Lecture 6
Equilibration Algorithms for a General Transportation Network

Dr. Anna Nagurney

John F. Smith Memorial Professor
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003
Assume that the user cost functions on the links are still linear and increasing, that is,

\[ c_a(f_a) = g_a f_a + h_a, \quad g_a, h_a > 0 \]

but now the transportation network need no longer look like this:
It can have any topology:
Some Real-World Transportation Network Examples

The Massachusetts Turnpike

Photo by: Central Artery / Tunnel Project [Massachusetts Turnpike Authority]
Some Real-World Transportation Network Examples

The Los Angeles I110-I105 Interchange
The Atlanta I85-I285 Highway
We will now present **General Equilibration Algorithms** (GEAs) for the determination of either the U-O or the S-O flow pattern on a transportation network. These algorithms can be applied to compute the flows for networks with any topology.

We assume that the user link travel costs are given by:

\[ c_a(f_a) = g_a f_a + h + a, \quad \forall a \in L. \]

The total link travel costs are then:

\[ \hat{c}_a(f_a) = g_a f_a^2 + h_a, \quad \forall a \in L. \]

These algorithms are iterative algorithms and are guaranteed to converge to the unique link flow pattern. They were developed by Dafermos and Sparrow (1969), where convergence was also established, and have an appealing intuitive interpretation as dynamic adjustment processes.
A General Equilibration Algorithm (U-O)

1. Initialization
Initialize with a feasible path flow and link flow pattern. Set $i = 1$.

2. Equilibration of O/D pair $w_i$:
Set
$r = \text{path } p \text{ with } F_p > 0 \text{ and maximum } C_p \text{ in } w_i,$
$q = \text{path } p \text{ with minimum } C_p \text{ in } w_i.$

Compute the reallocation of path flow

$$\Delta' = \frac{C_r(f) - C_q(f)}{\sum_{a \in L} g_a (\delta_{aq} - \delta_{ar})^2}.$$
The reallocation: \( \delta = \min\{\delta', F_r\} \).

Update: new \( F_q = F_q + \delta \),
new \( F_r = F_r - \delta \),
all other \( F_p \)s stay the same.

Continue Step 2 until O/D pair \( w_i \) satisfies the U-O conditions within a prespecified tolerance.
3. Go to next O/D pair:
Set \( i = i + 1 \); If \( i \leq J \), where \( J = \# \) of O/D pairs, goto Step 2; else goto Step 4.

4. Overall Convergence Verification
If all O/D pairs are equilibrated (within the prespecified convergence tolerance) then stop; the U-O pattern has been computed; else, set \( i = 1 \), and goto Step 2.
A General Equilibration Algorithm (S-O)

1. Initialization
Initialize with a feasible path flow and link flow pattern. Set $i = 1$.

2. Equilibration of O/D pair $w_i$:
Set
$r = \text{path } p \text{ with } F_p > 0 \text{ and maximum } \hat{C}'_p \text{ in } w_i,$
$q = \text{path } p \text{ with minimum } \hat{C}'_p \text{ in } w_i.$
Compute the reallocation of path flow

$$\Delta' = \frac{\hat{C}'_r(f) - \hat{C}'_q(f)}{2 \sum_{a \in L} g_a (\delta_{aq} - \delta_{ar})^2}.$$
The reallocation: $\delta = \min\{\delta', F_r\}$.

Update: new $F_q = F_q + \delta$,
new $F_r = F_r - \delta$,
all other $F_p$’s stay the same.

Continue Step 2 until O/D pair $w_i$ satisfies the S-O conditions within a prespecified tolerance.
3. Go to next O/D pair:
Set $i = i + 1$; If $i \leq J$, where $J = \#$ of O/D pairs, goto Step 2; else goto Step 4.

4. Overall Convergence Verification
If all O/D pairs are equilibrated (within the prespecified tolerance) then stop; the S-O pattern has been computed; else, set $i = 1$, and goto Step 2.
A Numerical Example

We now present an example for which we computed both the U-O and the S-O flow patterns using the respective general equilibration algorithm just given. The network topology is given below.

The user link cost functions are:

\[ c_a(f_a) = 6f_a + 1, \]
\[ c_b(f_b) = f_b + 4, \]
\[ c_c(f_c) = 2f_c + 3, \]
\[ c_d(f_d) = 3f_d + 1, \]
\[ c_e(f_e) = 2f_e + 1. \]
The U-O Solution

The O/D pairs are: \( w_1 = (1, 2) \) and \( w_2 = (1, 4) \), with demands \( d_{w_1} = 40 \) and \( d_{w_2} = 80 \).
The paths are: \( p_1 = a \), \( p_2 = (e, d) \), \( p_3 = b \), and \( p_4 = (e, c) \).
Both the U-O and the S-O general equilibration algorithms (GEAs) were implemented in software.
The computed U-O solution obtained via the U-O GEA: the equilibrium path flows were:

\[
F^*_{p_1} = 19.7059, \quad F^*_{p_2} = 20.2941, \quad F^*_{p_3} = 72.1176, \quad F^*_{p_4} = 7.8824,
\]
and the equilibrium link flows were:

\[
f^*_a = 19.7059, \quad f^*_b = 72.1176, \quad f^*_c = 7.8824, \quad f^*_d = 20.2941, \quad f^*_e = 28.1765.
\]

The user path costs for paths connecting the first O/D pair were:
\( C_{p_1} = C_{p_2} = 119.2353 \). The user path costs for paths connecting the second O/D pair were:
\( C_{p_3} = C_{p_4} = 76.1176 \).
The S-O Solution

The S-O solution, in turn, computed via the S-O GEA was: the optimal path flows were:

\[ F_{p_1} = 19.6569, \ F_{p_2} = 20.3431, \ F_{p_3} = 72.1373, \ F_{p_4} = 7.8627, \]

and the optimal link flows were:

\[ f_a = 19.6569, \ f_b = 72.1373, \ f_c = 7.8627, \ f_d = 20.3431, \ f_e = 28.2059. \]

The marginal total path costs evaluated at the S-O solution were as follows: For O/D pair \( w_1 \):

\[ \hat{C}'_{p_1} = \hat{C}'_{p_2} = 236.8824; \]

for O/D pair \( w_2 \):

\[ \hat{C}'_{p_3} = \hat{C}'_{p_4} = 148.2745. \]

