

Cost is a disutility - Cost is a function of travel time, probability of an accident, scenery of a link.

Assume that all such factors can be grouped together into a disutility.

Both economists and traffic engineers work on determining travel cost functions on the links.

In particular, we consider travel cost functions of a user exercised via links of the network.

Cost Structure

In the **first generation** model, travel cost of users was assumed constant and could be determined **a priori** - such networks are known as *uncongested* networks.

In the **second generation** model, the networks are *congested*, that is, the user's travel cost depends on the characteristics of the link, but also on the flow on that link.

Clearly, the quality of the road/link also affects the cost and travel time.

The Standard Model Link Cost Structure and the BPR Function

In what is known as the *standard* model of transportation, we assume that the user cost on a link (i.e., the cost as perceived by a traveler or user) is a function of the flow on the link, that is,

$$c_a = c_a(f_a), \quad \forall a \in L.$$

To capture congestion, the user link cost is an increasing function of the flow.
A well-known cost function, in practice, is:

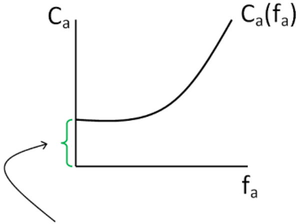
The Bureau of Public Roads (BPR) Cost Function

$$c_a = c_a^0 \; [\; 1 \; + \; \alpha \; (\frac{f_a}{t'_a})^{\beta} \;]$$

where c_a: travel time on link a, f_a: link flow on link a, c_a^0: free flow travel time, t'_α: "practical capacity" of link a, and α, β: model parameters (typically $\alpha = 0.15, \;\; \beta = 4$).

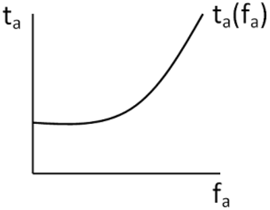
The Simplest Model

The Standard Model



uncongested cost (time)

Often one can substitute cost with time.



The Simplest User Link Cost Structure that Captures Congestion

Example: Simplest - Linear

$c_a(f_a) = g_a f_a + h_a$,

where $g_a, h_a > 0$ and constant.

g_a is the *congestion factor*.

h_a - is the *uncongested term* or the free-flow travel time on the link.

For example, we may have that:

$$c_a = 10f_a + 20.$$

Suppose now that we have 2 classes of users that perceive cost in different ways; this framework can also handle multiple modes of transportation.

More General Two Mode/Class Model Link Cost Structure

Link cost for link a as perceived by mode/class 1:

$$c_a^1 = c_a^1(f_a^1, f_a^2)$$

Link cost for link a as perceived by mode/class 2:

$$c_a^2 = c_a^2(f_a^1, f_a^2)$$

Can generalize the 2-class cost structure to k classes or modes.

$*$ But when you make the travel choice you choose paths, not links.

The Biggest Traffic Jam

The world's longest traffic jam in physical length and time almost closed the highway in the Inner-Mongolia region of China in August 2010.

According to the Chinese State News, the ongoing road work to expand the capacity of the highway was the primary cause for the delays. Trucks, including freight trucks, many of which carry coal, make up the majority of the traffic on this busy highway that feeds Beijing, China's capital city, and it is one of the busiest in the country and the world.

The traffic jam was over 100 kilometers/60 miles long, involved more than 10,000 cars, and lasted over 9 days!

Images from the Biggest Traffic Jam

Biggest Traffic Jam Ever - On the Beijing-Tibet Highway, August 2010

Path Cost Relationship to Link Costs

Let C_p denote the user's or personal travel cost along path p.

$$C_p = C_p(f) = \sum_{a \in L} c_a(f_a)\delta_{ap}, \quad \forall p \in P,$$

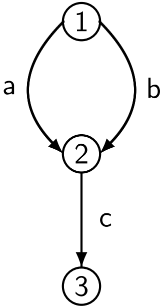
where f is a vector of all the link flows and

$$\delta_{ap} = \begin{cases} 1, & \text{if link } a \text{ is contained in path } p; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the user cost on a path is equal to the sum of user costs on links that make up the path.

Clearly, the time that it will take you to reach your destination from your point of origin will depend upon the sequence of roads that you take (and the traffic).

A Transportation Network Example



The O/D pair is: $w_1 = (1,3)$. The paths are:

$$p_1 = (a,c), \quad p_2 = (b,c).$$

The user link cost functions are:

$$c_a(f_a) = 10f_a + 5, \quad c_b(f_b) = f_b + 10, \quad c_c(f_c) = 5f_c + 5.$$

The travel demand is $d_{w_1} = 10$ and the path flows are: $x_{p_1} = 5, x_{p_2} = 5$. What is $C_{p_1} = ?$ What is $C_{p_2} = ?$

Another type of cost is the **social** or **total** cost.

The total cost on a link a is denoted by $\hat{c}_a$ and, for the standard model, is expressed as:

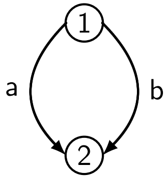
$$\hat{c}_a(f_a) = c_a(f_a) \times f_a.$$

If the user link cost c_a is linear, then the total link cost:

$$\hat{c}_a(f_a) = (g_a f_a + h_a) \times f_a = g_a f_a^2 + h_a f_a.$$

Hence, if the user cost function on a link is linear, then the total cost is *quadratic*.

Another Transportation Network Example



O/D pair $w_1 = (1, 2)$. The user link travel costs and the total link costs are:

$$c_a(f_a) = 10f_a + 5, \quad \hat{c}_a(f_a) = 10f_a^2 + 5f_a,$$

$$c_b(f_b) = 4f_b + 10, \quad \hat{c}_b(f_b) = 4f_b^2 + 10f_b.$$

Suppose that $p_1 = (a)$, $p_2 = (b)$, and $d_{w_1} = 20$, and

$$x_{p_1} = 10, \quad x_{p_2} = 10.$$

What are the user and total costs on the links a and b?

The Marginal Link Total Cost Structure

The marginal total cost $\hat{c}'_a(f_a) = \dfrac{\partial \hat{c}_a(f_a)}{\partial f_a}$,

where $\hat{c}_a(f_a) = c_a(f_a) \times f_a$.

In the uncongested model, the marginal total cost is a constant.

Hence, the marginal total cost in congested networks must be an increasing function of the link flows.

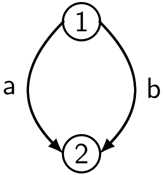
The Total Network Cost

Different ways to express the total network cost, denoted by S.

- $S(f) = \sum_{a \in L} \hat{c}_a(f_a)$
- $S(f) = \sum_{a \in L} c_a(f_a) \times f_a$
- $S(f, x) = \sum_{p \in P} C_p(f) \times x_p$

In the S-O problem, we seek to minimize the total cost in the network, subject to the conservation of flow equations.

An Example to Illustrate Concepts



The O/D pair $w_1 = (1, 2)$.
The user link travel costs and the total link costs are:

$$c_a(f_a) = 1f_a + 15, \quad \hat{c}_a(f_a) = 1f_a^2 + 15f_a,$$

$$c_b(f_b) = 2f_b + 5, \quad \hat{c}_b(f_b) = 2f_b^2 + 5f_b.$$

Hence, the marginal total costs on the links are:

$$\hat{c}'_a(f_a) = 2f_a + 15, \quad \hat{c}'_b(f_b) = 4f_b + 5.$$

The total cost expression for the network is:

$$S(f) = \hat{c}_a(f_a) + \hat{c}_b(f_b) = 1f_a^2 + 15f_a^2 + f_b^2 + 5f_b.$$

The U-O Conditions

Definition: U-O or Network Equilibrium – Fixed Demands

A path flow pattern x^**, with nonnegative path flows and O/D pair demand satisfaction, is said to be U-O or in equilibrium, if the following condition holds for each O/D pair* $w \in W$ *and each path* $p \in P_w$*:*

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0. \end{cases}$$

The S-O Conditions

Definition: S-O Conditions

A path flow pattern x with nonnegative path flows and O/D pair demand satisfaction, is said to be S-O, if for each O/D pair $w \in W$ and each path $p \in P_w$:

$$\hat{C}'_p(x) \begin{cases} = \mu_w, & \text{if} \quad x_p > 0, \\ \geq \mu_w, & \text{if} \quad x_p = 0, \end{cases}$$

where $\hat{C}'_p(x) = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}$, and μ_w is a Lagrange multiplier.

The Braess Paradox

The importance of behavior will now be illustrated through a famous example known as the Braess paradox which demonstrates what can happen under **U-O** as opposed to **S-O** behavior.

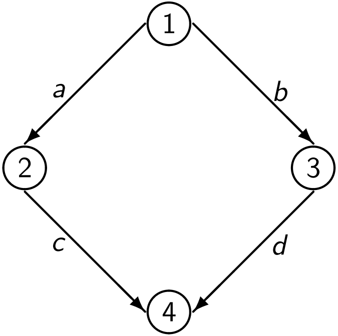
Although the paradox was presented in the context of transportation networks, it is relevant to other network systems in which decision-makers act in a noncooperative (competitive) manner.

Assume a network with a single O/D pair (1,4). There are 2 paths available to travelers: $p_1 = (a, c)$ and $p_2 = (b, d)$.

For a travel demand of **6**, the equilibrium path flows are $x^*_{p_1} = x^*_{p_2} = 3$ and

The equilibrium path travel cost is

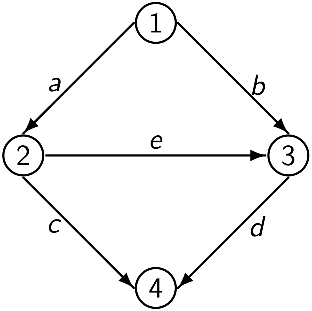
$C_{p_1} = C_{p_2} = 83.$



$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d.$$

Adding a Link Increases Travel Cost for All!



We now add a new link e with user cost: $c_e(f_e) = f_e + 10$. Adding a new link creates a new path $p_3 = (a, e, d)$.

The original flow distribution pattern is no longer an equilibrium pattern, since at this level of flow the cost on path p_3, $C_{p_3} = 70$. The new equilibrium flow pattern is: $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$.

The equilibrium path travel cost: $C_{p_1} = C_{p_2} = C_{p_3} = 92$.

The 1968 Braess article has been translated from German to English and appears as:

"On a Paradox of Traffic Planning,"

D. Braess, A. Nagurney, and T. Wakolbinger (2005) *Transportation Science* **39**, pp 446-450.

Über ein Paradoxon aus der Verkehrsplanung

Von D. BRAESS, Münster [1])

Eingegangen am 28. März 1968

Zusammenfassung: Für die Straßenverkehrsplanung möchte man den Verkehrsfluß auf den einzelnen Straßen des Netzes abschätzen, wenn die Zahl der Fahrzeuge bekannt ist, die zwischen den einzelnen Punkten des Straßennetzes verkehren. Welche Wege am günstigsten sind, hängt nun nicht nur von der Beschaffenheit der Straße ab, sondern auch von der Verkehrsdichte. Es ergeben sich nicht immer optimale Fahrzeiten, wenn jeder Fahrer nur für sich den günstigsten Weg heraussucht. In einigen Fällen kann sich durch Erweiterung des Netzes der Verkehrsfluß sogar so umlagern, daß größere Fahrzeiten erforderlich werden.

Summary: For each point of a road network let be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of the traffic flow. Whether a street is preferable to another one depends not only upon the quality of the road but also upon the density of the flow. If every driver takes that path which looks most favorable to him, the resultant running times need not be minimal. Furthermore it is indicated by an example that an extension of the road network may cause a redistribution of the traffic which results in longer individual running times.

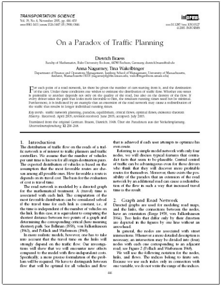
1. Einleitung

Für die Verkehrsplanung und Verkehrssteuerung interessiert, wie sich der Fahrzeugstrom auf die einzelnen Straßen des Verkehrsnetzes verteilt. Bekannt sei dabei die Anzahl der Fahrzeuge für alle Ausgangs- und Zielpunkte. Bei der Berechnung wird davon ausgegangen, daß von den möglichen Wegen jeweils der günstigste gewählt wird. Wie günstig ein Weg ist, richtet sich nach dem Aufwand, der zum Durchfahren nötig ist. Die Grundlage für die Bewertung des Aufwandes bildet die Fahrzeit.

Für die mathematische Behandlung wird das Straßennetz durch einen gerichteten Graphen beschrieben. Zur Charakterisierung der Bögen gehört die Angabe des Zeitaufwandes. Die Bestimmung der günstigen Stromverteilungen kann als gelöst betrachtet werden, wenn die Bewertung konstant ist, d. h., wenn die Fahrzeiten unabhängig von der Größe des Verkehrsflusses sind. Sie ist dann äquivalent mit der bekannten Aufgabe, den kürzesten Abstand zweier Punkte eines Graphen und den zugehörigen kritischen Pfad zu bestimmen [1], [5], [7].

Will man das Modell aber realistischer gestalten, ist zu berücksichtigen, daß die benötigte Zeit stark von der Stärke des Verkehrs abhängt. Wie die folgenden Untersuchungen zeigen, ergeben sich dann gegenüber dem Modell mit konstanter (belastungsunabhängiger) Bewertung z. T. völlig neue Aspekte. Dabei erweist sich schon eine Präzisierung der Problemstellung als notwendig; denn es ist zwischen dem Strom zu unterscheiden, der für alle am günstigsten ist,, und dem, der sich einstellt, wenn jeder Fahrer nur seinen eigenen Weg optimalisiert.

[1]) Priv.-Doz. Dr. Dietrich BRAESS, Institut für numerische und instrumentelle Mathematik, 44 Münster, Hüfferstr. 3 a.



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On a Paradox of Traffic Planning

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For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Key words: traffic network planning; paradox; equilibrium; critical flows; optimal flows; existence theorem
History: Received: April 2005; revision received: June 2005; accepted: July 2005.

Translated from the original German: Braess, Dietrich. 1968. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* **12** 258–268.

1. Introduction

The distribution of traffic flow on the roads of a traffic network is of interest to traffic planners and traffic controllers. We assume that the number of vehicles per unit time is known for all origin-destination pairs. The expected distribution of vehicles is based on the assumption that the most favorable routes are chosen among all possible ones. How favorable a route is depends on its travel cost. The basis for the evaluation of cost is travel time.

The road network is modeled by a directed graph for the mathematical treatment. A (travel) time is associated with each link. The computation of the most favorable distribution can be considered solved if the travel time for each link is constant, i.e., if the time is independent of the number of vehicles on the link. In this case, it is equivalent to computing the shortest distance between two points of a graph and determining the corresponding critical (here meaning shortest) path. See Bellman (1958), von Falkenhausen (1963), and Pollack and Wiebenson (1960).

In more realistic models, however, one has to take into account that the travel time on the links will strongly depend on the traffic flow. Our investigations will show that we will encounter new effects compared to the model with flow-independent costs. Specifically, a more precise formulation of the problem will be required. We have to distinguish between flow that will be optimal for all vehicles and flow that is achieved if each user attempts to optimize his own route.

Referring to a simple model network with only four nodes, we will discuss typical features that contradict facts that seem to be plausible. Central control of traffic can be advantageous even for those drivers who think that they will discover more profitable routes for themselves. Moreover, there exists the possibility of the paradox that an extension of the road network by an additional road can cause a redistribution of the flow in such a way that increased travel time is the result.

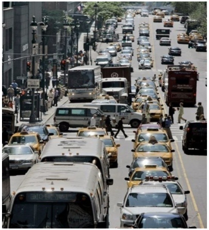
2. Graph and Road Network

Directed graphs are used for modeling road maps, and the links, the connections between the nodes, have an orientation (Berge 1958, von Falkenhausen 1966). Two links that differ only by their direction are depicted in the figures by one line without an arrowhead.

In general, the nodes are associated with street intersections. Whenever a more detailed description is necessary, an intersection may be divided into (four) nodes with each one corresponding to an adjacent road; see Figure 2 (Pollack and Wiebenson 1960).

We will use the following notation for the nodes, links, and flows. The indices belong to finite sets. Because we use each index only in connection with one variable, we do not write the range of the indices.

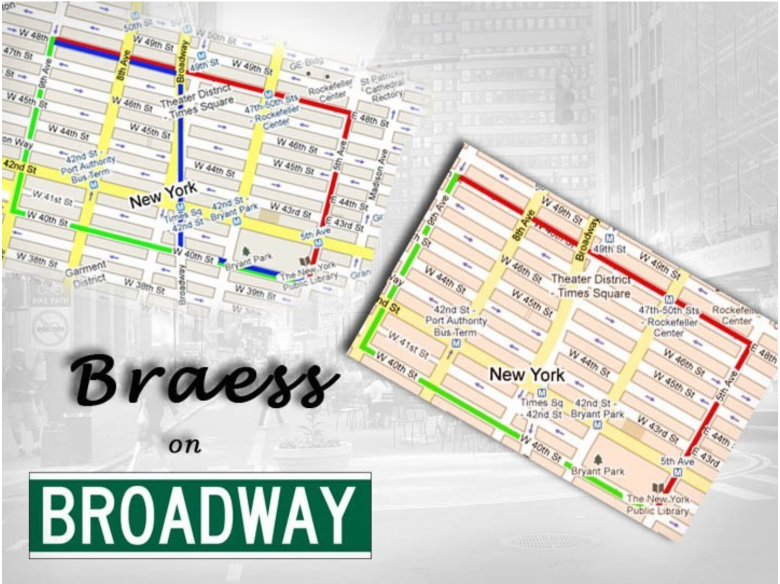
The Braess Paradox Around the World



1969 - Stuttgart, Germany - The traffic worsened until a newly built road was closed.

1990 - Earth Day - New York City - 42^{nd} Street was closed and traffic flow improved.

2002 - Seoul, Korea - A 6 lane road built over the Cheonggyecheon River that carried 160,000 cars per day and was perpetually jammed was torn down to improve traffic flow.

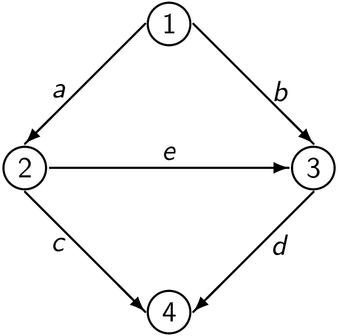

Braess
on
BROADWAY
New York
Theater District - Times Square

Interview on Broadway for *America Revealed* on March 15, 2011

http://video.pbs.org/video/2192347741/

Under S-O behavior, the total cost in the network is minimized, and the new route p_3, under the same demand, would not be used.

The Braess paradox never occurs in S-O networks.



Recall the Braess network with the added link e.

What happens as the demand increases?

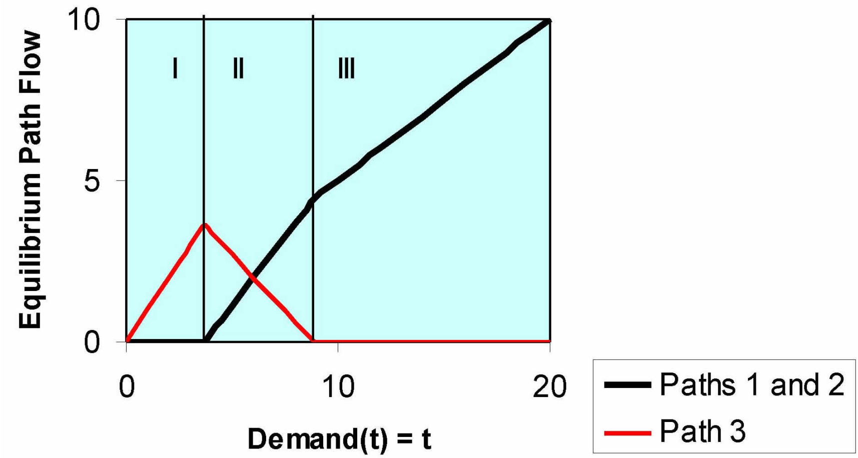
For Networks with Time-Dependent Demands

We Use Evolutionary Variational Inequalities

Radcliffe Institute for Advanced Study – Harvard University 2005-2006

Research with Professor David Parkes of Harvard University and Professor Patrizia Daniele of the University of Catania, Italy

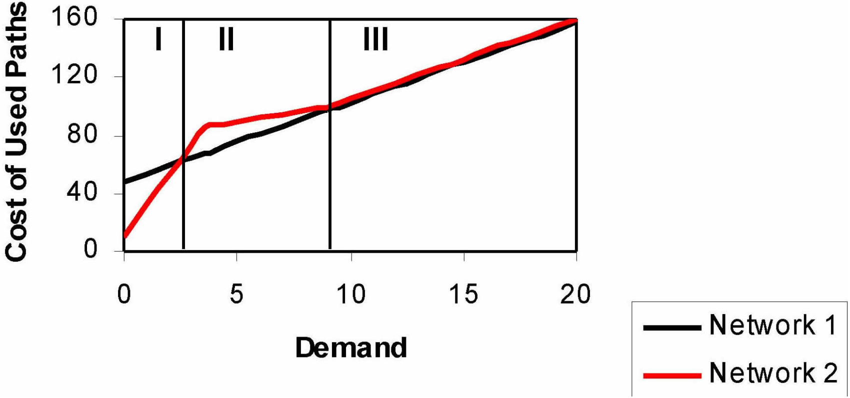
The U-O Solution of the Braess Network with Added Link (Path) and Time-Varying Demands Solved as an *Evolutionary Variational Inequality* (Nagurney, Daniele, and Parkes, *Computational Management Science* **4** (2007), pp 355-375).



In Demand Regime I, **Only the New Path is Used**.

In Demand Regime II, the travel demand lies in the range [2.58, 8.89], and **the Addition of a New Link (Path) Makes Everyone Worse Off**!

In Demand Regime III, when the travel demand exceeds 8.89, **Only the Original Paths are Used!**



The new path is never used, under U-O behavior, when the demand exceeds 8.89, even when the demand goes out to infinity!

Variational Inequalities

It was the identification by Dafermos (1980) that the Traffic Network Equilibrium conditions, as formulated by Smith (1979)), were, in fact, a variational inequality problem, that unveiled this theory for the formulation, analysis, and computations of solution to numerous equilibrium problems in operations research, economics, engineering, and other disciplines.

Background

To-date, problems which have been formulated and studied as variational inequality problems include:

- traffic network equilibrium problems
- spatial price equilibrium problems
- oligopolistic market equilibrium problems
- financial equilibrium problems
- migration equilibrium problems, as well as
- environmental network and ecology problems,
- knowledge network problems,
- electric power generation and distribution networks,
- supply chain network equilibrium problems, and even
- the Internet!

Observe that many of the applications explored to-date that have been formulated, studied, and solved as variational inequality problems are, in fact, **network problems**.

In addition, as we shall see, many of the advances in variational inequality theory have been spurred by needs in practice!

Variational inequality theory provides us with a tool for:

• formulating a variety of equilibrium problems;

• qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and

• providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

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It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

Also, as shown by Dupuis and Nagurney (1993), there is associated with a variational inequality problem, a *projected dynamical system*, which provides a natural underlying dynamics until an equilibrium state is achieved, under appropriate conditions.

This result further enriches the scope and reach of variational inequalities in terms of theory and especially applications!

The Variational Inequality Problem

Definition: Variational Inequality Problem

The finite - dimensional variational inequality problem, $\mathrm{VI}(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K} \subset R^N$*, such that*

$$F(X^*)^T \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{K},$$

or, equivalently,

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K} \tag{1}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in N-dimensional Euclidean space, as does "$\cdot$".*

Here we assume that all vectors are column vectors, except where noted.

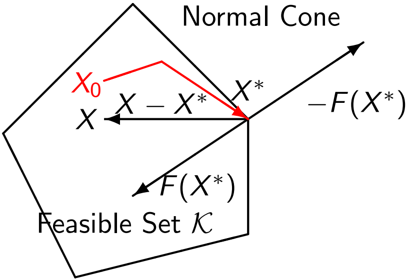
Another equivalent way of writing (1) is:

$$\sum_{i=1}^{N} F_i(X^*) \times (X_i - X_i^*) \geq 0, \quad \forall X \in \mathcal{K}. \tag{2}$$

$\mathcal{K}$ is the feasible set, X^* is the vector of solution values of the variables, and F is sometimes referred to as the function that enters the variational inequality.

Geometric Interpretation of VI($F, \mathcal{K}$) and a Projected Dynamical System (Dupuis and Nagurney, Nagurney and Zhang)

In particular, $F(X^*)$ is "orthogonal" to the feasible set $\mathcal{K}$ at the point X^*.



Associated with a VI is a Projected Dynamical System, which provides the natural underlying dynamics.

To model the **dynamic behavior of complex networks**, including supply chains, we utilize *projected dynamical systems* (PDSs) advanced by Dupuis and Nagurney (1993) in *Annals of Operations Research* and by Nagurney and Zhang (1996) in our book *Projected Dynamical Systems and Variational Inequalities with Applications*.

Such nonclassical dynamical systems are now being used in

evolutionary games (Sandholm (2005, 2011)),

ecological predator-prey networks (Nagurney and Nagurney (2011a, b)),

even **neuroscience** (Girard et al. (2008), and

dynamic spectrum model for cognitive radio networks (Setoodeh, Haykin, and Moghadam (2012)).

Optimization Problems

An optimization problem is characterized by its specific objective function that is to be maximized or minimized, depending upon the problem and, in the case of a constrained problem, a given set of constraints.

Possible objective functions include expressions representing profits, costs, market share, portfolio risk, etc. Possible constraints include those that represent limited budgets or resources, nonnegativity constraints on the variables, conservation equations, etc. Typically, an optimization problem consists of a single objective function.

Both **unconstrained** and **constrained** optimization problems can be formulated as variational inequality problems. The subsequent two propositions and theorem identify the relationship between an optimization problem and a variational inequality problem.

Proposition

Let X^* be a solution to the optimization problem:

$$\text{Minimize} \quad f(X) \tag{3}$$

$$\text{subject to:} \quad X \in \mathcal{K},$$

where f is continuously differentiable and $\mathcal{K}$ is closed and convex. Then X^* is a solution of the variational inequality problem:

$$\nabla f(X^*)^T \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{K}. \tag{4}$$

Proposition

If $f(X)$ is a convex function and X^* is a solution to $\mathrm{VI}(\nabla f, \mathcal{K})$, then X^* is a solution to the optimization problem (3).

If the feasible set $\mathcal{K} = R^N$, then the unconstrained optimization problem is also a variational inequality problem.

Relationship Between Optimization Problems and Variational Inequalities

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem.

In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then fix this relationship in a theorem.

Relationship Between Optimization Problems and Variational Inequalities

Theorem

Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix} \tag{5}$$

is symmetric and positive-semidefinite.

Then there is a real-valued convex function $f : \mathcal{K} \mapsto R^1$ *satisfying*

$$\nabla f(X) = F(X)$$

with X^* *the solution of* $\mathrm{VI}(F, \mathcal{K})$ *also being the solution of the mathematical programming problem:*

$$\text{Minimize} \quad f(X) \tag{6}$$

$$\text{subject to:} \quad X \in \mathcal{K}.$$

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, only when the symmetry condition and the positive-semidefiniteness condition hold.

The variational inequality, therefore, is the more general problem in that it can also handle a function $F(X)$ with an asymmetric Jacobian.

Consequently, variational inequality theory allows for the modeling, analysis, and solution of multimodal traffic network equilibrium problems, multicommodity spatial price equilibrium problems, general economic equilibrium problems, and numerous competitive supply chain network equilibrium problems since one no longer has to make a "symmetry" assumption of $F(X)$.

Fixed Point Problems

Fixed point theory has been used to formulate, analyze, and compute solutions to economic equilibrium problems. The relationship between the variational inequality problem and a fixed point problem can be made through the use of a projection operator. First, the projection operator is defined.

Lemma

Let $\mathcal{K}$ be a closed convex set in R^n. Then for each $x \in R^N$, there is a unique point $y \in \mathcal{K}$, such that

$$\|x - y\| \leq \|x - z\|, \quad \forall z \in K,$$

and y is known as the orthogonal projection of X on the set $\mathcal{K}$ with respect to the Euclidean norm, that is,

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \|X - z\|.$$

Projection Operator

A property of the projection operator which is useful both in qualitative analysis of equilibria and their computation is now presented.

Corollary

Let $\mathcal{K}$ be a closed convex set. Then the projection operator $P_{\mathcal{K}}$ is nonexpansive, that is,

$$\|P_{\mathcal{K}}X - P_{K}X'\| \leq \|X - X'\|, \quad \forall X, X' \in R^N.$$

Geometric Interpretation of Projection

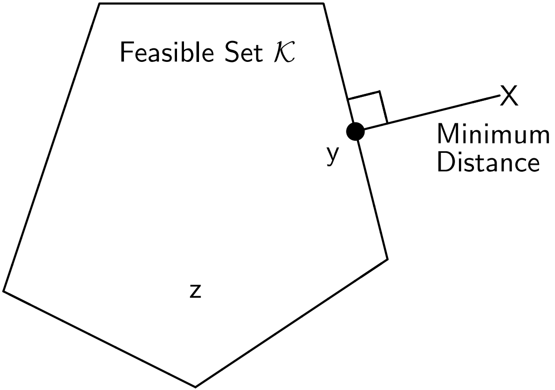


Figure: The projection y of X on the set $\mathcal{K}$

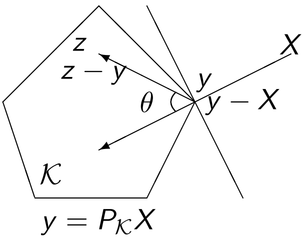


Figure: Geometric interpretation of $\langle (y - X), z - y \rangle \geq 0$, for $y = P_{\mathcal{K}}X$ and $y \neq P_{\mathcal{K}}X$

Relationship Between Fixed Point Problems and Variational Inequalities

The relationship between a variational inequality and a fixed point problem is as follows.

Theorem

Assume that $\mathcal{K}$ is closed and convex. Then $X^* \in \mathcal{K}$ is a solution of the variational inequality problem $\mathrm{VI}(F, \mathcal{K})$ if and only if for any $\gamma > 0$, X^* is a fixed point of the map

$$P_{\mathcal{K}}(I - \gamma F) : \mathcal{K} \mapsto \mathcal{K},$$

that is,

$$X^* = P_{\mathcal{K}}(X^* - \gamma F(X^*)).$$

Basic Existence and Uniqueness Results

Variational inequality theory is also a powerful tool in the qualitative analysis of equilibria. We now provide conditions for existence and uniqueness of solutions to $\mathrm{VI}(F, \mathcal{K})$ are provided.

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set $\mathcal{K}$ is compact. Indeed, we have the following:

Theorem: Existence Under Compactness and Continuity

If $\mathcal{K}$ is a compact convex set and $F(X)$ is continuous on $\mathcal{K}$, then the variational inequality problem admits at least one solution X^*.

Let VI_R denote the variational inequality problem:
Determine $x_R^* \in K_R$, such that

$$F(X_R^*)^T \cdot (y - X_R^*) \geq 0, \quad \forall y \in K_R.$$

Theorem

$\mathrm{VI}(F, \mathcal{K})$ *admits a solution if and only if there exists an* $R > 0$ *and a solution of* $\mathrm{VI_R}$, x_R^*, *such that* $\|x_R^*\| < R$.

Although $\|x_R^*\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition: Monotonicity

$F(X)$ *is monotone on* $\mathcal{K}$ *if*

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition: Strict Monotonicity

$F(X)$ *is strictly monotone on* $\mathcal{K}$ *if*

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) > 0, \quad \forall X^1, X^2 \in \mathcal{K},\ X^1 \neq X^2.$$

Basic Existence and Uniqueness Results

Definition: Strong Monotonicity

$F(X)$ is strongly monotone on $\mathcal{K}$ if for some $\alpha > 0$

$$\left[F(X^1) - F(X^2)\right]^T \cdot (X^1 - X^2) \geq \alpha \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition: Lipschitz Continuity

$F(X)$ is Lipschitz continuous on K if there exists an $L > 0$, such that

$$\|F(X^1) - F(X^2)\| \leq L\|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathcal{K}.$$

A uniqueness result is presented in the subsequent theorem.

Theorem: Uniqueness

Suppose that $F(X)$ is strictly monotone on $\mathcal{K}$. Then the solution is unique, if one exists.

Monotonicity is closely related to positive-definiteness.

Theorem

Suppose that $F(X)$ is continuously differentiable on $\mathcal{K}$ and the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix},$$

which need not be symmetric, is positive-semidefinite (positive-definite). Then $F(X)$ is monotone (strictly monotone).

Proposition

Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that $\nabla F(X)$ is strongly positive-definite. Then $F(X)$ is strongly monotone.

One obtains a stronger result in the special case where $F(X)$ is linear.

Corollary

Suppose that $F(X) = MX + b$, where M is an $N \times N$ matrix and b is a constant vector in R^N. The function F is monotone if and only if M is positive-semidefinite. F is strongly monotone if and only if M is positive-definite.

Proposition

Assume that $F : \mathcal{K} \mapsto R^N$ is continuously differentiable at $\bar{X}$. Then $F(X)$ is locally strictly (strongly) monotone at $\bar{X}$ if $\nabla F(\bar{X})$ is positive-definite (strongly positive-definite), that is,

$$v^T F(\bar{X}) v > 0, \quad \forall v \in R^N, v \neq 0,$$

$$v^T \nabla F(\bar{X}) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^N.$$

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set $\mathcal{K}$ is made.

Theorem: Existence and Uniqueness

Assume that $F(X)$ is strongly monotone. Then there exists precisely one solution X^* to $\mathrm{VI}(F, \mathcal{K})$.

Hence, in the case of an unbounded feasible set $\mathcal{K}$, strong monotonicity of the function F guarantees both existence and uniqueness. If $\mathcal{K}$ is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

A Contraction

Assume now that $F(X)$ is both strongly monotone and Lipschitz continuous. Then the projection $P_{\mathcal{K}}[X - \gamma F(X)]$ is a contraction with respect to X, that is, we have the following:

Theorem

Fix $0 < \gamma \leq \frac{\alpha}{L^2}$ where α and L are the constants appearing, respectively, in the strong monotonicity and the Lipschitz continuity condition definitions. Then

$$\|P_{\mathcal{K}}(X - \gamma F(X)) - P_{\mathcal{K}}(y - \gamma F(y))\| \leq \beta \|X - y\|$$

for all $X, y \in \mathcal{K}$, where

$$(1 - \gamma\alpha)^{\frac{1}{2}} \leq \beta < 1.$$

An immediate consequence of the Theorem and the Banach Fixed Point Theorem is:

Corollary

The operator $P_{\mathcal{K}}(X - \gamma F(X))$ *has a unique fixed point* X^*.

Variational Inequality Formulations of Traffic Network Equilibrium

Variational Inequality Formulations of Traffic Network Equilibrium

Theorem: Path Flow Formulation

A vector of path flows $x^* \in K^1$, where $K^1 \equiv \{x | x \geq 0, \text{ and } \sum_{p \in P_w} x_p = d_w, \forall w\}$ is a Traffic Network Equilibrium (U-O pattern) if and only if it satisfies the VI problem:

$$\sum_w \sum_{p \in P_w} C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K^1.$$

Variational Inequality Formulations of Traffic Network Equilibrium

Theorem: Link Flow Formulation

A vector of link flows $f^* \in K^2$, where
$K^2 \equiv \{\exists x | x \geq 0, \text{ and } \sum_{p \in P_w} x_p = d_w, \forall w, f_a = \sum_{p \in P} x_p \delta_{ap}, \forall a\}$
is a Traffic Network Equilibrium (U-O pattern) if and only if it satisfies the VI problem:

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2.$$

Nash Equilibrium and Oligopolies

Oligopoly theory dates to Cournot (1838), who investigated competition between two producers, the so-called duopoly problem, and is credited with being the first to study noncooperative behavior.

In an oligopoly, it is assumed that there are several firms, which produce a product and the price of the product depends on the quantity produced.

Examples of oligopolies include large firms in computer, automobile, chemical or mineral extraction industries, among others.

Nash Equilibrium

Nash (1950, 1951) subsequently generalized Cournot's concept of an equilibrium for a behavioral model consisting of n agents or players, each acting in his/her own self-interest, which has come to be called a noncooperative game.

The Nobel Laureate John F. Nash

www.search.tvnz.co.nz

Specifically, consider m players, each player i having at his/her disposal a strategy vector $X_i = \{X_{i1}, \ldots, X_{in}\}$ selected from a closed, convex set $K_i \subset R^n$, with a utility function $U_i : K \mapsto R^1$, where $K = K_1 \times K_2 \times \ldots \times K_m \subset R^{mn}$.

The rationality postulate is that each player i selects a strategy vector $X_i \in K_i$ that maximizes his/her utility level $U_i(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_m)$ given the decisions $(X_j)_{j \neq i}$ of the other players.

In this framework one then has:

Definition: Nash Equilibrium

A Nash equilibrium is a strategy vector

$$X^* = (X_1^*, \ldots, X_m^*) \in K,$$

such that

$$U_i(X_i^*, \hat{X}_i^*) \geq U_i(X_i, \hat{X}_i^*), \quad \forall X_i \in K_i, \forall i, \tag{7}$$

where $\hat{X}_i^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_m^*)$.

It has been shown (cf. Hartman and Stampacchia (1966) and Gabay and Moulin (1980)) that Nash equilibria satisfy variational inequalities. In the present context, under the assumption that each U_i is continuously differentiable on K and concave with respect to X_i, one has

Theorem: Variational Inequality Formulation of Nash Equilibrium

Under the previous assumptions, X^* is a Nash equilibrium if and only if $X^* \in K$ is a solution of the variational inequality

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K, \tag{8}$$

where $F(X) \equiv (-\nabla_{X_1} U_1(X), \ldots, -\nabla_{X_m} U_m(X))$ is a row vector and where $\nabla_{X_i} U_i(X) = (\frac{\partial U_i(X)}{\partial X_{i1}}, \ldots, \frac{\partial U_i(X)}{\partial X_{in}})$.

Proof: Since U_i is a continuously differentiable function and concave with respect to X_i, the equilibrium condition (7), for a fixed i, is equivalent to the variational inequality problem

$$-\langle \nabla_{X_i} U_i(X^*), X_i - X_i^* \rangle \geq 0, \quad \forall x_i \in K_i, \tag{9}$$

which, by summing over all players i, yields (8). □

If the feasible set K is compact, then existence is guaranteed under the assumption that each U_i is continuously differentiable. Rosen (1965) proved existence under similar conditions. Karamardian (1969), on the other hand, relaxed the assumption of compactness of K and provided a proof of existence and uniqueness of Nash equilibria under the strong monotonicity condition.

As shown by Gabay and Moulin (1980), the imposition of a coercivity condition on $F(X)$ will guarantee existence of a Nash equilibrium X^* even if the feasible set is no longer compact. Moreover, if $F(X)$ satisfies the strict monotonicity condition, uniqueness of X^* is guaranteed, provided that the equilibrium exists.

We now consider the classical oligopoly problem in which there are m producers involved in the production of a homogeneous commodity. The quantity produced by firm i is denoted by q_i, with the production quantities grouped into a column vector $q \in R^m$. Let f_i denote the cost of producing the commodity by firm i, and let p denote the demand price associated with the good. Assume that

$$f_i = f_i(q_i), \tag{10}$$

$$p = p(\sum_{i=1}^{m} q_i). \tag{11}$$

The profit for firm i, u_i, can then be expressed as

$$u_i(q) = p(\sum_{i=1}^{m} q_i)q_i - f_i(q_i). \tag{12}$$

Assuming that the competitive mechanism is one of noncooperative behavior, in view of the Theorem, one can write down the following Theorem.

Theorem: Variational Inequality Formulation of Classical Cournot-Nash Oligopolistic Market Equilibrium

Assume that the profit function $u_i(q)$ is concave with respect to q_i, and that $u_i(q)$ is continuously differentiable. Then $q^* \in R^m_+$ is a Nash equilibrium if and only if it satisfies the variational inequality

$$\sum_{i=1}^{m}\left[\frac{\partial f_i(q_i^*)}{\partial q_i} - \frac{\partial p(\sum_{i=1}^{m} q_i^*)}{\partial q_i} q_i^* - p(\sum_{i=1}^{m} q_i^*)\right] \times [q_i - q_i^*] \geq 0,$$

$$\forall q \in R^m_+. \qquad (13)$$

We now establish the equivalence between the classical oligopoly model and a network equilibrium model. For a graphic depiction, see the Figure below.

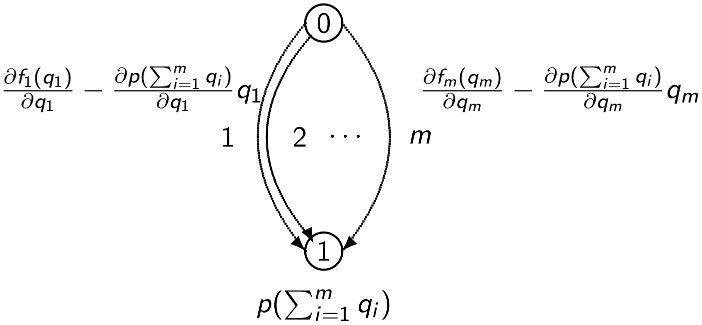


Figure: Network equilibrium representation of an oligopoly model

An Equivalence

Let 0 be the origin node and 1 the destination node. Construct m links connecting 0 to 1. The cost on a link i is then given by:

$$\left[\frac{\partial f_i(q_i)}{\partial q_i} - \frac{\partial p(\sum_{i=1}^m q_i)}{\partial q_i} q_i\right],$$

and the inverse demand associated with the origin/destination (O/D) pair $(0,1)$ is given by $p(\sum_{i=1}^m q_i)$. The flow on link i corresponds to q_i and the demand associated with the O/D pair to $\sum_{i=1}^m q_i$.

Hence, the classical oligopoly model is isomorphic to a network equilibrium model with a single O/D pair, m paths corresponding to the m links, and with elastic demand.

Indeed, there are a remarkable number of problems that are isomorphic to traffic network equilibrium problems.

Summary

In this lecture, the fundamental qualitative tools for the formulation and analysis of finite-dimensional variational inequalities have been provided.

In subsequent lectures, we will describe algorithms and a plethora of applications of variational inequalities and game theory models.

References

Below are many of the citations referenced in the lecture as well as other relevant ones.

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Thank You!

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